Directed Rectangle-Visibility Graphs have Unbounded Dimension

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Abstract

Visibility representations of graphs map vertices to sets in Euclidean space and express edges as visibility relations between these sets. One visibility representation in the plane that has been studied is one in which the vertices of the graph map to closed isothetic rectangles and the edges are expressed by horizontal or vertical visibility between the rectangles. Two rectangles are only considered to be visible to one another if there is a non-zero width horizontal or vertical band of sight between them. A graph that can be represented in this way is called a *rectangle-visibility graph*.

A rectangle-visibility graph can be directed by directing all edges towards the positive x and y directions, which yields a directed acyclic graph. A directed acyclic graph G has *dimension* d if d is the minimum integer such that the vertices of G can be ordered by d linear orderings, $<_1, \ldots, <_d$, and for vertices u and v there is a directed path from u to v if and only if $u <_i v$ for all $1 \leq i \leq d$. In this note we show that the dimension of the class of directed rectangle-visibility graphs is unbounded.

1 Rectangle-Visibility Graphs

The problem of determining a *visibility representation* of a graph, where the vertices of the graph map to sets in Euclidean space and the edges are expressed as visibility relations between these sets, has been widely studied (see [BETT93] for a survey). One representation in the plane that has been studied [Wis89, DH94, DH96] maps each vertex of the graph to a closed isothetic rectangle in E^2 and each edge to a horizontal or vertical band of sight between two rectangles.

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More formally, consider an arrangement of closed rectangles in E^2 such that the sides of the rectangles are parallel to the axes and the rectangles are pairwise interior disjoint. Two rectangles R_i and R_j are *ϵ-visible* if there is a non-degenerate rectangle R_{ij} with two opposite sides that are subsets of the boundaries of R_i and R_j, and R_{ij} intersects no other rectangle. Such an arrangement is a *rectangle-visibility representation* of a graph $G = (V, E)$. A graph admits such a representation provided that the following hold:

- There exists a 1-1 onto correspondence between the rectangles and the vertices in V.
- Vertices v_i and v_j are adjacent in G if and only if their corresponding rectangles R_i and R_j are ϵ-visible.

Rectangle-visibility graphs are an extension of *bar-visibility graphs*, which were defined independently by Wismath [Wis85] and Tamassia and Tollis [TT86]. In this representation vertices map to closed, disjoint, horizontal line segments in the plane, and two vertices are adjacent in the graph if and only if their corresponding segments are ϵ-visible in the vertical direction.

The class of all bar-visibility graphs was completely characterized by both Wismath [Wis85] and Tamassia and Tollis [TT86]. They independently proved that a graph has a bar-visibility representation if and only if it has a planar embedding such that all cut vertices lie on the external face.

The class of all rectangle-visibility graphs has not been completely characterized. However, Wismath [Wis89] proved that all planar graphs have a rectangle-visibility representation. Also, Dean and Hutchinson [DH94, DH96] proved that a complete bipartite graph $K_{p,q}$ has a rectangle-visibility representation if and only if $p \leq 4$.

Another extension of bar-visibility graphs that was studied by Bose et al. [BEF$^+$94] is the class of *VR-representable* graphs. In this representation each vertex of the graph maps to a closed rectangle in E^3 and edges are expressed by vertical visibility between rectangles. The rectangles representing vertices are disjoint, contained in planes perpendicular to the z-axis, and have sides parallel to the x or y axes. Two vertices are adjacent in the graph if and only if their corresponding rectangles are ϵ-visible in the z direction.

2 Dimension of Directed Acyclic Graphs

A directed acyclic graph G has *dimension* d if d is the minimum integer such that the vertices of G can be ordered by d linear orderings, $<_1, \ldots, <_d$, and for vertices u and v there is a directed path from u to v if and only if $u <_i v$ for all $1 \leq i \leq d$ [Tro92]. A class $\mathcal{G}$ of graphs has dimension d if d is the largest dimension of any graph in $\mathcal{G}$.

A bar-visibility representation of a graph can be directed by directing all edges towards the positive y direction. It has been shown ([BT88, RU88]) that any graph with a directed bar-visibility representation has dimension at most two.

Both extensions of bar-visibility graphs defined in the previous section can be directed, yielding directed acyclic graphs. A VR-representation of a graph is directed by directing all edges towards the positive z direction. A rectangle-visibility representation of a graph is directed by directing all edges towards the positive x and y directions.

Romanik [Rom94] proved that the dimension of the class of directed VR-representable graphs is unbounded. In this note we prove that the dimension of the class of directed rectangle-visibility graphs is unbounded.

Since a directed acyclic graph can be used to represent a partial order, work done by Rival and Urrutia [RU88, RU92] on representing partially ordered sets by moving convex objects in space is related to our study of the dimension of rectangle-visibility graphs.

3 Unbounded Dimension of Rectangle-Visibility Graphs

We show that the dimension of the class of directed rectangle-visibility graphs is unbounded by giving a class of graphs $\mathcal{G} = \{G_n \mid n \geq 1\}$ such that the dimension of G_n is at least n, and then giving a directed rectangle-visibility representation of G_n.

Let us denote by $K_{n,n} - M$ a complete bipartite graph with a perfect matching removed, where n is the size of both partitions. Note that both partitions must have the same size for there to be a perfect matching. It is well known that the directed $K_{n,n} - M$, where all edges are directed from one partition to the other, has dimension n.

The directed graph $G_n = (V, E)$ that we construct is similar to $K_{n,n} - M$, except that the edges are replaced by directed paths. It has $2n + 3n(n-1)$ vertices defined as follows:

$$V = \{a_1, \ldots, a_n, e_1, \ldots, e_n\} \cup \{b_{i,j}, c_{i,j}, d_{i,j} \mid 1 \leq i, j \leq n, i \neq j\}$$

The vertices $a_1, \ldots, a_n$ and $e_1, \ldots, e_n$ correspond to the two partitions of $K_{n,n} - M$. The following is a description of the edges of G_n:

- Each vertex a_i is a source and has edges $\{(a_i, b_{i,j}) \mid j \neq i\}$ coming out of it.
- The $b_{i,j}, c_{i,j}, d_{i,j}$ vertices are connected by edges $(b_{i,j}, c_{i,j})$ and $(c_{i,j}, d_{i,j})$.
- Each vertex e_j is a sink and has edges $\{(d_{i,j}, e_j) \mid j \neq i\}$ going into it.

See Figure 1 for an illustration of the subgraph of G_n with source vertex a_i. In graph G_n each vertex a_i has a directed path to each e_j, where $j \neq i$, but there is no path from a_i to e_i.

Lemma 3.1 *Graph G_n has dimension at least n.*

Proof: We consider only the relative order of the a_i and e_i vertices. Since there is a directed path from a_i to $e_j, j \neq i$, a_i must appear before $e_j, j \neq i$ (i.e. $a_i < e_j$) in each linear ordering of the vertices. Since there is no path from a_i to e_i, e_i must appear before a_i (i.e. $e_i < a_i$) in some linear ordering of the vertices. Consider a linear ordering $<_l$ in which $e_i <_l a_i$. For all other $a_j, j \neq i$, we must have $a_j <_l e_i$, and for all other $e_j, j \neq i$, we must have $a_i <_l e_j$. Thus in the ordering $<_l$, no other pair a_j, e_j can be reversed. Since each pair a_j, e_j must be reversed in some ordering, this requires at least n linear orderings. $\square$

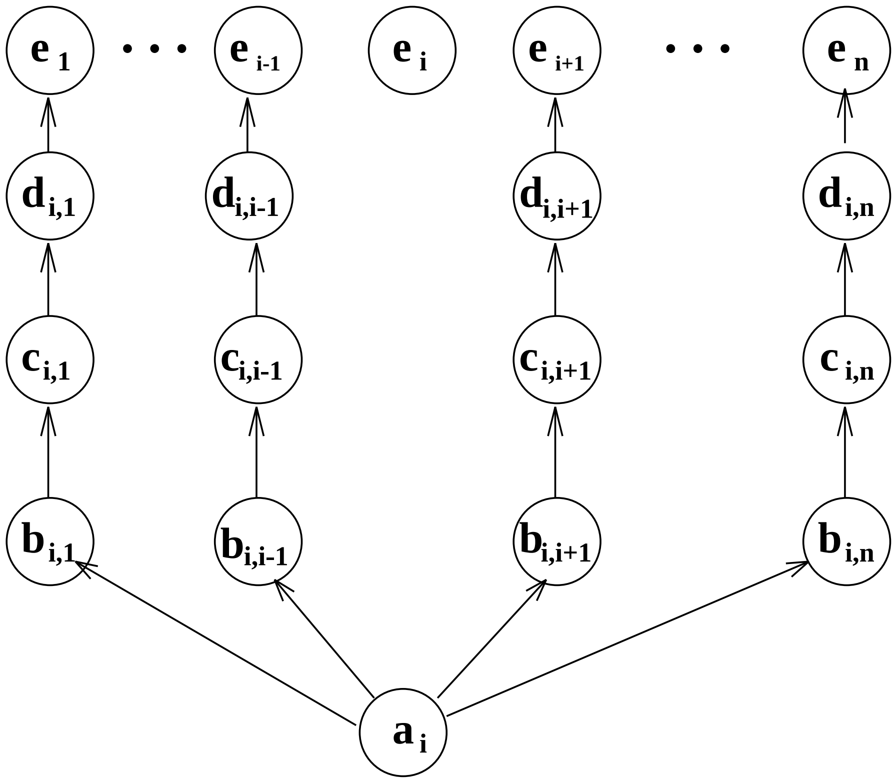

Figure 1: Subgraph of G_n with source vertex a_i

We now describe a directed rectangle-visibility representation for G_n. The rectangles for the a_i vertices are in a stairstep arrangement, as are the rectangles for the $b_{i,j}$ vertices, the $d_{i,j}$ vertices and the e_i vertices (see Figure 2). The vertical edges of the $b_{i,j}$ rectangles and the horizontal edges of the $d_{i,j}$ rectangles, if extended, intersect to form a grid pattern. The $c_{i,j}$ rectangles are formed by "filling in" squares of this grid. It is easy to see that any $b_{i,j}$ vertex can be connected to any $d_{l,k}$ vertex by filling in the appropriate square of the grid.

All of the $(a_i, b_{i,j})$ edges and $(c_{i,j}, d_{i,j})$ edges are horizontal, and the rest of the edges are vertical. Figure 2 illustrates the construction for G_4. It is easy to see how this construction can be extended for any $n \geq 1$. Using this rectangle-visibility representation of G_n we get the following theorem.

Theorem 3.1 *The dimension of the class of directed rectangle-visibility graphs is unbounded.*

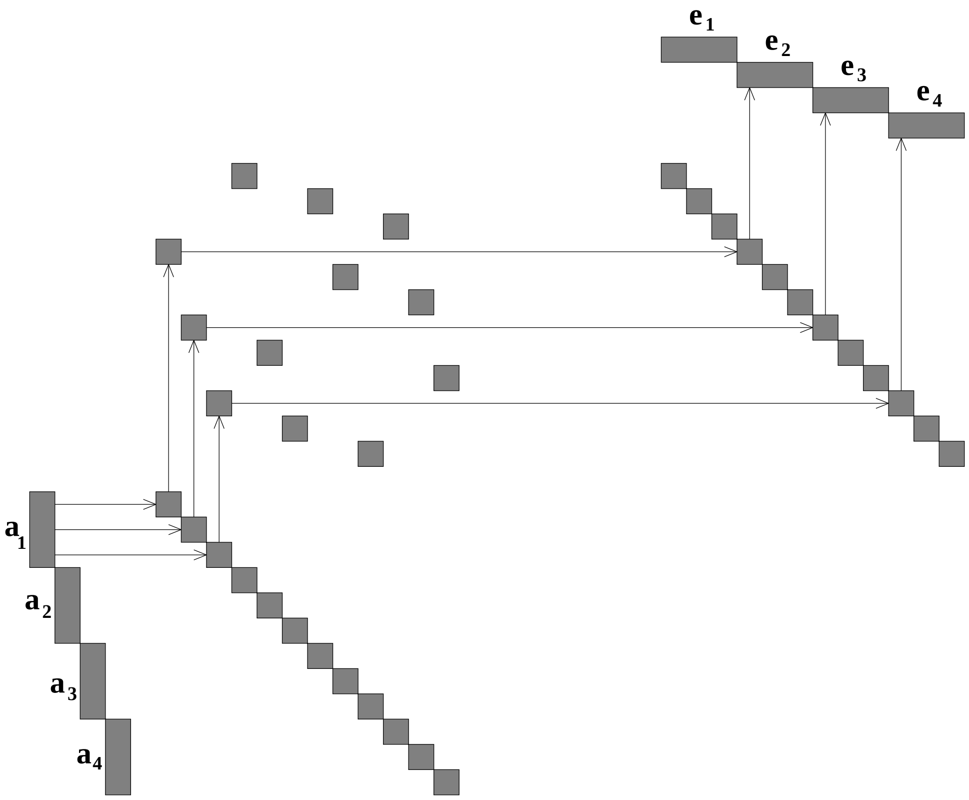

Figure 2: Rectangle-visibility representation for graph G_n

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