Testing Orthogonal Shapes

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Abstract

A *testing algorithm* takes a model and produces a set of points that can be used to test whether or not an unknown object is sufficiently similar to the model. A testing algorithm performs a complementary task to that performed by a learning algorithm, which takes a set of examples and builds a model that succinctly describes them. Testing can also be viewed as a type of geometric probing that uses *point probes* (i.e. test points) to verify that an unknown geometric object is similar to a given model. In this paper we examine the problem of verifying orthogonal shapes using test points. In particular, we give testing algorithms for sets of disjoint rectangles in two and higher dimensions and for general orthogonal shapes in 2-D and 3-D. This work is a first step towards developing efficient testing algorithms for objects with more general shapes, including those with non-orthogonal and curved surfaces.

Keywords: Testing, probing, helpful teacher learning

[1]Supported in part by NSF Grant CCR-9112976 and ONR Grant N00014-92-J-1254.
[2]Supported in part by NSF Grants IRI-9116843 and IRI-9223591.

1 Introduction

Many tasks in the physical world require that objects be distinguished from other objects. For some of these tasks, we would like to know how to distinguish objects strictly through the sense of touch. For example, suppose that a robot on a manufacturing assembly line must pick a specific part out of a bin, where all parts in the bin meet certain shape and size restrictions. Some of the objects in the bin may be quite similar to the object the robot needs, whereas others may be radically different. Given sufficient time, the robot can make a series of sensory or "finger" probes to determine completely the shape of any object in the bin. Because we want the robot to operate as fast as possible, we prefer an efficient strategy in which the robot makes the minimal number of probes necessary to determine whether it has found the object it is looking for.

Another important task is that of quality assurance, in which an inspector checks that all dimensions of a manufactured part fall within specified tolerances. If the part fails to meet the tolerances, then it is rejected. A robot designed to perform this task will check that every object passing through the assembly line meets the specifications. In a typical test situation, a part is placed in a fixed position and orientation in front of the robot for these tests. Efficiency is of the utmost importance, since objects must be tested in real time so as not to slow down the line.

The problem of how to identify or test geometric objects using probes has been explored by various researchers in the area of geometric probing [CY87, Ski88, RS90a, Kar91, ABM+93]. Although there are many different kinds of geometric probes, the type of probe we use in this paper is a "point probe" [RS90a]. The input to this probe is a point in Euclidean space and the output is either "positive", if the point is inside the object being probed, or "negative" if it is outside. We use these probes (called *test points*) to test whether an unknown object is similar to a given model.

The idea of using probes to distinguish objects is related to certain problems in machine learning. Typical machine learning algorithms attempt to infer a concept description from a set of examples, where each example is a vector of attribute values; i.e., a point in some d-dimensional attribute space. So, while learning involves deriving a concept description from a set of points, testing involves finding a set of points that distinguish a given object (or concept) from others in a class. Testing is most closely related to the "helpful teacher" learning model [SDHK91, GK91]. In the helpful teacher model, the teacher knows a concept and attempts to teach it using the fewest possible examples. The goal of finding the smallest set of examples to teach a concept is analogous to the goal in testing, which is to find the smallest possible set of points to test an object.

In this paper we devise testing algorithms for several classes of orthogonal shapes (i.e. shapes in which all the boundaries are axis-parallel). A *testing algorithm* takes as input a model for an object and an error bound, and it produces a set of test points that can be used to determine whether an unknown object is similar to the model. That is, the test points generated by the algorithm will detect if an object differs from the model by more than the error bound. We present testing algorithms for disjoint rectangles in 2-D and higher dimensions and for general orthogonal shapes in 2-D and 3-D. For a fixed dimension, the test sets generated by our algorithms for disjoint rectangles are linear in the number of rectangles in an object, and the test sets for general orthogonal shapes are linear in the number of vertices of an object.

2 Testability

We model the objects we wish to identify as subsets of Euclidean space, E^d. In this paper we are examining only orthogonal shapes; i.e., polyhedrals in which all edges are parallel to the axes. We first give a formal definition of a testable object class.

An *object class* is a set Q of objects. Given an object $p \in Q$ to be tested and a target object (or model) $q \in Q$, p is *consistent* with q on some finite set of test points $t = \{t_1, t_2, \ldots, t_m\}$ if p and q contain the same subset of t, i.e. $t \cap q = t \cap p$. The *error* of p with respect to q is given by $V(q \triangle p)$, where $V(p)$ denotes the d-dimensional volume of an object p and $q \triangle p$ denotes the symmetric difference of the sets. Thus, the error of the object p is measured as the volume of the region that forms the symmetric difference between p and q.

A testing algorithm takes as input a target object and an error bound, and it produces a test set that can be used to verify any other object with respect to the target.

Definition. T is a *testing algorithm for Q with test set size m* if for all $\epsilon > 0$ and for all $q \in Q$, T produces a finite set of points $T(q, \epsilon)$ in E^d of size no greater than $m(\epsilon)$ and these points have the property that for all $p \in Q$, if p is consistent with q on $T(q, \epsilon)$, then $V(q \triangle p) \leq \epsilon$. $T(q, \epsilon)$ is called a *test set for q with respect to the class Q*. For each $t_i \in T(q, \epsilon)$, if $t_i \in q$ then t_i is a *positive test point*; otherwise, t_i is a *negative test point*.

Thus given a target object $q \in Q$ and an error bound $\epsilon > 0$, T produces a test set for q such that any consistent object p has error no more than ϵ. If such a T and m exist, then the class Q is *testable with test set size m*. If T produces a constant size test set (i.e. if m is a constant k), then Q is *k-testable*.

3 Disjoint Rectangles

As a starting point, we consider orthogonal shapes that consist of a fixed number of disjoint hyperrectangles. In previous work [RS90b, RS90a, GK91] it was shown that one d-dimensional orthogonal hyperrectangle can be tested optimally with $2d + 2$ test points. A natural question to ask is whether this result can be generalized to $n(2d + 2)$ points for n disjoint rectangles. It turns out that it can, for two dimensions, as we demonstrate in the following theorem.

Theorem 1. Let R_n^2 be the class consisting of objects composed of n disjoint rectangles in E^2. R_n^2 is $6n$-testable.

Proof. Given $q \in R_n^2$ and $0 < \epsilon < 1$, let $w_{\min}$ be the width of the narrowest rectangle (in either the x or the y dimension) and let $w_{\max}$ be the width of the widest rectangle in q. Also, let each rectangle $r \in q$ be represented by its minimum and maximum corners, i.e. $r = ((x_{\min}, y_{\min}), (x_{\max}, y_{\max}))$. Let $\text{perp}_{\min}$ be the smallest perpendicular distance between 2 rectangles in q, where perpendicular distance is defined as the length of a line segment from one edge of one rectangle to an edge of the other rectangle, such that the line segment is perpendicular to both edges and does not pass through the interior of either rectangle. Finally, let $\alpha = \frac{1}{2}\min(w_{\min}, \text{perp}_{\min}, \frac{\epsilon}{4nw_{\max}})$. For each rectangle $r \in q$ choose 2 positive test points, one which is a distance α in each direction from the minimum corner of r (called a lower point) and one which is α from the maximum corner of r (called an upper point). Next choose 4 negative test points by reflecting each positive test point outside the rectangle in each direction by a distance of α. See Figure 1 for an illustration.

We will prove that the points selected above form a test set for q. First we will show that any set of n disjoint rectangles that is consistent with q on these points must partition the positive points into rectangles the same way q does.

Claim 1a. No rectangle r that is consistent with q on the selected points contains 3 or more positive points.

Proof. Suppose a rectangle r contains 3 or more positive points. Then it must contain 2 lower or 2 upper points; without loss of generality, assume it contains 2 upper points, $p_1 = (x_1, y_1), p_2 = (x_2, y_2)$. Let r' be the minimum rectangle containing p_1 and p_2. Rectangle r must contain r'. Since p_1 and p_2 are from disjoint rectangles, there exists a dimension such that p_1 and p_2 are a distance greater than 2α apart in this dimension; w.l.g. assume $x_2 - x_1 > 2\alpha$. Since p_1 is an upper point, there is a corresponding negative point $(x_1 + 2\alpha, y_1)$ that is inside r', and therefore inside r. This contradicts the fact that r is consistent on the selected points. Thus, the claim is proven. Claim 1a □

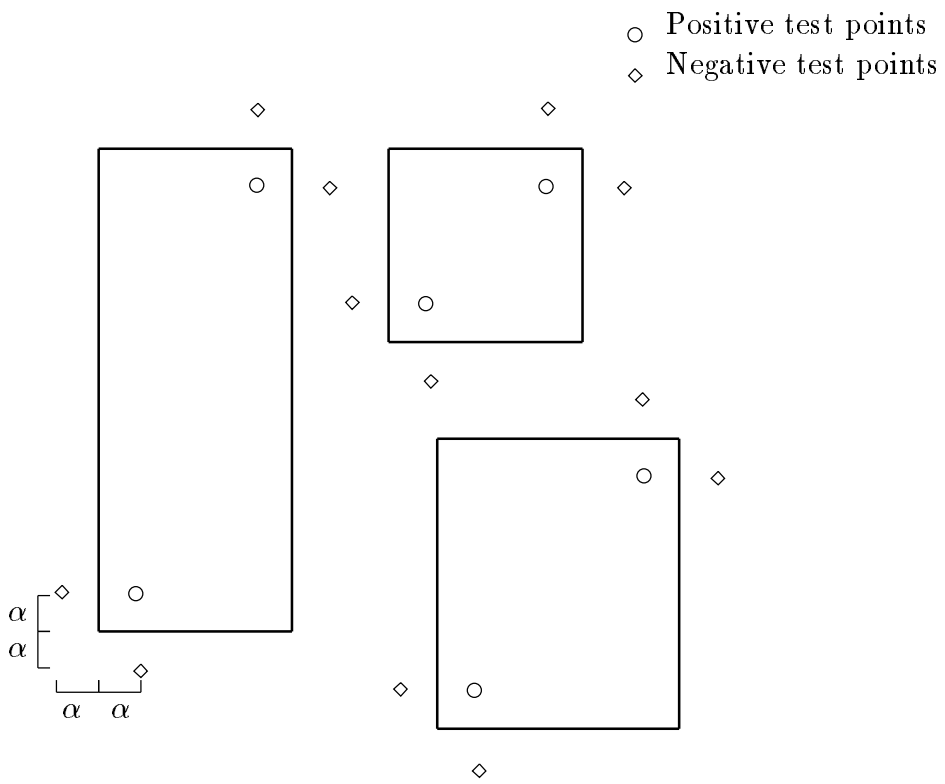


Figure 1: Set of disjoint rectangles with test points

Since no rectangle consistent with q on the selected points contains 3 or more positive points, the only way to divide the $2n$ positive points among n consistent rectangles is for each rectangle to contain exactly 2 positive points. Now we must show that the only way to do this is to partition the points the same way q does, for which we need the notion of a "top" rectangle. For a set of n disjoint rectangles, we define a rectangle $r = ((x_{\min}, y_{\min}), (x_{\max}, y_{\max}))$ as a *top rectangle* if the quadrant that has $(x_{\min}, y_{\min})$ as its minimum point contains no part of any other rectangle in the set. We show next that a top rectangle always exists.

Claim 1b. Any set of n disjoint rectangles contains a top rectangle.

Proof. Define a directed graph G that has a vertex for each rectangle in the set and an edge $r_i \to r_j$ for each pair of rectangles such that r_j is partially contained in the quadrant defined by the minimum corner of r_i. Edges $r_i \to r_i$ are not included in G. The edge relation on G is antisymmetric since if $r_i \to r_j$ and $r_j \to r_i$ were both edges in G, then the rectangles r_i and r_j would not be disjoint. Also, G is acyclic. If it were not, then it would contain a minimal length cycle $r_1 \to r_2 \to r_3 \ldots \to r_k \to r_1$ of length at least three. Since the rectangles are all disjoint, r_1 and r_2 must be nonoverlapping in at least one dimension; w.l.g. say the x dimension. This means $x_{1,\max} < x_{2,\min}$, since $r_1 \to r_2$ is an edge. Similarly, r_2

and r_3 must be nonoverlapping in the x dimension (i.e. $x_{2,\max} < x_{3,\min}$), or a contradiction results. That is, an overlapping x dimension would imply $y_{2,\max} < y_{3,\min}$ (since r_2 and r_3 would not overlap in the y dimension), but then (since edge $r_1 \rightarrow r_2$ implies $y_{1,\min} < y_{2,\max}$) $y_{1,\min} < y_{2,\max} < y_{3,\min} < y_{3,\max}$ and $x_{1,\min} < x_{1,\max} < x_{2,\min} < x_{3,\max}$, so $r_1 \rightarrow r_3$ would be an edge in G which contradicts the assumption that the cycle chosen was of minimal length. Using the same argument, all rectangles in this cycle must be nonoverlapping in the x dimension. Therefore, we have $x_{1,\min} < x_{1,\max} < x_{2,\min} \ldots < x_{k,\max} < x_{1,\min}$, which is a contradiction, so G must be acyclic. Since G is acyclic and its set of vertices is finite, there exists a rectangle with outdegree 0 in G. This rectangle is a top rectangle. Claim 1b □

Claim 1c. Any set of n disjoint rectangles consistent with q on the selected points partitions the positive points the same way as q does.

Proof (by induction). Base Case: For $n = 1$ there is obviously only one way to partition the positive points.
Inductive Step: Assume the claim is true for $n - 1$. Consider a top rectangle $r \in q$. (Note that a set of n rectangles might contain more than one top rectangle, but we only need the existence of one for our proof.) Any rectangle containing the minimum positive point in r that is consistent on the test set cannot contain a positive point in any rectangle other than r. Therefore, the only way it can contain 2 positive points is if it also contains the maximum positive point in r. By the induction hypothesis, any set of $n-1$ disjoint rectangles consistent with the remaining $n - 1$ rectangles in q on the test points partitions the positive points in the same way as these rectangles do. Therefore, n disjoint, consistent rectangles partition the positive points the same way as q does. Claim 1c □

Now we return to the proof of Theorem 1. Since a set of n disjoint rectangles that is consistent with q on the selected points must partition the positive points the same way as q does, the error of this set is bounded by the test points around each rectangle. That is, the error of each rectangle is bounded by the test points of the rectangle in q whose positive test points it contains, and the total error of the set is the sum of the errors of each rectangle. A rectangle that is consistent with the test points of one of the rectangles of q cannot have an error area greater than $4\alpha w_{\max} + 4\alpha^2 = 4\alpha(w_{\max} + \alpha)$. Therefore, the total error of the set is no more than $4n\alpha(w_{\max} + \alpha)$. Since $\alpha < w_{\max}$, $4n\alpha(w_{\max} + \alpha) < 8nw_{\max}\alpha$. Since $\alpha \leq \frac{1}{2}\left(\frac{\epsilon}{4nw_{\max}}\right)$,

$$\text{Total error} < 8nw_{\max}\alpha \leq 8nw_{\max}\left(\frac{1}{2}\right)\left(\frac{\epsilon}{4nw_{\max}}\right) = \epsilon.$$

Theorem 1 □

It is easy to see that $6n$ is also a lower bound on the number of test points needed to test an object in R_n^2. For ϵ chosen sufficiently small, each rectangle must contain two positive test points, otherwise a rectangle that is smaller by more than ϵ would be consistent. Also, each rectangle must contain four negative test points, otherwise a rectangle that expands in the direction of a side where no negative test point is placed would be ϵ larger than the model.

4 Higher Dimensional Objects

Surprisingly, the testing procedure used for Theorem 1 cannot be generalized to d dimensions. This is because the notion of a top rectangle does not generalize to higher dimensions. In fact, even the method of using test points at opposite corners does not work in general to test n disjoint rectangles in d dimensions.

Figure 2 illustrates a configuration of three boxes (i.e., orthogonal rectangles in E^3), $r_1 = ((0,0,0),(3,4,3)), r_2 = ((-2,0,4),(3,1,5)), r_3 = ((-2,2,0),(-1,3,5))$, for which there is no top box (rectangle). This configuration cannot be tested using points near the minimum and maximum corner of each box, since the three boxes $r_1' = ((0,0,0),(3,1,5)), r_2' = ((-2,0,4),(-1,3,5)), r_3' = ((-2,2,0),(3,4,3))$ will be consistent with it on these points but will have error greater than ϵ for small enough ϵ.

Guibas and Yao [GY80] showed a similar result by proving that, given any set of disjoint orthogonal rectangles in the plane and a direction vector θ, there exists an ordering of the rectangles such that for any distance d, moving the rectangles one by one according to the ordering by d in the direction of θ will not cause any rectangle to intersect another one. However, they show that such an ordering does not always exist for three-dimensional orthogonal rectangles.

Fortunately, we can still test d-dimensional disjoint rectangles with $O(nd)$ test points using a different algorithm. Note that for an automated manufacturing application, $d = 2$ or $d = 3$, so the algorithm we present below is linear for this application.

Theorem 2. Let R_n^d be the class consisting of objects composed of n disjoint rectangles in E^d. R_n^d is $n(4d+2)$-testable.

Proof. Given $q \in R_n^d$ and $0 < \epsilon < 1$, let $w_{\min}$ be the width of the narrowest rectangle (in any dimension) and let $w_{\max}$ be the width of the widest rectangle in q. Also, let each rectangle $r \in q$ be represented by its minimum and maximum corners, i.e. $r = ((x_{1,\min}, \ldots, x_{d,\min}), (x_{1,\max}, \ldots, x_{d,\max}))$. Let $\text{perp}_{\min}$ be the smallest perpendicular distance between 2 rectangles in q and let $\alpha = \frac{1}{2}\min(w_{\min}, \text{perp}_{\min}, \frac{\epsilon}{2^{d-1}ndw_{\max}^{d-1}})$. As in the case of two dimensions, for each rectangle $r \in q$ choose 2 positive test points, one that is a distance α in

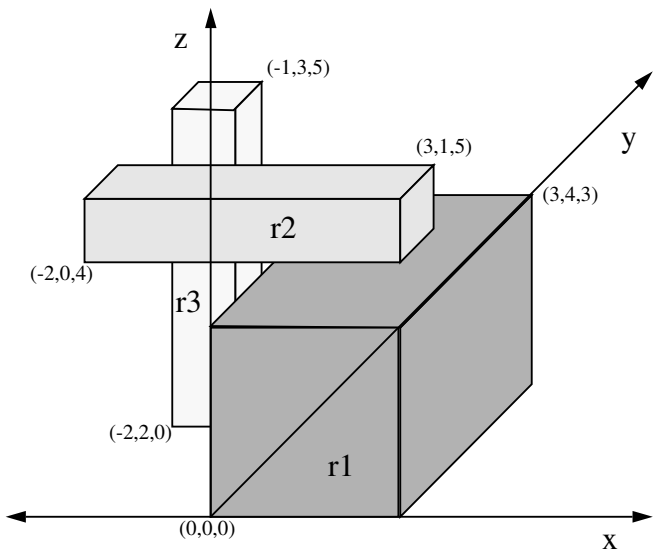


Figure 2: Set of disjoint rectangles without a top rectangle

each direction from the minimum corner of r (called a lower point) and one that is α from the maximum corner of r (called an upper point). Next for each positive test point x in r and each dimension i, choose 2 negative test points on the line through x that runs parallel to the i axis, each a distance α outside r. This yields a total of $4d$ negative test points for the rectangle r, and a total of $n(4d+2)$ test points for q. See Figure 3 for an illustration of this configuration of test points for an object in E^2.

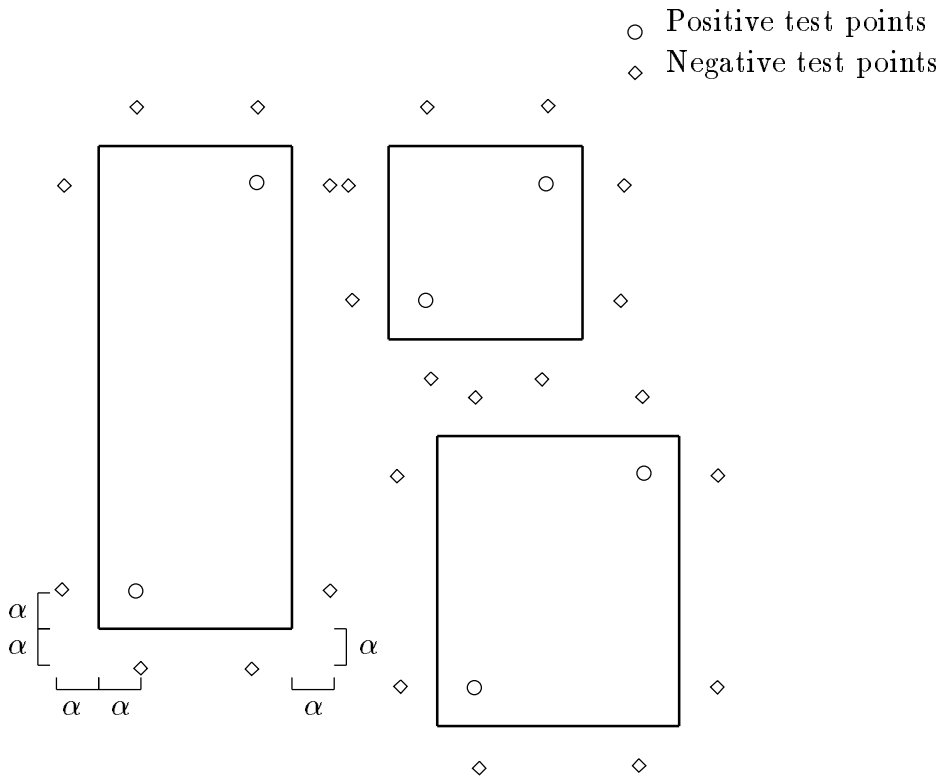


Figure 3: Configuration of test points for disjoint rectangles in higher dimensions

As in the case of two dimensions, we will prove that the points selected above form a test set for q. First we will show that any set of n disjoint rectangles that is consistent with q on these points must partition the positive points into rectangles the same way q does.

Claim 2a. No rectangle r that is consistent with q on the selected points contains positive points from more than one rectangle of q.

Proof. Suppose a consistent rectangle r contains positive points from 2 different rectangles of q. Call the two rectangles r_x and r_y, and let the two positive points be $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. Let r' be the minimum rectangle containing x and y. Rectangle r must contain r'. Since r_x and r_y are disjoint rectangles, there exists a dimension i along which they are separated by some positive distance. In other words, either $x_i < r_{y,i,\min} - \alpha \leq y_i - 2\alpha$

or $x_i > r_{y,i,\max} + \alpha \geq y_i + 2\alpha$, where $r_{y,i,\min}, r_{y,i,\max}$ are the minimum and maximum i^{th} coordinates, respectively, of r_y. However, there are 2 negative points in the test set that differ from point y only in the i^{th} coordinate. These 2 negative points have i^{th} coordinates $r_{y,i,\min} - \alpha$ and $r_{y,i,\max} + \alpha$. Since one of these coordinates falls between the i^{th} coordinates of x and y, one of these points falls inside r', and therefore inside r. This contradicts the fact that r is consistent on the selected points. Thus, the claim is proven. Claim 2a □

Since no rectangle consistent with q on the selected points contains positive points from more than one rectangle of q, the only way to divide the $2n$ positive points among n consistent rectangles is for each rectangle to contain exactly 2 positive points, both from the same rectangle of q. In other words, the n consistent rectangles must partition the positive points the same way as q does. Therefore, the error of each rectangle is bounded by the test points of the rectangle in q whose positive test points it contains, and the total error of the set is the sum of the errors of each rectangle. A rectangle that is consistent with the test points of one of the rectangles of q can only differ from the rectangle in q by a distance of α in each of $2d$ directions (i.e. positively and negatively along each dimension). Therefore, it cannot have an error greater than $2d\alpha(w_{\max} + 2\alpha)^{d-1}$. Thus the total error of the set is no more than $2nd\alpha(w_{\max} + 2\alpha)^{d-1}$. Since $\alpha \leq \frac{1}{2} w_{\max}$,

$$2nd\alpha(w_{\max} + 2\alpha)^{d-1} \leq 2nd\alpha(2w_{\max})^{d-1} = 2^d nd\alpha w_{\max}^{d-1}.$$

Since $\alpha \leq \frac{1}{2} \frac{\epsilon}{2^{d-1} nd w_{\max}^{d-1}}$,

$$2^d nd\alpha w_{\max}^{d-1} \leq 2^d nd w_{\max}^{d-1} \frac{\epsilon}{2^d nd w_{\max}^{d-1}} = \epsilon.$$

Therefore, the total error of a consistent set of n disjoint rectangles is no greater than ϵ. Theorem 2 □

5 General Orthogonal Shapes

Suppose an object that we want to test cannot be represented as a union of disjoint orthogonal rectangles. It may still be possible to represent it as a union of orthogonal rectangles, yielding a general orthogonal shape that might contain holes. Many common objects such as bookshelves, desks and chairs can be modeled by such orthogonal shapes. Also, many parts that are machined by automated manufacturing systems have orthogonal shapes as well. In this section we examine how to test such objects.

As we did in the case of disjoint rectangles, we want to be able to distinguish one object in a set from all other objects in that set. Obviously some restrictions must be made on objects belonging to the set. For orthogonal shapes we use the restriction that all objects in the set have the same number of vertices, where a *vertex* for a d-dimensional object is defined as the intersection point of d or more $(d-1)$-dimensional boundary faces of the object.

Also we assume that no objects contain degenerate boundary faces. A boundary face is *degenerate* if part of the face is bordered on both sides by exterior (or interior) points. We define this more formally as follows:

Definition. A boundary face f of an orthogonal object is an *exterior degenerate face (interior degenerate face)* if there exists a one dimensional line l that has a perpendicular intersection with f and there exists a distance $\delta > 0$ such that for all ϵ, $0 < \epsilon < \delta$, the two points on l that are a distance ϵ from f are both outside (inside) the object.

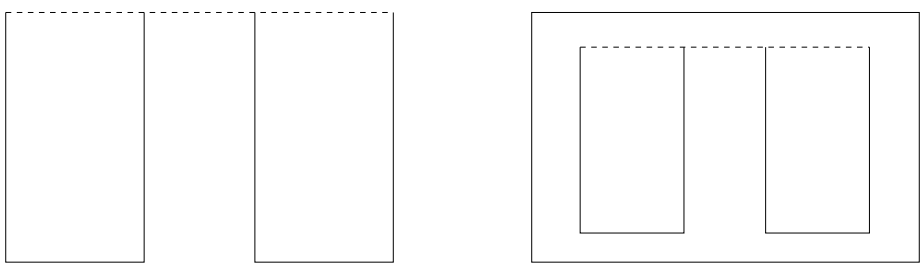


Figure 4: An exterior and an interior degenerate face are shown as dotted lines.

Figure 4 shows two orthogonal objects in E^2 with degenerate boundary faces. The object on the left has an exterior degenerate face and the object on the right has an interior degenerate face.

First we consider orthogonal objects in two dimensions. There are three ways a vertex can result from the intersection of two lines in E^2. If only one of the four quadrants formed by the intersection is interior to the object, then it is a *convex vertex*. If three quadrants are interior, then it is a *concave vertex*. If two diagonal quadrants are interior, then we consider it to be the meeting point of two convex vertices (see Figure 5).

Theorem 3. Let O_n^2 be the class consisting of orthogonal objects (with or without holes) in E^2 with n vertices. O_n^2 is $3n$-testable.

Proof. Given $o \in O_n^2$ and $0 < \epsilon < 1$, let $l_{\min}$ be the length of the shortest boundary face of o, let $l_{\max}$ be the length of the longest boundary face of o, let $d_{\min}$ be the minimum distance between any two parallel lines containing, non-colinear faces of o, and let $d_{\max}$ be the maximum distance between any two parallel lines containing faces of o. Let $\alpha =$

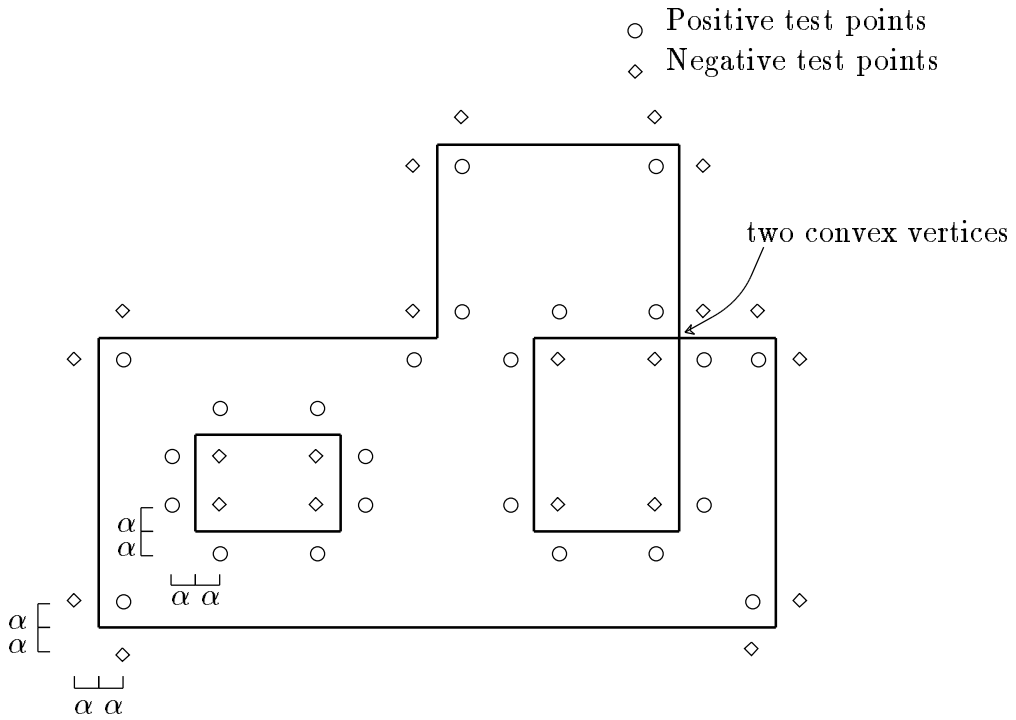


Figure 5: A 2D object in O^2_{16} with its test points

$\frac{1}{2}\min(l_{\min}, d_{\min}, \frac{2\epsilon}{n(l_{\max}+d_{\max}-l_{\min})})$. For each convex vertex of o, choose one positive test point a distance of α in each direction from the vertex, and choose two negative test points by reflecting the positive test point outside the object in each direction by a distance of α. Similarly, for each concave vertex choose one negative test point and two positive test points. See Figure 5 for an illustration of an object in O^2_{16} with its test points.

This method of choosing test points will yield a number of test points no greater than three times the number of vertices of the object; that is, no more than $3n$ points.

To show that any other object in O^2_n that is consistent on the given test points is within the error bound of ϵ, we first show that any consistent object must contain a vertex "close" to every vertex of o. At each vertex of o there are two test points, one positive and one negative, with the same x coordinate, and there are two test points, one positive and one negative, with the same y coordinate. Any object consistent with o on these points must have at least one horizontal boundary that passes between the two test points with the same x coordinate (otherwise it would be impossible for one test point to be positive and the other negative). Similarly, it must have at least one vertical boundary that passes between the two test points with the same y coordinate. Since there is both a vertical and a horizontal boundary inside the rectangle defined by the test points around the vertex, there must also be a vertex of the consistent object inside this rectangle.

By the above argument, any object consistent with o must contain a vertex within each rectangle defined by the test points around a vertex of o. In the case where two convex vertices meet at a point, a consistent object must contain two vertices inside the rectangle defined by the test points around the "double vertex." Also, by the choice of α used to place the test points, these rectangles defined by test points around a vertex are pair-wise disjoint. Therefore, since every object in O_n^2 contains exactly n vertices, a consistent object cannot have any other vertices besides these vertices and can only have one vertex in each of these rectangles, except for the case of double vertices where it will have two vertices.

A vertex of a consistent object that occurs within the rectangle defined by the test points around a vertex of o will not necessarily have the same orientation (i.e. convex or concave) as the corresponding vertex of o. If it has a different orientation, then the consistent object may contain a "strip" of width at most 2α that does not occur in o (see Figure 6). These strips can be thought of as fibers (if our object is made of fabric) that our algorithm cannot detect. The length of such a strip is bounded by $(d_{\max} - 2l_{\min})$. This is because a strip must both begin and end near vertices of o – otherwise a consistent object containing such a strip would have more than n vertices. Also, a strip extends away from a vertex, so the vertices where it begins and ends cannot be farther apart than $(d_{\max} - 2l_{\min})$. In addition, the number of such strips is bounded above by $\frac{n}{2}$ since each strip originates and ends at a vertex of o and two strips cannot originate from the same vertex (otherwise the consistent object would have more than one vertex near the origin of the two strips). Therefore, the total area of all strips that a consistent object can contain is bounded above by

$$\frac{n}{2}(2\alpha)(d_{\max} - 2l_{\min}) = n\alpha(d_{\max} - 2l_{\min}).$$

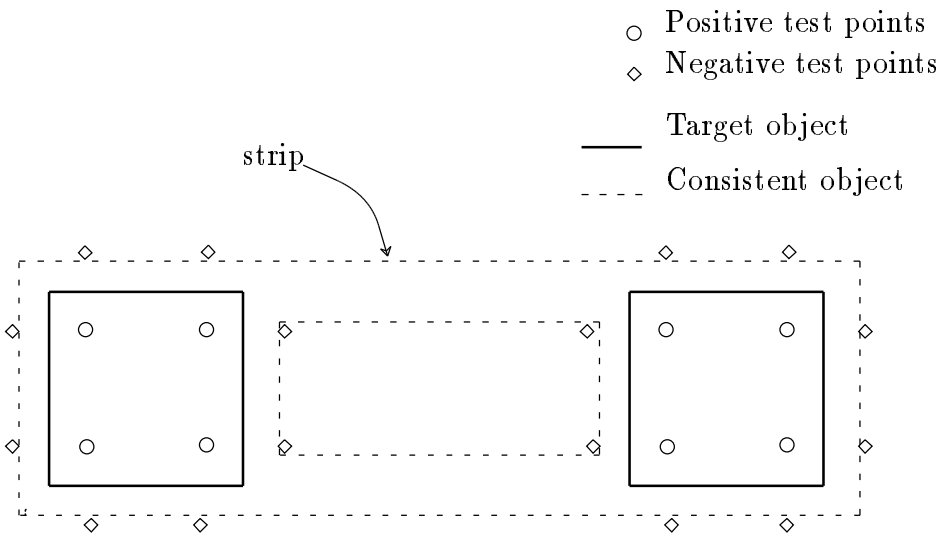


Figure 6: A target object and an object consistent on the test points that contains strips

Because the vertices of an object consistent with o must occur near the true vertices of o, and the object cannot contain any additional vertices, it can only differ from o by a distance of α along each face of o and by the total area of any strips it may contain. Therefore, the error of a consistent object is no more than the sum of $n\alpha(l_{\max} + \alpha)$ (the total amount it can differ from o along the faces) and $n\alpha(d_{\max} - 2l_{\min})$ (the total area possible for all strips). Thus the total error of an object consistent with o on the test points is no greater than

$$n\alpha(l_{\max} + \alpha) + n\alpha(d_{\max} - 2l_{\min}) = n\alpha(l_{\max} + \alpha + d_{\max} - 2l_{\min}).$$

Since $\alpha < l_{\min}$, this quantity is less than $n\alpha(l_{\max} + d_{\max} - l_{\min})$. Thus, since $\alpha \leq \frac{1}{2}\frac{2\epsilon}{n(l_{\max}+d_{\max}-l_{\min})}$,

$$n\alpha(l_{\max} + d_{\max} - l_{\min}) \leq n(l_{\max} + d_{\max} - l_{\min})\frac{\epsilon}{n(l_{\max} + d_{\max} - l_{\min})} = \epsilon.$$

Therefore, the total error of an object consistent with o on the test points is less than ϵ. Theorem 3 □

When we consider testing three-dimensional orthogonal objects, the task becomes more difficult. First, since a vertex is defined as the intersection point of 3 or more planar (2D) boundary faces of an object, and the intersection of 3 orthogonal planes produces 8 quadrants, there are $2^8 = 256$ possible combinations of interior and exterior quadrants that can form a vertex. Of course, not all of these combinations form true vertices and other combinations are symmetric with each other, but there are still many different kinds of vertices in 3D.

Another difficulty in testing three-dimensional objects is how to define a face. In addition to the restriction that faces not be degenerate, we make the restriction that a face cannot pass through an edge or a vertex of an object. For example, in Figure 8 a and b are distinct faces because any path from a to b that remains on the surface of the object and on the plane containing a and b must pass through a vertex of the object. Similarly, in Figure 7 face a and the face on the opposite side of the cube as b are coplanar but are considered distinct faces since any path from one to the other on this plane must pass through the edge $\overline{xy}$. By making this restriction on faces we eliminate the possibility that a face will have two orientations (i.e., that the interior of the object will lie on one side of the face at one place and will lie on the other side of the face at another place).

Restricting a class of objects by the number of faces that an object may have is not sufficient for testing the class. That is, for $n \geq 26$, the class F_n of three-dimensional orthogonal objects with n faces is not testable using a finite number of test points. For example, the shape in Figure 8(a) has 26 faces, where faces a, b, c, d and e are distinct faces. No matter how close negative test points are chosen to the shape, a consistent object, such as the one

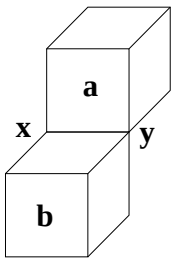
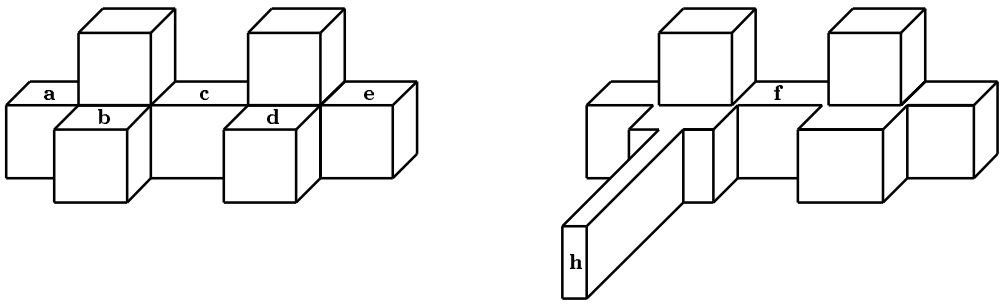


Figure 7: Face a cannot pass through edge $\overline{xy}$

in Figure 8(b), can be found that expands the sides of faces b and d so that the five faces a, b, c, d and e are merged into one face f. Since this consistent object now has four less faces than the target object, four additional faces can be added to it to form the protrusion ending with face h. If this protrusion is extended sufficiently far, it will cause the consistent object to have error greater than ϵ. Since it is impossible to use test points to prevent the merging of the five faces a, b, c, d and e into one face in a consistent object, and since it is always possible to add a protrusion that extends to infinity between the negative test points of any supposed test set, no finite set of test points can distinguish this target object from all other objects in F_{26} that have error greater than ϵ.



(a) A target object in F_{26} (b) A consistent object in F_{26}

Figure 8: Test points cannot determine if faces a, b, c, d and e are merged

Despite these difficulties, three-dimensional orthogonal shapes can still be tested efficiently, if we use the number of vertices to define the class of objects we want to test.

Theorem 4. Let O_n^3 be the class consisting of orthogonal objects in E^3 with n vertices. O_n^3 is $8n$-testable.

Proof. Given $o \in O_n^3$ and $0 < \epsilon < 1$, let $w_{\min}$ be the width of the narrowest face of o, let $w_{\max}$ be the width of the widest face of o (i.e., $w_{\max}$ is the largest distance between two parallel sides of a face), let $d_{\min}$ be the minimum distance between any two parallel

planes containing non-coplanar faces of o, let $d_{\max}$ be the maximum distance between any two parallel planes containing faces of o, and let f be the number of faces of o. Let $\alpha = \frac{1}{2}\min(w_{\min}, d_{\min}, \frac{\epsilon}{2(fw_{\max}^2+nd_{\max}^2)})$. For each vertex of o, choose 8 test points, each a distance of α in each direction from the vertex. For a vertex (x, y, z) the 8 test points would be $(x+\alpha, y+\alpha, z+\alpha), (x-\alpha, y+\alpha, z+\alpha), (x+\alpha, y-\alpha, z+\alpha), (x+\alpha, y+\alpha, z-\alpha), (x-\alpha, y-\alpha, z+\alpha), (x-\alpha, y+\alpha, z-\alpha), (x+\alpha, y-\alpha, z-\alpha), (x-\alpha, y-\alpha, z-\alpha)$.

This method of choosing test points will yield a number of test points no greater than eight times the number of vertices of the object. Thus this testing algorithm uses $8n$ points.

To show that any other object in O_n^3 that is consistent on the given test points is within the error bound of ϵ, we first show that any consistent object must contain a vertex "close" to every vertex of o. For each face of o that touches a vertex, there are at least two test points, one positive and one negative, on either side of the face. These test points insure that any object consistent on the test points will also have a face with the same orientation that passes through these two test points. Since the consistent object will have at least 3 orthogonal faces passing into the cube defined by the eight test points, it must contain a vertex within this cube. By the choice of α used to place the test points, the cubes defined by test points around a vertex are pair-wise disjoint. Therefore, since every object in O_n^3 contains exactly n vertices, a consistent object cannot have any other vertices besides the vertices occurring in these cubes and can only have one vertex in each of these cubes.

As in the case of two dimensional objects, a vertex of a consistent object that occurs within the cube defined by the test points around a vertex of o will not necessarily be of the same type as the corresponding vertex of o. However, since a test point is placed in each quadrant around a vertex, a consistent object can only have a vertex that differs from the target by the addition of strips passing between test points. No more than two strips can meet at a vertex of o, or a new vertex will be formed by them, causing the consistent object to have more than n vertices. Also, each strip has at least four vertices that must occur near vertices of o, otherwise the consistent object will contain more than n vertices. Therefore, there are no more than $\frac{n}{2}$ strips. In addition, a strip has length and width no greater than $d_{\max} + 2\alpha$ and thickness no greater than 2α. Therefore, the volume of a strip is bounded by $2\alpha(d_{\max} + 2\alpha)^2$, and the total volume of strips is bounded by $n\alpha(d_{\max} + 2\alpha)^2$.

Since the vertices of an object consistent with o must occur near the vertices of o, and the object cannot contain any additional vertices, it can only differ from o by a distance of α along each face of o and by the total volume of any strips it may contain. Therefore, the error of a consistent object is no more than the sum of $f\alpha(w_{\max} + 2\alpha)^2$ (the total amount it can differ from o by expanding parallel to the faces of o) and $n\alpha(d_{\max} + 2\alpha)^2$ (the total volume possible for all strips). Thus the total error of an object consistent with o on the test

points is no greater than

$$\alpha \left(f(w_{\max} + 2\alpha)^2 + n(d_{\max} + 2\alpha)^2 \right).$$

Since $\alpha \leq \frac{w_{\max}}{2}$ and $\alpha < \frac{d_{\max}}{2}$, this quantity is less than

$$\alpha \left(f \left(w_{\max} + 2\frac{w_{\max}}{2} \right)^2 + n \left(d_{\max} + 2\frac{d_{\max}}{2} \right)^2 \right) = 4\alpha \left(f w_{\max}^2 + n d_{\max}^2 \right).$$

Since $\alpha \leq \frac{1}{2} \left(\frac{\epsilon}{2(f w_{\max}^2 + n d_{\max}^2)} \right)$,

$$4\alpha \left(f w_{\max}^2 + n d_{\max}^2 \right) \leq 4 \frac{\epsilon}{4(f w_{\max}^2 + n d_{\max}^2)} \left(f w_{\max}^2 + n d_{\max}^2 \right) = \epsilon.$$

Therefore, the total error of an object consistent with o on the test points is less than ϵ. Theorem 4 □

Although it is not always necessary to use 8 test points at every vertex of an object in O_n^3, it is not sufficient to just place test points on either side of every boundary face that meets at a vertex. For example, if this strategy were used on the object from O_{32}^3 in Figure 9(a), then four points would be placed around each vertex - a positive point inside the block and three negative points obtained by reflecting the positive one outside each face. However, the object in Figure 9(b), which is also in O_{32}^3, would be consistent with this object on these points, but it would differ by more than ϵ. Therefore, these points do not form a test set.

(a) A target object in O_{32}^3 (b) A consistent object in O_{32}^3

Figure 9: Four points around each vertex are not sufficient to test a 3D orthogonal object

It is an open problem to determine a lower bound on the number of test points needed to test an object in O_n^3.

6 Conclusion

Our current results have applications both in automated manufacturing, for inspecting parts, and in computer vision, for distinguishing objects in a scene. Thus far, these results only

apply to objects defined by axis-parallel planes. In future work we would like to consider the problem of recognizing orthogonal shapes whose faces are not axis-parallel. This corresponds to the problem of a robot finding a part in a bin where the parts may have been thrown into the bin haphazardly. Naturally, this problem is somewhat harder than that of identifying parts that have been properly aligned with the axes. A logical extension beyond that will be to consider objects that are not orthogonal; i.e., objects of any polygonal shape. Another interesting question is how to test objects that have the same shape as the model, but that may differ in scale.

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