Directed VR-Representable Graphs have Unbounded Dimension

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Abstract

Visibility representations of graphs map vertices to sets in Euclidean space and express edges as visibility relations between these sets. A three-dimensional visibility representation that has been studied is one in which each vertex of the graph maps to a closed rectangle in $\mathbb{R}^3$ and edges are expressed by vertical visibility between rectangles. The rectangles representing vertices are disjoint, contained in planes perpendicular to the z-axis, and have sides parallel to the x or y axes. Two rectangles R_i and R_j are considered visible provided that there exists a closed cylinder C of non-zero length and radius such that the ends of C are contained in R_i and R_j, the axis of C is parallel to the z-axis, and C does not intersect any other rectangle. A graph that can be represented in this way is called *VR-representable*.

A VR-representation of a graph can be directed by directing all edges towards the positive z direction. A directed acyclic graph G has *dimension* d if d is the minimum integer such that the vertices of G can be ordered by d linear orderings, $<_1, \ldots, <_d$, and for vertices u and v there is a directed path from u to v if and only if $u <_i v$ for all $1 \leq i \leq d$. In this note we show that the dimension of the class of directed VR-representable graphs is unbounded.

1 VR-Representation of Graphs

The problem of determining a *visibility representation* of a graph, where the vertices of the graph map to sets in Euclidean space and the edges are expressed as visibility relations between these sets, has been widely studied (see [BETT93] for a survey). A three-dimensional visibility representation has been studied by Bose et al.[BEF+94], in

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which each vertex of the graph maps to a closed rectangle in $\mathbb{R}^3$ and edges correspond to vertical visibility between rectangles.

More formally, consider an arrangement of closed, disjoint rectangles in $\mathbb{R}^3$ such that the planes determined by the rectangles are perpendicular to the z (vertical) axis, and the sides of the rectangles are parallel to the x or y axes. Two rectangles R_i and R_j are *ϵ-visible* in the vertical direction if between the two rectangles there is a closed cylinder C of non-zero length and radius such that the ends of C are contained in R_i and R_j, the axis of C is parallel to the z-axis, and the intersection of C with any other rectangle in the arrangement is empty. Such an arrangement is a *Visibility Representation by Isothetic Rectangles in 3-Space* or simply a *VR-Representation* of a graph $G = (V, E)$. A graph admits such a representation provided that the following hold:

- There exists a 1-1 onto correspondence between the rectangles and the vertices in V.
- Vertices v_i and v_j are adjacent in G if and only if their corresponding rectangles R_i and R_j are ϵ-visible in the vertical direction.

If a graph can be represented in this way, the graph is *VR-representable.*

A similar representation in two-dimensions, called an *ϵ-visibility representation*, was defined independently by Wismath [Wis85] and Tamassia and Tollis [TT86]. In this representation vertices map to closed, disjoint, horizontal line segments in the plane, and two vertices are adjacent in the graph if and only if their corresponding segments are ϵ-visible in the vertical direction. Both Wismath [Wis85] and Tamassia and Tollis [TT86] independently showed that any 2-connected planar graph has an ϵ-visibility representation.

2 Dimension of Directed Acyclic Graphs

A directed acyclic graph G has *dimension* d if d is the minimum integer such that the vertices of G can be ordered by d linear orderings, $<_1, \ldots, <_d$, and for vertices u and v there is a directed path from u to v if and only if $u <_i v$ for all $1 \leq i \leq d$ [Tro92]. A class $\mathcal{G}$ of graphs has dimension d if d is the largest dimension of any graph in $\mathcal{G}$.

An ϵ-visibility representation of a graph can be directed by directing all edges towards the positive y direction. It has been shown ([BT88, RU88]) that any graph with a directed ϵ-visibility representation has dimension at most two.

A VR-representation of a graph can be directed by directing all edges towards the positive z direction, yielding a directed acyclic graph. What is the maximum dimension of any graph with a directed VR-representation?

Let us denote a complete bipartite graph by $K_{m,n}$, where m and n are the sizes of each partition, and a complete bipartite graph with a perfect matching removed by $K_{n,n} - M$, where n is the size of both partitions. Note that both partitions must have the same size for there to be a perfect matching. It is well known that the directed $K_{n,n} - M$, where all edges are directed from one partition to the other, has dimension n. In [BEF$^+$94] a VR-representation was given for the directed $K_{4,4} - M$, so the class of directed VR-representable graphs has dimension at least four.

Since a directed acyclic graph can be used to represent a partial order, work done by Rival and Urrutia [RU92] on representing ordered sets by moving convex objects in $\mathbb{R}^3$ is related to our study of the dimension of VR-representable graphs.

3 Unbounded Dimension of VR-Representable Graphs

We now show that the dimension of the class of directed VR-representable graphs is unbounded. We show this by giving a class of graphs $\mathcal{G} = \{G_n \mid n \geq 1\}$ such that the dimension of G_n is at least n, and then giving a directed VR-representation of G_n.

The directed graph $G_n = (V, E)$ that we construct is similar to $K_{n,n} - M$, except that some edges are replaced by directed paths. It has $4n-2$ vertices: $V = \{a_1, \ldots, a_n, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}, d_1, \ldots, d_n\}$. The following is a description of the edges of G_n:

- Each vertex a_i is a source and has edges $\{(a_i, d_j) \mid j < i\}$ coming out of it. If $i < n$, then edge (a_i, b_i) is also in the graph.
- Each vertex b_i has one edge (b_i, c_i) coming out of it.
- Each vertex c_i has edges $\{(c_i, d_j) \mid j > i\}$ coming out of it.
- Each vertex d_i is a sink.

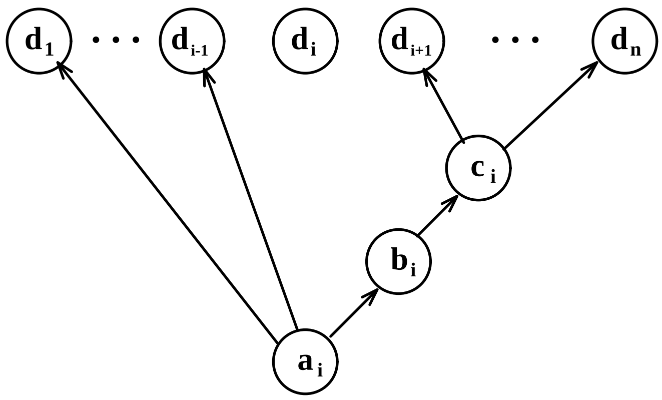

Figure 1: Subgraph of G_n with source vertex a_i

See Figure 1 for an illustration of the subgraph of G_n with source vertex a_i. In graph G_n each vertex a_i has an edge to each d_j, where $j < i$, and a directed path to each d_j, where $j > i$, but there is no path from a_i to d_i.

Lemma 3.1 *Graph G_n has dimension at least n.*

Proof: We consider only the relative order of the a_i and d_i vertices. Since there is a directed path from a_i to $d_j, j \neq i$, a_i must appear before $d_j, j \neq i$ (i.e. $a_i < d_j$) in each linear ordering of the vertices. Since there is no path from a_i to d_i, d_i must appear before a_i (i.e. $d_i < a_i$) in some linear ordering of the vertices. Consider a linear ordering $<_l$ in which $d_i <_l a_i$. For all other $a_j, j \neq i$, we must have $a_j <_l d_i$, and for all other $d_j, j \neq i$, we must have $a_i <_l d_j$. Thus in the ordering $<_l$, no other pair a_j, d_j can be reversed. Since each pair a_j, d_j must be reversed in some ordering, this requires at least n linear orderings. □

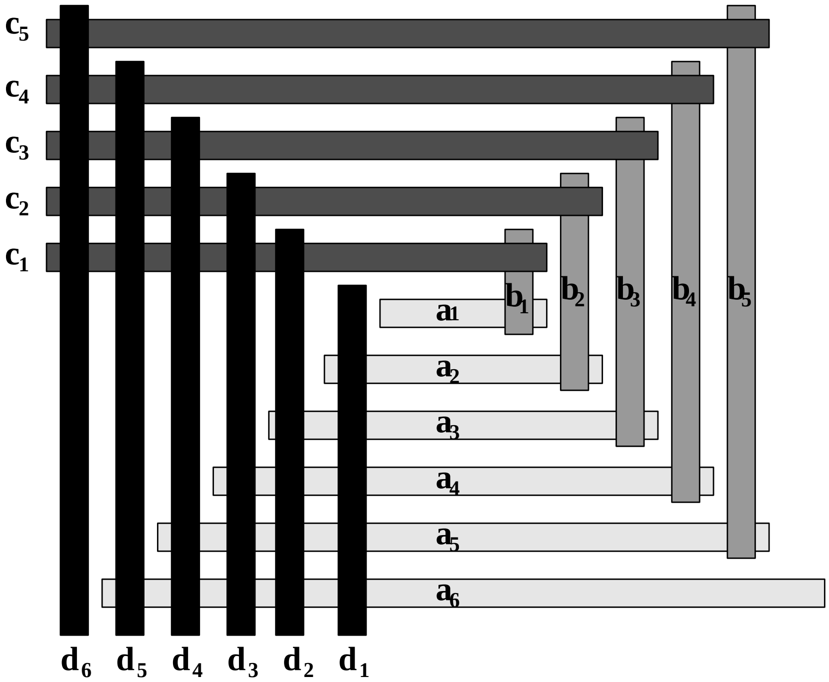

Figure 2: VR-representation for graph G_n

We now describe a directed VR-representation for G_n. The rectangles for the a_i vertices are contained in the plane $z = 0$, those for the b_i's are contained in $z = 1$, those for the c_i's are contained in $z = 2$, and those for the d_i's are contained in $z = 3$. Figure 2 illustrates the construction for G_6, where the representation is being viewed from above. It is easy to see how this construction can be extended for any $n \geq 1$. Using this VR-representation of G_n we get the following theorem.

Theorem 3.1 *The dimension of the class of directed VR-representable graphs is unbounded.*

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