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Classes of 3-regular graphs that are $(7, 2)$-edge-choosable

by

Daniel W. Cranston[1]
DIMACS
Rutgers University
Piscataway, New Jersey 08854

Douglas B. West[2]
Dept. of Mathematics
University of Illinois
Urbana, Illinois 61801

[1]Postdoctoral Associate

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ABSTRACT

A graph is $(7,2)$-edge-choosable if, for every assignment of lists of size 7 to the edges, it is possible to choose two colors for each edge from its list so that no color is chosen for two incident edges. We show that every 3-edge-colorable graph is $(7,2)$-edge-choosable and also that many non-3-edge-colorable 3-regular graphs are $(7,2)$-edge-choosable.

1 Introduction

An *s-tuple coloring* of a graph is an assignment of s-sets (of colors) to the vertices such that adjacent vertices receive disjoint sets. This notion was introduced by Stahl [7]. A graph G is (r, s)*-colorable* if it has an s-tuple coloring using altogether at most r colors, and the *s-set chromatic number* is the least r such that G is (r, s)-colorable. The *fractional chromatic number* $\chi^*(G)$ (also called *set chromatic number*, *multicoloring number*, or *ultimate chromatic number*) is $\inf\{r/s\colon G$ is (r, s)-colorable$\}$. Since the ordinary chromatic number is achievable as such a ratio (with $s = 1$), always $\chi^*(G) \le \chi(G)$.

The parameter χ^* is worthy of study for various reasons. It is the linear programming relaxation of the ordinary chromatic number of a graph (and hence the infimum is a minimum); many papers have studied it from this viewpoint. Furthermore, Klostermeyer and Zhang [5] observed that $\chi^*(G)$ is the minimum ratio r/s such that there is a homomorphism from G to the Kneser graph $K(r, s)$, where $K(r, s)$ is the graph on the s-subsets of an r-element set in which adjacency is defined by disjointness.

A modern variation for coloring problems restricts the colors available at vertices. A *list assignment* L gives each vertex v a list $L(v)$ of available colors. A good coloring must satisfy the additional requirement that any colors used at v must lie in $L(v)$. A graph G is k*-choosable* if an ordinary proper coloring can be chosen from the lists whenever the list have size at least k, and the *choosability* (or *list chromatic number*) $\chi_\ell(G)$ is the least k such that G is k-choosable. More generally, G is *s-set k-choosable* if an s-set coloring can be chosen from any assignment of lists of size k, and the *s-set choosability* is the minimum k such that G is s-set k-choosable.

When coloring edges, the sets chosen for incident edges must be disjoint, so the edge-coloring problems reduce to the vertex coloring problems mentioned above on the line graph. In particular, in this paper we study the 2*-set edge-choosability* of 3-regular graphs. Since extra colors can be discarded without making the choosing of a proper coloring easier, it suffices to study *r-uniform* list assignments, where each list has size r. Given a list assignment L on the edges of G, a selection of s colors from each list so that incident edges have disjoint lists is an *s-set L-coloring* of $E(G)$.

Question. *What is the least r such that, for every 3-regular graph G, every r-uniform list assignment L on the edges of G admits a 2-set L-coloring?*

That is, we seek the least r such that every 3-regular graph is $(r, 2)$-edge-choosable. On his website, in a "Problem of the Month", Bojan Mohar [6] asked for this value. He conjectured that every 3-regular graph is $(7, 2)$-edge-choosable.

A generalization of Brooks' Theorem implies that every 3-regular graph is $(8, 2)$-edge-choosable. Tuza and Voigt [9] proved that if a connected graph G is not complete and is not an odd cycle, then G is $(\Delta(G)m, m)$-choosable whenever $m \ge 1$. Since the line graph of a 3-regular graph has maximum degree 4, every 3-regular graph is thus $(8, 2)$-edge-choosable.

On the other hand, it is also easy to construct a 3-regular graph that is not $(6, 2)$-edge-choosable (in fact, not even $(6, 2)$-edge-colorable). It is well known that the smallest 3-regular graph G that is not 3-edge-colorable is formed from two copies of K_4 by subdividing one edge in each and making the two new vertices adjacent. A $(6,2)$-edge-coloring of G would put a total of 30 colors on the 15 edges. With only six colors available, each color class would have to be a perfect matching. Since every perfect matching in G uses the central cut-edge, this contradicts that we choose only two colors on it.

Thus Mohar's conjecture on the 2-set choosability of 3-regular graphs is sharp if true.

In this paper, we show that every 3-edge-colorable graph is $(7,2)$-edge-choosable and that many 3-regular graphs that are not 3-edge-colorable are also $(7,2)$-edge-choosable.

Some classes of 3-regular graphs are known to be $(6,2)$-edge-choosable. Ellingham and Goddyn [2] showed that planar d-regular d-edge-colorable multigraphs are d-edge-choosable; by doubling the edges, this implies that planar 3-regular graphs are $(6,2)$-edge-choosable. Haxell and Naserasr [3] showed that the Petersen graph is $(6,2)$-edge-choosable. Both of these results use the Alon-Tarsi Theorem and thus provide only existence proofs. Our proofs of $(7,2)$-edge-choosability provide a fairly simple algorithm for choosing a 2-set edge-coloring from lists of size 7 on the edges.

An edge e of a 3-regular graph is incident to four other edges. Choosing colors for them could forbid eight colors from usage on e. Our main idea is to show that we can choose the four colors on two of these incident edges using only three colors from $L(e)$. Any time we reduce the number of available colors at e by less than the number of colors we choose on edges incident to e, we say that we have *saved a color on e*. In particular, when we save a color on e while choosing four colors on two edges incident to e, we retain a list of (at least) four colors available for e. We will apply this repeatedly to choose colors on some edges so that the remaining graph is $(4,2)$-edge-choosable and retains lists of size 4.

2 The Key Idea

Our first lemma is a simple form of our main tool. It is a generalization of the well-known result [8] that even cycles are $(2m,m)$-edge-choosable. To understand the proof, it may be helpful to picture the case when $A \cup B$ forms an even cycle. In general, however, B need not be a matching. We will be saving a color on edges of B.

Lemma 1. *Let A and B be sets of k edges in a graph, with $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$. Suppose that A is a matching and that b_i is incident to a_i and a_{i+1} but not to any other edge in A (the indices are viewed modulo k). From a uniform list assignment L on the edges, one can choose one color at each edge of A so that for each i, together a_i and a_{i+1} receive at most one color from $L(b_i)$.*

Proof. We choose a color $\phi(e)$ for each edge e in A. If the lists for all edges in $A \cup B$ are identical, then use the same color on each edge of A. If they are not all identical, then they differ for two consecutive edges in the cyclic list $a_1, b_1, a_2, b_2, \ldots a_k, b_k$. We may index the edges so that these are a_1 and b_k.

Since the lists have the same size, we may choose $\phi(a_1) \in L(a_1) - L(b_k)$. If $\phi(a_1) \notin L(b_1)$, then choose $\phi(a_2)$ from $L(a_2)$ arbitrarily. If $\phi(a_1) \in L(b_1) \cap L(a_2)$, then let $\phi(a_2) = \phi(a_1)$. Finally, if $\phi(a_1) \in L(b_1) - L(a_2)$, then choose $\phi(a_2) \in L(a_2) - L(b_1)$. In each case, at most one of the colors chosen for the edges a_1 and a_2 incident to b_1 is in $L(b_1)$.

Continue in the same manner choosing colors for edges $a_3, \ldots a_k$ so that at most one color from $L(b_i)$ is used on a_i and a_{i+1}. At the end, at most one color from $L(b_k)$ appears in $\{\phi(a_k), \phi(a_1)\}$, since $\phi(a_1) \notin L(b_k)$. □

Corollary 2. (Tuza and Voigt [8]) *Even cycles are* $(2m, m)$*-edge-choosable.*

Proof. Partition the edge set into two matchings, A and B. Choose one color for each edge of A as guaranteed by Lemma 1. Repeat this step m times. (With each repetition, we may need to discard colors from some lists so that the remaining lists have equal sizes.) Now each edge of B has at least m remaining available colors, which we use on those edges. □

It is not immediately obvious that Lemma 1 implies anything more than Corollary 2. Its power lies in carefully choosing the edge sets A and B, as we show in the next proof. Theorem 3 is a special case of Theorem 7, but it is useful to prove Theorem 3 independently because it is much simpler than the general result and yet illustrates our main technique.

Theorem 3. *Every* 3*-edge-colorable graph is* $(7, 2)$*-edge-choosable.*

Proof. Let G be a 3-edge-colorable graph. We can raise the degree of each vertex that has degree less than 3 by taking two disjoint copies of G and making the two copies of each deficient vertex adjacent. A proper 3-edge-coloring of G extends to a proper 3-edge-coloring of the new graph. If G has minimum degree k, then we obtain a 3-regular 3-edge-colorable supergraph of G after $3 - k$ iterations of this transformation. We may thus assume that G is 3-regular, since every subgraph of a $(7, 2)$-edge-choosable graph is $(7, 2)$-edge-choosable.

Let J, K, and L be the three color classes in a proper 3-edge-coloring of G. Since G is 3-regular, $|J| = |K| = |L|$. We apply Lemma 1 in two phases. The first phase uses even cycles in $J \cup K$, with the portion in J as A and the portion in K as B. It chooses one color for each edge of J. The lists of colors remaining available have size at least 6 for edges of J and K and size at least 5 for edges of L.

The second phase applies Lemma 1 using even cycles in $J \cup L$, with the portion in J as A and the portion in L as B. Since the lemma requires equal list sizes, discard colors from the lists on edges of J and L to reduce to size 5 before applying the lemma. We have now chosen two colors for each edge in J but none for any edge in K or L. There remain at least four colors available at each edge of $K \cup L$.

Since the edges of $K \cup L$ form vertex-disjoint even cycles, applying Lemma 2 to each cycle completes the desired coloring. □

Like the iteration in Corollary 2, iterating the argument of Theorem 3 shows that every 3-edge-colorable graph is $(\lceil 7m/2 \rceil, m)$-edge-choosable, for each positive integer m.

3 The Main Result

The ideas in Theorem 3 can be applied to prove that many other graphs are (7,2)-edge-choosable. Some of these are "snarks". A *snark* is a 2-edge-connected 3-regular graph with edge-chromatic number 4, girth at least 5, and cyclic-connectivity at least 4. The smallest snark is the Petersen graph. The drawing of it in Figure 1 has exactly one vertical edge c and two horizontal edges labeled c' and b. Initial interest in snarks was due to the equivalence of the Four Color Theorem with the statement that no planar snarks exist. They remain interesting because many important conjectures reduce to the special case of snarks.

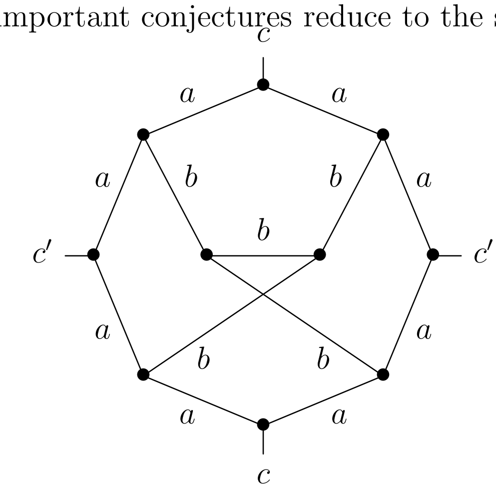

Figure 1: The Petersen graph.

In this paper, we use the term *double-star* for a 6-vertex tree having two adjacent vertices of degree 3. This is slightly non-standard; the term is often used for any tree having two non-leaves. Each edge of a double-star incident to a leaf is a *leaf edge*; the other edge is the *center edge*. Given a subgraph F whose components are pairwise disjoint double-stars in a graph G, we say that F *consists of independent double-stars* if the set of leaves of components of F is an independent set of vertices in G.

In Figure 1, we draw the Petersen graph on the torus with one crossing. The edge c wraps top-to-bottom, and c' wraps right-to-left. The graph decomposes into a matching (c and c'), an 8-cycle (a), and a double-star (b). In view of Theorem 7, this decomposition proves that the Petersen graph is $(7, 2)$-edge-choosable.

Definition 4. *A* MED decomposition *of a 3-regular graph is a decomposition of it into subgraphs G_1, G_2, G_3, where G_1 is a Matching, the components of G_2 are Even cycles, and G_3 consists of independent Double-stars. Given a MED decomposition, let H be the graph formed from G_3 by deleting the center edge from each component of G_3.*

In a MED decomposition, no vertex lies in both G_1 and G_3, since the third edge at that vertex could not belong to any G_i. Also, every vertex not in G_2 belongs to the center edge of a component of G_3. Finally, each component of H is a 3-vertex path.

To simplify the proof of Theorem 7, we prove two lemmas. The first establishes the sufficiency of an intermediate coloring by showing how to complete the job of choosing a 2-set L-coloring. A *partial 2-set L-coloring* is a selection of at most 2 colors from the list at each edge so that the sets chosen at incident edges are disjoint.

Lemma 5. *Let G be a graph with maximum degree 3 that has a MED decomposition. Given a 7-uniform list assignment L on $E(G)$, let ϕ be a partial 2-set L-coloring that chooses two colors for each edge in G_1 and one color for each edge in H. Each edge of G_2 on which ϕ does not save a color is* needy. *Each edge of H incident to a needy edge is a* sponsor. *If the needy edges form a matching, the sponsors form a matching, and each center edge incident to two sponsors has at least four remaining colors, then ϕ extends to a 2-set L-coloring.*

Proof. We will choose one additional color for each edge of H and two colors for each center edge in G_3. We must choose these colors to save one color on each needy edge. If this is done, then Corollary 2 allows us to choose colors for the edges of G_2 to complete the extension of ϕ to a 2-set L-coloring of $E(G)$. For each edge $e \in E(G)$, let $L'(e)$ denote what remains of $L(e)$ when the colors chosen by ϕ on edges incident to e are deleted; this assignment L' remains fixed throughout the proof.

Let F be a double-star in G_3 with center edge e, leaf edges f_1 and f_2 at one end of e, and leaf edges f_3 and f_4 at the other end. We have $|L'(e)| \geq 3$. Since G_1 and G_3 share no vertices, we have $|L'(f_i)| \geq 5$ for all i. Since the sponsors form a matching, there is at most one sponsor in $\{f_1, f_2\}$ and at most one in $\{f_3, f_4\}$.

Suppose first that one leaf edge of F is a sponsor. We may assume by symmetry that it is f_1; let g be its incident needy edge. Since G_3 consists of independent double-stars, the other endpoint of g is incident to an edge of G_1. Thus, we have already chosen three colors for edges incident to g. By discarding additional colors if necessary, we may assume that $|L'(g)| = 4$. Since $|L'(f_1)| \geq 5$, we save a color on g by choosing a color in $L'(f_1) - L'(g)$ for f_1. We choose any two remaining colors for e (at least two remain) and any remaining color for each of f_2, f_3, and f_4. Since $|L'(f_i)| \geq 5$, such choices are available.

Suppose instead that we need to save a color both on g incident to f_1 and on g' incident to f_3. As above, we have $|L'(f_1)| \geq 5$ and may assume that $|L'(g)| = 4$, and we can choose the second color for f_1 to save a color on g. Similarly, we can choose the second color for f_3 to save on g'. By hypothesis, $|L'(e)| \geq 4$ (before choosing for f_1 and f_3); hence e still has at least two available colors, and we choose two of them. Finally, since $|L'(f_i)| \geq 4$, we can choose a second available color for each of f_2 and f_4.

Now it remains only to choose two colors for each edge of G_2. Since completing the choices on G_3 produced a partial 2-set L-coloring that saves a color on each edge of G_2, these edges retain lists of size at least 4, and Corollary 2 completes the coloring. $\square$

The next lemma describes a step that may be repeated often in obtaining a partial coloring to use as input to the procedure in Lemma 5. The proof is similar to that of Lemma 1.

Lemma 6. *Given a 3-regular graph G with a MED decomposition, let C be a cycle in G_2 with edges $a_1, \ldots, a_k$ in order. Let b_i be the other edge incident to a_i and a_{i+1} (viewing indices modulo k). Let L be a list assignment such that for each i, it holds that $|L(b_i)| \ge |L(a_i)|$ or $b_i \in E(H)$. If $L(b_1) \not\subseteq L(a_1)$, or if $b_1 \in E(H)$ and $|L(b_1)| < |L(a_1)|$, then a color can be chosen for each edge in $b_1, \ldots, b_k$ so that for each i, either (1) a color is saved on a_i or (2) $b_i \in E(H)$ and either $|L(b_i)| < |L(a_i)|$ or some earlier b_j is incident to b_i in H.*

Proof. If $L(b_1) \not\subseteq L(a_1)$, then choose $c \in L(b_1) - L(a_1)$ for b_1; this saves a color on a_1. Otherwise, $b_1 \in H$ and $|L(b_1)| < |L(a_1)|$, and we choose any $c \in L(b_1)$ for b_1 without saving a color on a_1. In either case, remove c from $L(a_2)$; if $c \notin L(a_2)$, then remove any color from $L(a_2)$. Also remove c from the lists on edges incident to the other end of b_1; these edges may be on C, on another even cycle in G_2, or consist of another edge of H and a central edge in a component of G_3.

Let $L'(e)$ denote the current remaining list at e. In contrast to the proof of Lemma 5, here we update L' as we go along. We proceed inductively for increasing i. After choosing a color for b_{i-1}, we have $b_i \in E(H)$ or $|L'(b_i)| > |L'(a_i)|$, since we deleted a color from the list at a_i when choosing a color for b_{i-1}.

In the latter case, we choose a color for b_i from $L'(b_i) - L'(a_i)$ and save on a_i. This includes the possibility that b_i was visited earlier as b_j, because in that case we reduced the available lists at both b_j and a_i by choosing a color for b_j. It also includes the possibility that $b_i \in E(H)$ and $|L(b_i)| = |L(a_i)|$, which will be helpful later.

The remaining case is $b_i \in E(H)$ and $|L(b_i)| < |L(a_i)|$; the color deleted from the list at a_i does not guarantee $L'(b_i) - L'(a_i) \neq \emptyset$. In this case, choosing a color from $L'(b_i)$ for b_i may not save on a_i.

After choosing a color for b_k by these rules, the proof is complete. □

In the case of Lemma 6 where choosing a color on b_i does not save a color on a_i, the edge a_i will be designated needy, and the edge b_i will be its sponsor. In applying the lemma, we will sometimes exercise more care in choosing the color for b_i in that case.

We will use Lemma 6 to reach a state where Lemma 5 applies. We would like to save a color on each edge of G_2 while choosing a partial 2-set L-coloring with two colors chosen on each of the other edges, as in Lemma 3. However, the double-stars outside these even cycles cause complications and force us to leave needy edges and use Lemma 5.

Theorem 7. *Every 3-regular graph G having a MED decomposition is $(7, 2)$-edge-choosable.*

Proof. Before applying Lemma 5, we must save one color each on all edges of G_2 except a matching, while confining the sponsors also to a matching and choosing two colors for each edge of G_1 and one color for each edge of H. We do this in each component of $G_1 \cup G_2 \cup H$ independently; the central edges of G_3 tie the graph together in the step performed later by Lemma 5. Let J be a component of $G_1 \cup G_2 \cup H$.

Case 1. *J does not contain two incident edges, one in G_2 and one in $G_1 \cup H$, that have distinct lists.* In this case, for each component R of $J \cap (G_1 \cup G_2)$, the lists are identical on all edges of R and all edges of H incident to it. (There may be more than one such component

R; they may be joined by components of H.) Choose any two colors c and c' from this list, and choose $\{c, c'\}$ for each edge of $R \cap G_1$. For each edge of H incident to $V(R)$, choose c or c', avoiding the color on the neighboring edge of H if it has already been chosen by another such component.

Since G_1 and H share no vertices, these choices are available. Each edge e of G_2 is incident to two edges of $G_1 \cup H$, at least one in G_1. We have made at least three color assignments on edges incident to e using only two colors, thus saving at least one color on e.

Case 2. *J has incident edges e_1 in G_2 and e_2 in $G_1 \cup H$ with distinct lists.* Let C denote the even cycle in $J \cap G_2$ that contains e_1. Let $a_1, \ldots, a_k$ be the edges of C in order, with $a_1 = e_1$ and a_2 incident to e_1. Let b_i be the edge incident to a_i and a_{i+1}, so $b_1 = e_2$ and $L(b_1) \neq L(a_1)$. Note that $b_i \in E(G_1 \cup H)$. An edge may be labeled as both b_i and b_j with $i \neq j$; this holds for each edge of G_1 that is a chord of C.

We have chosen no colors on edges incident to J, so all lists have size 7. Since $L(b_1) \neq L(a_1)$, Lemma 6 applies.

Step 1. Choosing colors for $b_1, \ldots, b_k$. Using Lemma 6, we choose a color for each b_i that saves on a_i unless b_i is incident to an edge of H for which we have already chosen a color. That edge may for example be b_j with $j < i$. When we reach b_i, the remaining list at b_i may be smaller than that at a_i.

In this case, a_i remains needy and b_i is its sponsor. This only happens when b_i is the second edge in its component of H for which we are choosing a color. Let e be the center edge incident to b_i. If e has six remaining available colors when we choose for b_i, meaning that we have not chosen a color for any edge incident to e other than the other edge of H incident to b_i, then we choose any color for b_i among its six available colors.

If at most five colors remain available at e, then we choose a color for b_i to save a color on e. This ensures that each center edge incident to two sponsors will have at least four remaining colors, as needed for Lemma 5. With this refinement, we choose colors as in Lemma 6.

Let L' denote the assignment of remaining available colors after this step, and let $B = \{b_1, \ldots, b_k\}$. For any edge e incident to B, each color c chosen on an incident edge e' of B eliminates one element from $L'(e)$ (which can be any element if c was not in the list); it also eliminates c from $L'(e')$. If $e \in E(G_2)$, then e may be incident to another edge of B. Thus the list for e decreases by at least as much as the list for e'.

Step 2. Handling additional components of G_2 in J. If we have not yet chosen colors on all of J, then $J \cap G_2$ contains another unprocessed cycle C' with length k', reachable from an already processed cycle C via an edge of G_1 or via a component (two edges) of H (or possibly reachable in both ways). Suppose that $b_1, \ldots, b_k$ were the edges incident to C; we chose a color for each of these while processing C.

If some b_i is incident to C' such that $L'(b_i)$ has an element outside the list of some edge in C' incident to b_i, then let $b'_1 = b_i$; in this case $b'_1 \in E(G_1)$. Alternatively, if f_1 and f_2 are incident edges in H, with $f_1 = b_i$ and f_2 incident to C', then let $b'_1 = f_2$. In either case, index the edges of C' as $a'_1, \ldots, a'_{k'}$ in order, starting with edges incident to b'_1 as a'_1 and a'_2, choosing a'_1 so that $L'(b'_1) \not\subseteq L'(a'_1)$ if possible.

We selected $b'_1 \in E(G_1)$ only when $L'(b'_1) \not\subseteq L'(a'_1)$. Also, previous applications of Lemma 6 have left $|L'(b'_i)| \geq |L'(a'_i)|$ unless $b'_i \in E(H)$ and the edge incident to b'_i in H already had a color chosen while processing an earlier cycle. This requires $b'_1 \in E(H)$ and $|L'(b'_1)| < |L'(a'_1)|$. Hence in either case the hypotheses for Lemma 6 hold. We return to Step 1 to process C', treating C', k', b'_i, a'_i as C, k, b_i, a_i.

Continuing in this way, we process all components of $J \cup G_2$ unless we reach a state where a remaining cycle C in $J \cup G_2$ is incident only to edges of G_1, and the remaining list on each such edge is the same as the remaining list on the incident edges of C. Since each vertex of C has an incident edge in G_1, all the lists on C and its incident edges are the same.

That list contains six colors, since we have only chosen one color on each incident edge in G_1. Since each edge of C has six remaining colors in its list, and we have already chosen one color on each incident edge in G_1, we have already saved a color on each edge of C. We choose an additional color from this list as the second color for all the edges of G_1 incident to C.

Under the algorithm used in Lemma 6, only the second edge of a component of H on which a color is chosen can become a sponsor. Hence the sponsors form a matching. Furthermore, an edge a_i of G_2 becomes needy only when the edge incident to a_i at its endpoint incident to the subsequent edge (a_{i+1} when processing its cycle in G_2) is a sponsor. This prevents two incident edges of G_2 from being left needy, since G_3 consists of independent double-stars. We have therefore produced a partial 2-set L-coloring that satisfies the hypotheses of Lemma 5, and Lemma 5 completes the 2-set L-coloring. □

We have used Theorem 7 to check that all snarks with at most 24 vertices are $(7, 2)$-edge-choosable, as are the double star snark, the Szekeres snark, the Goldberg snark, the Watkins snarks of orders 42 and 50, all cyclically 5-edge-connected snarks of order 26, and the infinite family of flower snarks. Cavicchioli et al. [1] presented drawings of the 67 pairwise non-isomorphic snarks with at most 24 vertices as well as all cyclically 5-edge-connected snarks of order 26. Their drawings illustrate that the snarks have cycles omitting two adjacent vertices. Since a snark has girth at least five, the leaves of any single double-star form an independent set. Thus every snark having a cycle of length $n - 2$ has a MED decomposition using this cycle. All snarks listed above have such decompositions.

Example 8. Isaacs [4] discovered the flower snarks. For odd k with $k \geq 5$, the flower snark F_k consists of $4k$ vertices, labeled w_i, x_i, y_i, z_i for $1 \leq i \leq k$. The edge set consists of a cycle $[z_1, \dots, z_k]$ of length k, a cycle $[x_1, \dots, x_k, y_1, \dots, y_k]$ of length $2k$, and the k stars with center w_i and leaves x_i, y_i, z_i. To prove that F_k is (7,2)-edge-choosable, we provide a MED decomposition; as above, it suffices to find a cycle of length $4k-2$ that omits adjacent vertices. For example, $[x_1, x_2, w_2, y_2, y_3, w_3, x_3, \dots, x_{k-1}, w_{k-1}, y_{k-1}, y_k, w_k, z_k, \dots, z_1, w_1]$ omits only x_k and y_1.

We do not know whether every snark has a cycle that omits just two adjacent vertices. We conjecture at least that all snarks have MED decompositions. In fact, snarks seem to be easy graphs in which to find MED decompositions, and we believe the following.

Conjecture 9. *Every 2-connected graph with maximum degree 3 has a MED decomposition (a decomposition into a matching, a union of disjoint even cycles, and an independent set of double-stars).*

If Conjecture 9 is true, then by Theorem 7 all 2-connected graphs with maximum degree 3 are $(7,2)$-edge-choosable. The argument does not extend to graphs with cut-vertices.

Example 10. *There are* 3*-regular graphs having no MED decomposition.* A small example is the standard small 3-regular graph having no perfect matching. Let H be the graph obtained from K_4 by subdividing one edge. Form G^* from $3H$ by adding one vertex v made adjacent to the 2-valent vertices of $3H$. Let x_1, x_2, x_3 be the neighbors of v.

Suppose that G^* has a MED decomposition. Putting an x_i on an even cycle leaves a claw, which cannot be decomposed. Hence the decomposition must have each x_i as a central vertex in a double-star. Since a triangle can't be decomposed, the other central vertex must be v. Now the three required double-stars share v, yielding a contradiction.

Nevertheless, G^* is $(7,2)$-edge-choosable and hence is not a counterexample to Mohar's conjecture. The proof unfortunately requires some case analysis.

Theorem 11. *The graph G^* is $(7,2)$-edge-choosable.*

Proof. Again we reduce the problem to $(4,2)$-edge-choosability of even cycles.

Let H' be the graph obtained from K_4+K_1 by subdividing one edge and adding a pendant edge joining the new vertex to the isolated vertex. Let a be the pendant edge, incident to edges b_1 and b_2. Let d be the edge not incident to b_1 or b_2. Let c_1, c_2, c_3, c_4 be the edges of the remaining cycle C in order, with c_1 and c_2 incident to b_1.

Given a 7-uniform list assignment L on the edges of H', a partial 2-set L-coloring can be chosen with one color each on b_1 and b_2 and two on d so that a color is saved on each edge of C. If all the lists except that on a are the same, then we pick colors for b_1 and b_2 and use the same two colors on d. Otherwise, since all the lists have size 7, the argument of Lemma 6 finds such a coloring. After doing this, only one remaining color at b_2 is forbidden (by b_1), so we have at least five choices for a second color there.

Now let H'_1, H'_2, H'_3 denote the three copies of H' in G^*, and let a_i denote edge a in H'_i. First, choose colors on H'_1 as specified by the claim above. When choosing colors on H'_2 in this way, we pause before making the second choice on b_2. There are at least five choices for this color, and at least four choices remain at a_1, so we can choose color α for a_1 and β for b_2 to save a color on a_2.

Let $L'(e)$ denote the list of available colors for e at this time. Since we have chosen a color on a_1 and three colors incident to a_2 in H'_2, but saved a color on a_2, we may assume that $|L'(a_2)| = 4$. Similarly, from the colors chosen on H'_1 we may assume that $|L'(a_1)| = 3$.

If we can choose a 2-set coloring for $E(H_3)$ so that a_3 receives a color not in $L'(a_2)$, then we can safely pick one more color on a_1 and two on a_2. We can then return to pick the last extra color needed on H'_1 and H'_2 outside the copies of C, after which we can color each copy of C since we save a color on each of its edges.

Hence it remains to choose the colors on H_3 so that a_3 has a color outside a specified set of size 5 (avoiding also the color on a_1). There remain two desirable colors in $L(a_3)$; call them c and c'. We show that H_3' can be colored using c or c' on a_3.

Let $L_0(e) = L(e) - \{c, c'\}$ for $e \in E(H_3)$. If $L_0(b_1)$ has a color β outside $L(c_1) \cap L(c_2)$, then our original claim about H' using the argument of Lemma 6 allows us to pick β for b_1, two colors for d, and two colors for b_2 that are not both in $\{c, c'\}$ (while saving on each edge of C). Now we can use c or c' and one of three other colors on a_3 and then return to complete the coloring on H_3.

The same outcome occurs if $L_0(d)$ has a color not in $L(c_1) \cap L(c_2)$. Hence $L(c_1) \cap L(c_2)$ contains $L_0(b_1) \cup L_0(d)$. The restricted sets have size at least 5, and $|L(c_1) \cap L(c_2)| \leq 7$, so $|L_0(b_1) \cap L_0(d)| \geq 3$. The symmetric argument switching the roles of b_1 and b_2 shows that $|L_0(b_2) \cap L_0(d)| \geq 3$.

Hence we may choose $\alpha \in L_0(b_1) \cap L_0(d)$ and $\beta \in L_0(b_2) \cap L_0(d) - \{\alpha\}$, putting color α on b_1, β on b_2, and both on d. Choose another color on b_2, then c or c' on a_3, then a color that remains for b_1. This leaves lists of size 4 on C, and we can complete the 2-set L-coloring. $\square$

Mohar's conjecture remains open, and Example 10 shows that something stronger than Theorem 7 will be needed to prove it. There may be a generalization of MED decomposition that suffices for $(7, 2)$-edge-choosability, permitting more complicated structures than double-stars in addition to the matching and even cycles. Whether this is true or not, it would still be interesting to show that every 2-connected graph with maximum degree 3 has a MED decomposition.

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