# Reconnect '04
# A Couple of General Classes of Cutting Planes

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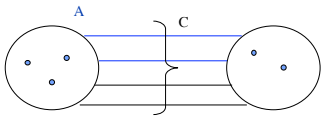
## Knapsack Cover (KC) Inequalities




$$u(A) = \sum_{e \in A} u_A < D(C)$$

residual $D(A) = D - u(A)$

$u_A(e) = \min(u_e, D(A))$

KC: $\sum_{e \in C-A} u_A(e) x_e \geq D(A)$



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## Moving Away from Graphs

The cuts apply to more general $a^T x \geq b$

For this discussion, assume $a \geq 0$ and $x \in \{0,1\}^n$

Let $I$ be a set of variable indices such that $\sum_{i \in I} a_i < b$

residual $r = b - \sum_{i \in I} a_i$

$a'_i = \min(a_i, r)$

cover cut: $\sum_{j \in N-I} a'_j x_j \geq r$



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## Cover Cuts

We can remove the assumption that $a \geq 0$

Consider a general inequality $a^+x^+ - a^-x^- \geq b,\ a^+, a^- \geq 0$

Set $y^- = 1 - x^- \geq 0$

$$a^+x^+ - a^-(1 - y^-) \geq b$$

$$a^+x^+ + a^-y^- \geq b + a^-$$

Apply a regular cover cut to $(a^+, y^-)$ and substitute $1 - x^-$ for $y^-$



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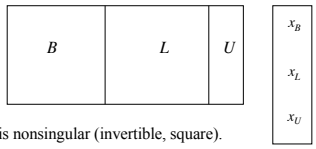
## Review: Linear Programming Basis

What does a corner look like algebraically?

$Ax=b$

Partition $A$ matrix into three parts




where B is nonsingular (invertible, square).

Reorder $x$: $(x_B, x_L, x_U)$

We have $Bx_B + Lx_L + Ux_U = b$



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## A Basic Solution

We have $Bx_B + Lx_L + Ux_U = b$

Set all members of $x_L$ to their lower bound.

Set all members of $x_U$ to their upper bound.

Let $b' = b - Lx_L - Ux_U$ (this is a constant because bounds $\ell$ and $u$ are)

Thus we have $Bx_B = b'$

Set $x_B = B^{-1}b'$

So we can express each basic variable in the current optimal LP solution x* as a function of the nonbasic variables.



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## Gomory Cuts

Assume we have a pure integer program (not necessarily binary)

Express each basic variable in the current optimal LP solution x* as a function of the nonbasic variables (tableau):

$$x_i = \sum_{x_j \in x_L} g_j \left(x_j - \ell_j\right) + \sum_{x_j \in x_U} g_j \left(u_j - x_j\right) + x_i^*$$

fr($g_j$) is the fractional part of $g_j$

Split $g_j$ into integral and fractional pieces:

$$x_i = \sum_{x_j \in x_L} \lfloor g_j \rfloor \left(x_j - \ell_j\right) + \sum_{x_j \in x_L} \mathrm{fr}\left(g_j\right)\left(x_j - \ell_j\right) + \sum_{x_j \in x_U} \lfloor g_j \rfloor \left(u_j - x_j\right) + \sum_{x_j \in x_U} \mathrm{fr}\left(g_j\right)\left(u_j - x_j\right) + x_i^*$$



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## Gomory Cuts

$$x_i = \sum_{x_j \in x_L} \lfloor g_j \rfloor \left(x_j - \ell_j\right) + \sum_{x_j \in x_L} \mathrm{fr}\left(g_j\right)\left(x_j - \ell_j\right) + \sum_{x_j \in x_U} \lfloor g_j \rfloor \left(u_j - x_j\right) + \sum_{x_j \in x_U} \mathrm{fr}\left(g_j\right)\left(u_j - x_j\right) + x_i^*$$

$$x_i - \sum_{x_j \in x_L} \lfloor g_j \rfloor \left(x_j - \ell_j\right) - \sum_{x_j \in x_U} \lfloor g_j \rfloor \left(u_j - x_j\right) = \sum_{x_j \in x_L} \mathrm{fr}\left(g_j\right)\left(x_j - \ell_j\right) + \sum_{x_j \in x_U} \mathrm{fr}\left(g_j\right)\left(u_j - x_j\right) + x_i^*$$

$\geq 0$ because $\ell_j \leq x_j \leq u_j$

$$x_i - \sum_{x_j \in x_L} \lfloor g_j \rfloor \left(x_j - \ell_j\right) - \sum_{x_j \in x_U} \lfloor g_j \rfloor \left(u_j - x_j\right) \geq x_i^*$$



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## Gomory Cuts

$$x_i - \sum_{x_j \in x_L} \lfloor g_j \rfloor (x_j - \ell_j) - \sum_{x_j \in x_U} \lfloor g_j \rfloor (u_j - x_j) \geq x_i^*$$

In a feasible solution $x_i$ is integral (pure integer program), so the whole left side is integral. Thus the right side must be as well:

$$x_i - \sum_{x_j \in x_L} \lfloor g_j \rfloor (x_j - \ell_j) - \sum_{x_j \in x_U} \lfloor g_j \rfloor (u_j - x_j) \geq \lceil x_i^* \rceil$$

This is (one type of) **Gomory Cut.**





## Global Validity

Cuts like the TSP subtour elimination cuts are globally valid (apply to all subproblems).

- Can be shared

Recall the key step for Gomory cuts:

$$x_i - \sum_{x_j \in x_L} \lfloor g_j \rfloor (x_j - \ell_j) - \sum_{x_j \in x_U} \lfloor g_j \rfloor (u_j - x_j) =$$
$$\sum_{x_j \in x_L} \mathrm{fr}(g_j)(x_j - \ell_j) + \sum_{x_j \in x_U} \mathrm{fr}(g_j)(u_j - x_j) + x_i^*$$

$\geq 0$ because $\ell_j \leq x_j \leq u_j$





## Global Validity

We require $\ell_j \leq x_j \leq u_j$ for the $\ell_j$ and $u_j$ in effect at the subproblem where the Gomory cut was generated.

- Gomory cuts are globally valid for binary variables
  - Need fixed at 1 to be fixed at upper and fixed at 0 to be at lower
- Gomory cuts are not generally valid for general integer variables