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The Cyclic Sieving Phenomenon for Faces of Generalized Cluster Complexes

by

Sen-Peng Eu[1]
Department of Applied Mathematics
National University of Kaohsiung
Kaohsiung 811, Taiwan, ROC

Tung-Shan Fu[2]
Mathematics Faculty
National Pingtung Institute of Commerce
Pingtung 900, Taiwan, ROC

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ABSTRACT

The notion of cyclic sieving phenomenon is introduced by Reiner, Stanton, and White as a generalization of Stembridge's $q = -1$ phenomenon. The generalized cluster complexes associated to root systems are given by Fomin and Reading as a generalization of the cluster complexes found by Fomin and Zelevinsky. In this paper, the faces of various dimensions of the generalized cluster complexes in type A_n, B_n, D_n, and $I_2(a)$ are shown to exhibit the cyclic sieving phenomenon under a cyclic group action. For the cluster complexes of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, a verification for such a phenomenon on the facets is given.

1 Introduction

In [8], Reiner, Stanton, and White introduced the notion of cyclic sieving phenomenon as a generalization of Stembridge's $q = -1$ phenomenon for generating functions of a set of combinatorial structures with a cyclic group action. Namely, a triple $(X, X(q), C)$ consisting of a finite set X, a polynomial $X(q) \in \mathbb{Z}[q]$ with the property that $X(1) = |X|$, and a cyclic group C that acts on X is said to exhibit the *cyclic sieving phenomenon* if for every $c \in C$,

$$[X(q)]_{q=\omega} = |\{x \in X : c(x) = x\}|, \tag{1}$$

where the complex number ω is a root of unity of the same multiplicative order as c. Equivalently, if $X(q)$ is expanded as $X(q) \equiv a_0 + a_1 q + \cdots + a_{n-1} q^{n-1} \pmod{q^n - 1}$, where n is the order of C, then a_k counts the number of orbits on X under C, the stabilizer-order of which divides k. In particular, a_0 counts the total number of orbits, a_1 counts the number of free orbits, and $a_2 - a_1$ is the number of orbits that have a stabilizer of order 2. This paper is motivated by the following concrete example. Here we use the notation $\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{[n]!_q}{[i]!_q [n-i]!_q}$, where $[n]!_q = [1]_q [2]_q \cdots [n]_q$ and $[i]_q = 1 + q + \cdots + q^{i-1}$.

Theorem 1.1 ([8, Theorem 7.1]) *Let X be the set of dissections of a regular $(n+2)$-gon using k noncrossing diagonals $(0 \le k \le n-1)$. Let*

$$X(q) := \frac{1}{[k+1]_q} \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{2}$$

Let the cyclic group C of order $n+2$ act on X by cyclic rotation of the polygon. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Note that $X(1) = |X|$ is the well-known Kirkman-Cayley number. In [4], Fomin and Zelevinsky introduced a simplicial complex $\Delta(\Phi)$, called the *cluster complex*, associated to a root system Φ, which can be realized by a combinatorial structure constructed in terms of polygon-dissections. In fact, Theorem 1.1 proves the cyclic sieving phenomenon for the k-faces of the cluster complex $\Delta(\Phi)$ in type A_{n-1}, under a cyclic group generated by a deformation Γ (defined in Section 2) of Coxeter element of Φ. This connection will be explained in next section.

From [4, Theorem 1.9], the number of *facets* (i.e., maximal faces) of $\Delta(\Phi)$ can be expressed uniformly as

$$\mathrm{Cat}(\Phi) := \prod_{i=1}^{n} \frac{h + e_i + 1}{e_i + 1}, \tag{3}$$

known as the *generalized Catalan numbers* $\mathrm{Cat}(\Phi)$, where h is the Coxeter number and $e_1, \ldots, e_n$ are the exponents of Φ. In particular, $\mathrm{Cat}(A_{n-1}) = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number. The cyclic group generated by Γ is of order $h+2$. Along with the q-analogue of (3) defined by

$$\mathrm{Cat}(\Phi, q) := \prod_{i=1}^{n} \frac{[h + e_i + 1]_q}{[e_i + 1]_q}, \tag{4}$$

one of our main results is to prove the following theorem.

Theorem 1.2 *Let X be the set of facets of the cluster complex $\Delta(\Phi)$. Let $X(q) := Cat(\Phi, q)$ be defined in (4). Let the cyclic group C of order $h+2$ generated by Γ act on X. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

Moreover, in [3], Fomin and Reading defined the *generalized cluster complexes* $\Delta^s(\Phi)$ associated to a root system Φ and a positive integer s, which specializes at $s=1$ to $\Delta(\Phi)$. The purpose of this paper is to study the cyclic sieving phenomenon for the faces of $\Delta^s(\Phi)$, along with a q-analogue $X(q)$ of face numbers, under a cyclic group action. Making use of Fomin and Reading's results, we prove the cyclic sieving phenomenon by a combinatorial approach for $\Delta^s(\Phi)$ in type A_n, B_n, D_n, and $I_2(a)$. For Φ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, although a systematic method is not available, we verify such a phenomenon by computer for the facets of $\Delta^s(\Phi)$ when $s=1$.

This paper is organized as follows. We review backgrounds of cluster complexes and generalized cluster complexes in Section 2. As the main results of this paper, the cyclic sieving phenomenon for $\Delta^s(\Phi)$ in type A_n, B_n, D_n, and $I_2(a)$ are given in Sections 3, 4, 5, and 6, respectively. The cases of exceptional type are given in Section 7. Finally, a discussion regarding the polynomial $X(q)$ is given in Section 8.

2 Backgrounds

In this section, we review basic facts of cluster complexes from [4] and interpret Theorem 1.1 in terms of cluster complexes (type A_{n-1}). Then we review the definition of generalized cluster complexes from [3] and introduce the main purpose of this paper. Most of this section follows the material in [3, Sections 2 and 3]

2.1 Cluster complexes

Let Φ be an irreducible root system of rank n. Let $\Phi_{>0}$ denote the set of positive roots in Φ, and let $\Pi = \{\alpha_i : i \in I\}$ denote the set of simple roots in Φ, where $I = \{1, \dots, n\}$. Accordingly, $-\Pi = \{-\alpha_i : i \in I\}$ is the set of negative simple roots. The set $S = \{s_i : i \in I\}$ of reflections corresponding to simple roots α_i generates a finite reflection group W that naturally acts on Φ. The pair (W, S) is a Coxeter system.

Let $I = I_+ \cup I_-$ be a partition of I such that each of sets I_+ and I_- is totally disconnected in the Coxeter diagram. Let $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$. Define the involutions $\tau_\pm : \Phi_{\geq -1} \to \Phi_{\geq -1}$ by

$$\tau_\epsilon(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_i, \text{ for } i \in I_{-\epsilon}, \\ \left(\prod_{i \in I_\epsilon} s_i\right)(\alpha) & \text{otherwise,} \end{cases}$$

for $\epsilon \in \{+, -\}$. The product $\Gamma = \tau_- \tau_+$ generates a cyclic group $\langle \Gamma \rangle$ that acts on $\Phi_{\geq -1}$. For example, let Φ be the root system of type A_2, with $I = \{1, 2\}$. The set $\Phi_{\geq -1} =$

$\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, -\alpha_1, -\alpha_2\}$ of roots is shown in Figure 1. Setting $I_+ = \{1\}$ and $I_- = \{2\}$, one can check that Γ acts on $\Phi_{\geq -1}$ by $\alpha_1 \to -\alpha_1 \to \alpha_1+\alpha_2 \to -\alpha_2 \to \alpha_2 \to \alpha_1$.

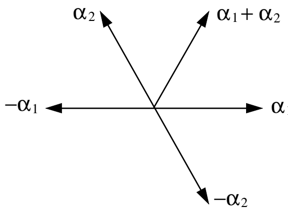

Figure 1: The set $\Phi_{\geq -1}$ in type A_2.

From [4, Section 3.1], a relation of *compatibility* on $\Phi_{\geq -1}$ is defined such that (i) $\alpha, \beta \in \Phi_{\geq -1}$ are compatible if and only if $\Gamma(\alpha)$ and $\Gamma(\beta)$ are compatible; (ii) $-\alpha_i \in -\Pi$ and $\beta \in \Phi_{>0}$ are compatible if and only if the simple root expansion of β does not involve α_i. Following [4, p. 983], the *cluster complex* $\Delta(\Phi)$ is defined to be the simplicial complex whose faces are subsets of roots in $\Phi_{\geq -1}$, which are pairwise compatible. For Φ in type A_n, B_n, and D_n, the cluster complex $\Delta(\Phi)$ can be realized by dissections of a regular polygon such that $\langle\Gamma\rangle$ corresponds to a group action on the dissections by rotation of the given polygon.

Specifically, consider the root system Φ of type A_n. Then $\Phi_{\geq -1}$ consists of positive roots of the form $\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, for $1 \leq i \leq j \leq n$, and negative simple roots $-\alpha_i$, for $1 \leq i \leq n$. Let P be a regular polygon with $n+3$ vertices, labeled by $\{1, 2, \ldots, n+3\}$ counterclockwise. The roots in $\Phi_{\geq -1}$ are identified with the diagonals of P as follows. For $1 \leq i \leq \frac{n+1}{2}$, the root $-\alpha_{2i-1} \in -\Pi$ is identified with the diagonal connecting vertices i and $n+3-i$. For $1 \leq i \leq \frac{n}{2}$, the root $-\alpha_{2i} \in -\Pi$ is identified with the diagonal connecting vertices $i+1$ and $n+3-i$. These diagonals form a 'snake' of negative simple roots. Figure 2 shows this snake in type A_5. The positive roots of $\Phi_{>0}$ are identified with the remaining diagonals as follows. Each α_{ij} is identified with the unique diagonal that intersects the diagonals $-\alpha_i, -\alpha_{i+1}, \ldots, -\alpha_j$ and no other diagonals in the snake. For example, Figure 3 is a realization of the set $\Phi_{\geq -1}$ in type A_2.

Under this bijection, every pair of compatible roots is carried to a pair of noncrossing diagonals. Hence each k-face (i.e., k-element simplex) of the cluster complex $\Delta(A_n)$ corresponds to a dissection of P using k noncrossing diagonals. Moreover, the map Γ corresponds to a clockwise rotation of P that carries point 2 to point 1, etc. Therefore, Theorem 1.1 can be interpreted in terms of $\Delta(A_{n-1})$, i.e., let X be the set of k-faces of $\Delta(A_{n-1})$, let $X(q)$ be the polynomial defined by (2). Then $(X, X(q), \langle\Gamma\rangle)$ exhibits the cyclic sieving phenomenon.

2.2 Generalized cluster complexes

Let s be a positive integer. For each $\alpha \in \Phi_{>0}$, let $\alpha^1, \ldots, \alpha^s$ denote the s 'colored' copies of α. Define

$$\Phi^s_{\geq -1} = \{\alpha^k : \alpha \in \Phi_{>0}, 1 \leq k \leq s\} \cup \{(-\alpha_i)^1 : i \in I\},$$

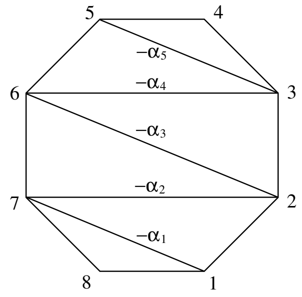

Figure 2: The snake in type A_5.

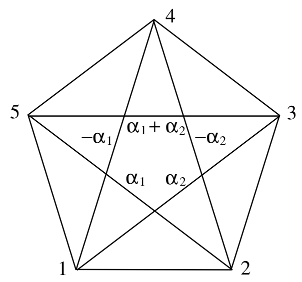

Figure 3: A representation for the set $\Phi_{\geq -1}$ of roots in type A_2.

i.e., $\Phi^s_{\geq -1}$ consists of s copies of the positive roots and one copy of the negative simple roots. The relation of compatibility on $\Phi^s_{\geq -1}$ can be defined by an s-analogue of Γ. For $\alpha^k \in \Phi^s_{\geq -1}$, define

$$\Gamma_s(\alpha^k) = \begin{cases} \alpha^{k+1} & \text{if } \alpha \in \Phi_{>0} \text{ and } k < s, \\ (\Gamma(\alpha))^1 & \text{otherwise.} \end{cases}$$

By [3, Theorem 3.4], the compatibility is defined by the following two conditions (i) α^k and β^l are compatible if and only if $\Gamma_s(\alpha^k)$ and $\Gamma_s(\beta^l)$ are compatible; (ii) $(-\alpha_i)^1$ and β^l are compatible if and only if the simple root expansion of β^l does not involve α_i. The *generalized cluster complex* $\Delta^s(\Phi)$ associated to a root system Φ is defined to be the simplicial complex whose faces are subsets of roots in $\Phi^s_{\geq -1}$, which are pairwise compatible.

A feasible q-analogue $X(q)$ of face numbers plays an essential role in the cyclic sieving phenomenon. Fomin and Reading derived a formula for the face numbers of various dimensions of $\Delta^s(\Phi)$ in terms of Coxeter numbers and exponents for every Φ (see [3, Theorem 8.5]). Let $f_k(\Phi, s)$ denote the *k-face number* of $\Delta^s(\Phi)$. For Φ of type A_n, B_n, D_n, and $I_2(a)$, the k-face numbers can be expressed explicitly as follows.

Theorem 2.1 *For Φ of type A_n, B_n, D_n, and $I_2(a)$, the face numbers of the generalized cluster complex $\Delta^s(\Phi)$ are given by*

(i) $f_k(A_n, s) = \dfrac{1}{k+1}\dbinom{s(n+1)+k+1}{k}\dbinom{n}{k}$,

(ii) $f_k(B_n, s) = \dbinom{sn+k}{k}\dbinom{n}{k}$,

(iii) $f_k(D_n, s) = \dbinom{s(n-1)+k}{k}\dbinom{n}{k} + \dbinom{s(n-1)+k-1}{k}\dbinom{n-2}{k-2}$,

(iv) $f_1(I_2(a), s) = sa+2$, *and* $f_2(I_2(a), s) = \dfrac{(sa+2)(s+1)}{2}$.

Case (i) of Theorem 2.1 is due to J. H. Przytycki and A. S. Sikora [7] and case (ii) is due to E. Tzanaki [11]. We remark that Fomin and Reading's formula is given as a polynomial involving s, which is in the form of the unique factorization of that polynomial into irreducibles. However, the obvious q-analogue of this formula is not always the feasible option for the polynomials $X(q)$, except for $k = n$. In some cases it is not even a polynomial. Readers may refer to the last section for further discussions. The q-analogues $X(q)$ that serve our purpose are derived case by case from Theorem 2.1. (See Theorems 3.1, 4.1, 5.1, and 6.1).

For the special case $k = n$, from [3, Theorem 8.4], the number of facets of $\Delta^s(\Phi)$ can be expressed uniformly as

$$\mathrm{Cat}^{(s)}(\Phi) := \prod_{i=1}^{n} \frac{sh+e_i+1}{e_i+1}, \tag{5}$$

where the *Coxeter number* h and the *exponents* $e_1, \ldots, e_n$ of Φ are listed in Figure 4. Note that the expression (5) specializes at $s = 1$ to the generalized Catalan numbers (3).

Φ	h	$e_1, \ldots, e_n$
A_n	$n+1$	$1, 2, 3, \ldots, n$
B_n	$2n$	$1, 3, 5, \ldots, 2n-1$
D_n	$2(n-1)$	$1, 3, 5, \ldots, 2n-3, n-1$
E_6	12	$1, 4, 5, 7, 8, 11$
E_7	18	$1, 5, 7, 9, 11, 13, 17$
E_8	30	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	12	$1, 5, 7, 11$
H_3	10	$1, 5, 9$
H_4	30	$1, 11, 19, 29$
$I_2(a)$	a	$1, a-1$

Figure 4: The Coxeter number and exponents of Φ.

In this paper, we aim to prove the cyclic sieving phenomenon for the generalized cluster complexes $\Delta^s(\Phi)$ in the framework that X is the set of k-faces of $\Delta^s(\Phi)$, C is the cyclic group of order $sh+2$ generated by Γ_s, and $X(q)$ is a q-analogue of the k-face numbers. For Φ of type A_n, B_n, and D_n, our results rely on Fomin and Reading's realization constructed in terms of polygon-dissections [3, Section 5]. Under this realization, the cyclic group C corresponds to rotation of the given polygon. In type $I_2(a)$, we make use of the graph-representation of $\Delta^s(I_2(a))$ given in [3, Example 4.4]. For Φ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4, a complete verification of such a phenomenon is given only for the facets of $\Delta^s(\Phi)$ and only when $s=1$.

3 The cyclic sieving phenomenon for $\Delta^s(A_{n-1})$

Let P be a regular polygon with $sn+2$ vertices labeled by $\{1,2,\dots,sn+2\}$ counterclockwise. Consider the set of dissections of P into $(sj+2)$-gons $(1\le j\le n-1)$ by noncrossing diagonals. Such dissections are called *s-divisible.* For convenience, a diagonal in an s-divisible dissection is called *s-divisible*. Consider a root system Φ of type A_{n-1}. Following [3, Section 5.1], the roots of $\Phi^s_{\ge -1}$ can be identified with the s-divisible diagonals of P as follows. For $1\le i\le \frac{n}{2}$, the root $-\alpha_{2i-1}$ is identified with the diagonal connecting points $s(i-1)+1$ and $s(n-i)+2$. For $1\le i\le\frac{n-1}{2}$, the root $-\alpha_{2i}$ is identified with the diagonal connecting points $si+1$ and $s(n-i)+2$. These $n-1$ diagonals form an s-snake of negative simple roots. For each positive root $\alpha_{ij}=\alpha_i+\cdots+\alpha_j$ $(1\le i\le j\le n-1)$, there are exactly s diagonals, which are s-divisible, intersecting the diagonals $-\alpha_i,\dots,-\alpha_j$ and no other diagonals in the s-snake. This collection of diagonals is of the form $D,\Gamma^1_s(D),\dots,\Gamma^{s-1}_s(D)$, for some diagonal D. For $1\le k\le s$, we identify α^k_{ij} with $\Gamma^{k-1}_s(D)$. Figure 5 shows the s-snake for $s=3$ and $n=4$, along with the diagonals identified with the colored roots α^1_{23}, α^2_{23}, and α^3_{23}. Under this bijection, the k-faces of $\Delta^s(A_{n-1})$ correspond to the s-divisible dissections of P using k noncrossing diagonals, and Γ_s corresponds to clockwise rotation of P carrying point 2 to point 1, etc.

A feasible polynomial for $X(q)$ is the natural q-analogue of Theorem 2.1(i). Define

$$G(s,n,k;q)=\frac{1}{[k+1]_q}\begin{bmatrix} sn+k+1\\ k\end{bmatrix}_q\begin{bmatrix} n-1\\ k\end{bmatrix}_q, \tag{6}$$

for $0\le k\le n-1$. Note that $G(s,n,k;1)=f_k(A_{n-1},s)$. As a generalization of Theorem 1.1, we prove that the k-faces of $\Delta^s(A_{n-1})$ exhibit the cyclic sieving phenomenon under the group action $\langle\Gamma_s\rangle$.

Theorem 3.1 *For positive integers s and n, let X be the set of s-divisible dissections of an $(sn+2)$-gon using k noncrossing diagonals. Let the cyclic group C of order $sn+2$ act on X by cyclic rotation of the polygon. Let $X(q):=G(s,n,k;q)$. Then $(X,X(q),C)$ exhibits the cyclic sieving phenomenon.*

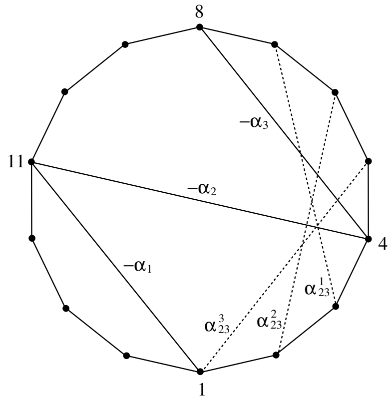

Figure 5: The s-snake in type A_3.

For example, take $s = 2$, $n = 3$, and $k = 2$. Then $X(q) \equiv 2+q+2q^2+q^3+2q^4+q^5+2q^6+q^7 \pmod{q^8 - 1}$. As shown in Figure 6, there are 12 2-divisible dissections of an octagon using 2 noncrossing diagonals. These dissections are partitioned into two orbits under a group action by cyclic rotation, one of which is free and the other has a stabilizer of order 2.

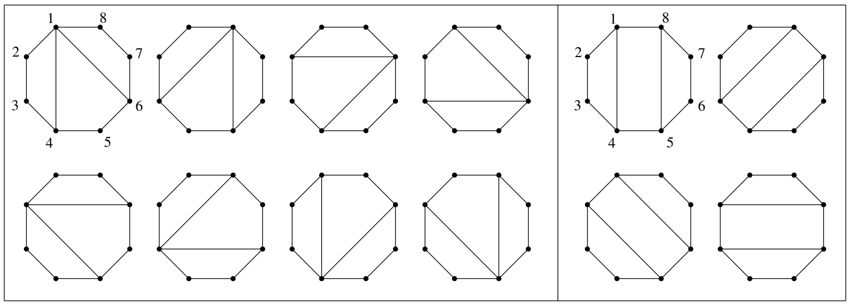

Figure 6: The 2-divisible dissections of an octagon using 2 noncrossing diagonals.

We shall prove Theorem 3.1 by verifying condition (1) mentioned in the introduction. Recall that a complex number ω is a primitive d-th root of unity if d is the least integer such that $\omega^d = 1$. First, we evaluate $X(q)$ at primitive roots of unity (Proposition 3.4). To do this, we make use of the following facts and the q-Lucas theorem (see [5, Proposition 2.1]).

Lemma 3.2 *Let m, m_1, m_2, k, and d be positive integers. Let ω be a primitive d-th root of unity. Then*

(i) $[m]_{q=\omega} = 0$ *if and only if d divides m ($d \neq 1$).*

(ii) *If $m_1 \equiv m_2$ (mod d), then*

$$\lim_{q\to\omega} \frac{[m_1]_q}{[m_2]_q} = \begin{cases} \frac{m_1}{m_2} & \text{if } m_1 \equiv m_2 \equiv 0 \pmod{d}, \\ 1 & \text{if } m_1 \equiv m_2 \not\equiv 0 \pmod{d}. \end{cases}$$

(iii) (q-Lucas Theorem) *If $m = ad + b$ and $k = rd + s$, where $0 \le b, s \le d-1$, then*

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q=\omega} = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_{q=\omega}.$$

As a special case of Lemma 3.2(iii), the following result is repeatedly used.

Corollary 3.3 ([1]) *Let m, k, and d be positive integers. Let ω be a primitive d-th root of unity. For $d \ge 2$ a divisor of m, then*

$$\begin{bmatrix} m+k-1 \\ k \end{bmatrix}_{q=\omega} = \begin{cases} \binom{\frac{m+k}{d}-1}{\frac{k}{d}} & \text{if } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

We begin by evaluating $G(s, n, k; q)$ at $q = \omega$.

Proposition 3.4 *For $d \ge 2$ a divisor of $sn+2$, let ω be a primitive d-th root of unity. Then*

$$[G(s,n,k;q)]_{q=\omega} = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \text{if } d = 2,\ k \text{ odd, and } n \text{ even} \\ \binom{\frac{sn+2+k}{d}-1}{\frac{k}{d}} \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}} & \text{if } d \ge 2 \text{ and } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, we have

$$\begin{aligned} [G(s,n,k;q)]_{q=\omega} &= \lim_{q\to\omega} \frac{1}{[k+1]_q} \begin{bmatrix} sn+k+1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \\ &= \lim_{q\to\omega} \frac{([sn+k+1]_q \cdots [sn+2]_q)([n-1]_q \cdots [n-k]_q)}{[k+1]!_q [k]!_q}. \end{aligned} \tag{7}$$

For $d = 2$, making use of (i) and (ii) of Lemma 3.2 and examining parities of the factors in the numerator and denominator in (7), we have

$$[G(s,n,k;q)]_{q=\omega} = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \text{if } k \text{ odd and } n \text{ even,} \\ \binom{\frac{sn+k}{2}}{\frac{k}{2}} \binom{\lfloor \frac{n-1}{2} \rfloor}{\frac{k}{2}} & \text{if } k \text{ even,} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

For $d \geq 3$, by Corollary 3.3, we have

$$\lim_{q\to\omega} \begin{bmatrix} sn+k+1 \\ k \end{bmatrix}_q = \begin{cases} \binom{\frac{sn+2+k}{d}-1}{\frac{k}{d}} & \text{if } d|k, \\ 0 & \text{otherwise.} \end{cases}$$

For the rest of $[G(s,n,k;q)]_{q=\omega}$, we have

$$\lim_{q\to\omega} \frac{1}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \lim_{q\to\omega} \frac{[n-1]_q \cdots [n-k]_q}{[k+1]!_q}. \tag{9}$$

Note that $d \nmid n$ since d divides $sn+2$. For $0 \leq k \leq n-1$, one can check that there are at least as many factors $[t]_q$ in the numerator as the denominator in (9) such that d divides t, and hence (9) does not tend to infinity. Moreover, for $d|k$, $\lim_{q\to\omega}[k+1]_q = 1$ and

$$\lim_{q\to\omega} \frac{1}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}}.$$

It follows that, for $d \geq 3$, we have

$$[G(s,n,k;q)]_{q=\omega} = \begin{cases} \binom{\frac{sn+2+k}{d}-1}{\frac{k}{d}} \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}} & \text{if } d|k, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The assertion follows from (8) and (10). □

Let X be the set of s-divisible dissections of an $(sn+2)$-gon P using k noncrossing diagonals. Let C be the cyclic group of order $sn+2$ acting on X by cyclic rotation of P. For $d \geq 2$ a divisor of $sn+2$, let $U(s,n,k,d) \subseteq X$ denote the set of dissections that are invariant under d-fold rotation (i.e., a subgroup C_d of order d of C generated by a $\frac{2\pi}{d}$-rotation of P). In the following, we shall enumerate $U(s,n,k,d)$ to complete the proof of Theorem 3.1.

For any centrally symmetric dissection, we observe that there is either a diameter or an $(sj+2)$-gon in the center. Hence if $U(s,n,k,d)$ is nonempty, then either $d=2$ and k is odd, or $d \geq 2$ and d divides k. These two cases are treated in Propositions 3.5 and 3.6, respectively.

Proposition 3.5 *For positive integers s and n, with sn even and k odd, we have*

$$|U(s,n,k,2)| = \begin{cases} \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} & \textit{if } n \textit{ even}, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Note that a dissection with an odd number of diagonals, which is invariant under 2-fold rotation, contains a unique diameter. This diameter dissects the polygon into two

$(sm+2)$-gons, where $m = \frac{n}{2}$. Hence n is even. Since there are $\frac{sn+2}{2}$ diameters and the dissection is completely determined by the subdivision of the $(sm+2)$-gon on either side of the diameter using $\frac{k-1}{2}$ diagonals, by (6) the number of such dissections is

$$\frac{sn+2}{2} \cdot G\left(s, \frac{n}{2}, \frac{k-1}{2}; 1\right) = \binom{\frac{sn+k+1}{2}}{\frac{k+1}{2}}\binom{\frac{n-2}{2}}{\frac{k-1}{2}},$$

as required. □

For the latter case, the result relies on a bijection (Proposition 3.6), which is inspired by work of Tzanaki [11]. In fact, for the special case $d = 2$ and n even, the result has been obtained by Tzanaki in [11, Corollary 3.2] by a bijection similar to the one given by Przytycki and Sikora in [7, Theorem 1]. We extend this method to enumerate d-fold rotationally symmetric dissections for all $d \geq 2$.

Proposition 3.6 *For $d \geq 2$ a common divisor of $sn+2$ and k, there is a bijection between $U(s,n,k,d)$ and the cartesian product of the set of sequences $\{1 \leq a_1 \leq a_2 \leq \cdots \leq a_{\frac{k}{d}} \leq \frac{sn+2}{d}\}$ and the set of sequences $(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$ with exactly $\frac{k}{d}$ entries equal to 1, where $m = \lfloor \frac{n-1}{d} \rfloor$.*

Proof. Fixing s and d, we define $N_r := \{n \in \mathbb{Z} : n-1 \equiv r \pmod{d}$, and $d|(sn+2)\}$, for $0 \leq r \leq d-1$. Given an N_r and an $n \in N_r$, let $b = \frac{sn+2}{d}$. The vertices of an $(sn+2)$-gon are partitioned into d *sectors* of size b by a vertex-labeling $\{1_1, 2_1, \ldots, b_1\}$, $\{1_2, 2_2, \ldots, b_2\}, \ldots, \{1_d, 2_d, \ldots, b_d\}$ in the counterclockwise order. For each $\pi \in U(s,n,k,d)$, there is no diameter in π since d divides k. Every diagonal of π is oriented in such a way that if you travel along it in this direction then the center of the polygon is on the left. These diagonals are partitioned into $\frac{k}{d}$ orbits of size d under the d-fold rotation C_d such that each orbit R contains a unique diagonal that leaves from the first sector $\{1_1, 2_1, \ldots, b_1\}$. We associate R with the label $i \in [b]$ if that diagonal leaves from i_1.

Let $m = \frac{n-1-r}{d}$. Define

$$A(m,k) = \{(a_1, \ldots, a_{\frac{k}{d}}) : 1 \leq a_1 \leq \cdots \leq a_{\frac{k}{d}} \leq b\},$$
$$B(m,k) = \{(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m : \text{ exactly } \tfrac{k}{d} \text{ entries } \epsilon_j \text{ are equal to } 1\}.$$

We shall establish a bijection $\Omega_m : U(s,n,k,d) \to \{(\mu,\nu) | \mu \in A(m,k), \nu \in B(m,k)\}$ by induction on m. Recall that $k \leq n-1$ and d divides k. If $m = 1$ then $\frac{k}{d} < 2$. For $k = 0$, the map Ω_1 sends the trivial dissection to $(\emptyset, (\epsilon_1))$, where $\epsilon_1 = 0$, and for $\frac{k}{d} = 1$, Ω_1 sends each dissection π to $((a_1), (\epsilon_1))$, where a_1 is the label of the unique diagonal-orbit in π and $\epsilon_1 = 1$.

For the inductive step, assume that we have established the bijection Ω_{m-1}, for $0 \leq \frac{k}{d} \leq m-1$. To determine Ω_m, let P be an $(sn+2)$-gon, and let $\pi \in U(s,n,k,d)$. The result is trivial for $k = 0$. For $\frac{k}{d} \geq 1$, let $\mu = (a_1, \ldots, a_{\frac{k}{d}})$ be the sequence of labels associated to the diagonal-orbits in π, where $1 \le a_1 \le \cdots \le a_{\frac{k}{d}} \le b$. Locate the first a_j $(1 \le j \le \frac{k}{d})$ such that $a_j + s + 1 \le a_{j+1}$ (where $a_{\frac{k}{d}+1}$ is understood to mean $a_1 + b$). Such a_j exists by pigeon hole principle. With abuse of notation, we identify a_i with the vertex labeled by $(a_i)_1$ and $a_i + k$ with the vertex at distance k from a_i in the counterclockwise order. We set $\epsilon_1 = 1$ if $a_j(a_j+s+1)$ is a diagonal in π, and $\epsilon_1 = 0$ otherwise. Let P' be the truncated $(sn+2-sd)$-gon obtained from P by removing the vertex-orbit of the set $\{a_j+1, \ldots, a_j+s\}$ under C_d, which is dissected by the remaining $\frac{k}{d} - \epsilon_1$ orbits of diagonals, and let π' be the dissection determined by these orbits. Note that $sn+2-sd \in N_r$ and that the labels associated to these diagonal-orbits in π' are the same, whether considered in P or P'. Therefore, by induction, Ω_{m-1} carries π' to a pair $(\mu', \nu') \in A(m-1,k) \times B(m-1,k)$, where $\mu' = \{a_1, \ldots, a_{\frac{k}{d}}\}$ if $\epsilon_1 = 0$ and $\mu' = \{a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{\frac{k}{d}}\}$ otherwise, and $\nu' = (\epsilon_2, \ldots, \epsilon_m)$ with $\frac{k}{d} - \epsilon_1$ entries ϵ_j equal to 1. We define $\Omega_m(\pi)$ to be the pair $((a_1, \ldots, a_{\frac{k}{d}}), (\epsilon_1, \ldots, \epsilon_m))$.

To define Ω_m^{-1}, we proceed by induction. The case $m = 1$ is obvious. Assume that Ω_{m-1}^{-1} has been determined. Given the sequences $\mu = (a_1, \ldots, a_{\frac{k}{d}})$, where $1 \le a_1 \le \cdots \le a_{\frac{k}{d}} \le b$, and $\nu = (\epsilon_1, \ldots, \epsilon_m)$ with $\frac{k}{d}$ entries equal to 1, we shall retrieve the dissection $\Omega_m^{-1}((\mu, \nu))$ of an $(sn+2)$-gon P from (μ, ν) by recovering the set of diagonal-orbits. For $k = 0$, the element $(\emptyset, (0, \ldots, 0))$ is carried to the trivial dissection. For $\frac{k}{d} \ge 1$, locate the first a_j $(1 \le j \le \frac{k}{d})$ such that $a_j + s + 1 \le a_{j+1}$. Then there is an orbit R containing the diagonal $a_j(a_j+s+1)$ if $\epsilon_1 = 1$ and there is not otherwise. Let P' be the $(sn+2-sd)$-gon the i-th sector of which consists the vertices $\{1_i, \ldots, (a_j)_i, (a_j+s+1)_i, \ldots, b_i\}$, for $1 \le i \le d$. Let $\mu' = (a_1, \ldots, a_{\frac{k}{d}})$ if $\epsilon_1 = 0$ and $\mu' = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{\frac{k}{d}})$ otherwise, and let $\nu' = (\epsilon_2, \ldots, \epsilon_m)$ with $\frac{k}{d} - \epsilon_1$ entries ϵ_j equal to 1. By induction, the set $\mathcal{R}$ of diagonal-orbits in $\mathcal{P}'$ is recovered from (μ', ν') by Ω_{m-1}^{-1}. Therefore, we define $\Omega_m^{-1}((\mu, \nu))$ to be the dissection determined by $\mathcal{R}$ if $\epsilon_1 = 0$ and by $\mathcal{R} \cup \{R\}$ if $\epsilon_1 = 1$.

Hence we prove the assertion for the $(sn+2)$-gons, where $n \in N_r$. Since the argument works well for all N_r $(0 \le r \le d-1)$, the proof is completed. □

For illustration, Figure 7(a) is a dissection of a polygon with 24 vertices ($s = 2$, $n = 11$) using 6 noncrossing diagonals, which is invariant under 3-fold rotation C_3. These diagonals are partitioned into two orbits under C_3 with labels 3 and 8, respectively. The bijection Ω_3 in Proposition 3.6 carries the dissection to the pair of sequences $(a_1, a_2) = (3, 8)$ and $(\epsilon_1, \epsilon_2, \epsilon_3) = (0, 1, 1)$ as shown in (b)-(d) of Figure 7.

Corollary 3.7 *For $d \ge 2$ a common divisor of $sn+2$ and k, we have*

$$|U(s,n,k,d)| = \binom{\frac{sn+2+k}{d} - 1}{\frac{k}{d}} \binom{\lfloor \frac{n-1}{d} \rfloor}{\frac{k}{d}}.$$

Since the results of Proposition 3.5 and Corollary 3.7 agree with that of Proposition 3.4, the proof of Theorem 3.1 is completed.

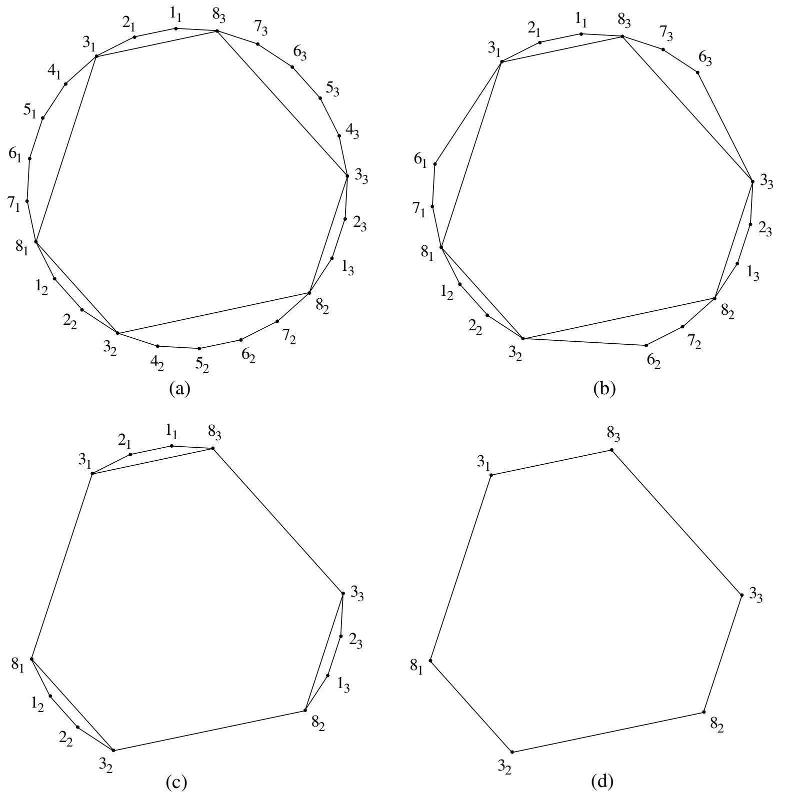

Figure 7: A dissection of a 24-gon carried to the pair $((3,8),(0,1,1))$ by Ω_3.

Of independent interest, we notice that the structure of noncrossing trees appears to admit the cyclic sieving phenomenon, which can be recovered immediately from the one for quadrangulations of a $(2n+2)$-gon. A *noncrossing tree* is a tree drawn on a circle of $n+1$ points numbered counterclockwise from 1 to $n+1$ such that the edges lie entirely within the circle and do not cross. We shall establish a bijection Λ between the set X of noncrossing trees with $n+1$ vertices and the set Y of quadrangulations of a $(2n+2)$-gon P, with the cyclic sieving phenomenon preserved.

Given a $\pi \in X$, we associate the edge set $\{E_1,\ldots,E_n\}$ of π with a set of noncrossing lines $\{L_1,\ldots,L_n\}$ within P such that if E_i connects points $j,k \in [n+1]$ of π then L_i connects points $2j-1$, $2k-1$ of P, for $1 \le i \le n$. Then $\Lambda(\pi)$ is defined to be the unique quadrangulation of P that has $\{L_1,\ldots,L_n\}$ as the set of diagonals of all quadrilaterals. Conversely, given a $\pi' \in Y$, we observe that each quadrilateral of π' has a diagonal the endpoints of which are labeled with odd numbers. Hence Λ^{-1} is obtained simply by a reverse procedure. Under this bijection, rotation on a noncrossing tree π by one point clockwise corresponds to rotation on $\Lambda(\pi)$ by two points clockwise.

For example, on the left of Figure 8 is a noncrossing tree with 5 points. The corresponding quadrangulation of a 10-gon is shown on the right, the set of diagonals of all quadrilaterals of which is shown in the center.

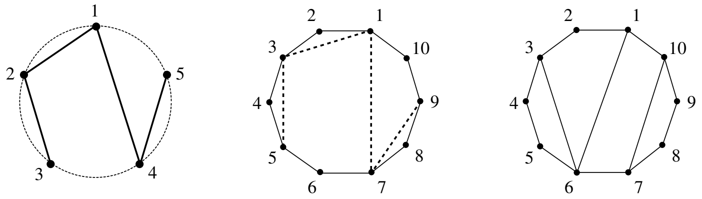

Figure 8: A noncrossing tree on 5 points, and the corresponding quadrangulation of a 10-gon.

By Theorem 3.1 with $(s, k) = (2, n-1)$ and bijection Λ, we have the following result.

Theorem 3.8 *Let X be the set of noncrossing trees with $n+1$ points, and let the cyclic group C of order $n+1$ act on X by cyclic rotation of the points. Let $X(q) := \frac{1}{[2n+1]_q}\begin{bmatrix}3n\\ n\end{bmatrix}_q$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

We remark that $X(1)$ recovers a known formula, due to M. Noy [6], for the number of noncrossing trees with $n+1$ points.

4 The cyclic sieving phenomenon for $\Delta^s(B_n)$

Following [3, Section 5.2], the generalized cluster complex $\Delta^s(\Phi)$ for type B_n can be realized as follows (see also [10, 11]). Let P be a regular polygon with $2sn+2$ vertices. The vertices are labeled by $\{1, 2, \ldots, sn+1, \overline{1}, \overline{2}, \ldots, \overline{sn+1}\}$ counterclockwise. A *B-diagonal* of P is either (i) a *diameter*, i.e., a diagonal that connects a pair of antipodal points $i, \overline{i}$, for some $1 \le i \le sn+1$, or (ii) a pair of s-divisible diagonals ij, $\overline{ij}$, for two distinct $i, j \in \{1, 2, \ldots, sn+1, \overline{1}, \overline{2}, \ldots, \overline{sn+1}\}$, nonconsecutive around the boundary of P. (It is understood that if $a = \overline{i}$, then $\overline{a} = i$). Note that a B-diagonal dissects P into a pair of $(sm+2)$-gons and a centrally symmetric $(2s(n-m)+2)$-gon $(1 \le m \le n)$. The vertices of $\Delta^s(B_n)$ correspond to the B-diagonals of P, and the faces of $\Delta^s(B_n)$ correspond to s-divisible dissections of P using B-diagonals. The facets correspond to the centrally symmetric dissections of P into $(s+2)$-gons. For $s = 1$, this complex is the dual complex of the n-dimensional *cyclohedron*, or *Bott-Taubes polytope* (see [2, Lecture 3]). Under this bijection, the map Γ_s corresponds to clockwise rotation of P carrying point 2 to point 1, etc.

Taking Gaussian coefficients with base q^2 in Theorem 2.1(ii), we define

$$H(s, n, k; q) = \begin{bmatrix}sn+k\\ k\end{bmatrix}_{q^2}\begin{bmatrix}n\\ k\end{bmatrix}_{q^2}, \tag{11}$$

for $0 \le k \le n$. Note that $H(s, n, k; 1) = f_k(B_n, s)$. We prove that the faces of $\Delta^s(B_n)$ exhibit the cyclic sieving phenomenon under the group action $\langle\Gamma_s\rangle$.

Theorem 4.1 *For positive integers s and n, let X be the set of s-divisible dissections of a $(2sn+2)$-gon using k noncrossing B-diagonals. Let the cyclic group C of order $2sn+2$ act on X by cyclic rotation of the polygon. Let $X(q) := H(s,n,k;q)$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

For example, take $s=1$, $n=3$, and $k=1$. $X(q) \equiv 3+3q^2+3q^4+3q^6 \pmod{q^8-1}$. As shown in Figure 9, there are 12 1-divisible dissections of an octagon using 1 B-diagonal. These dissections are partitioned into three orbits under a group action by cyclic rotation, all of which have a stabilizer of order 2.

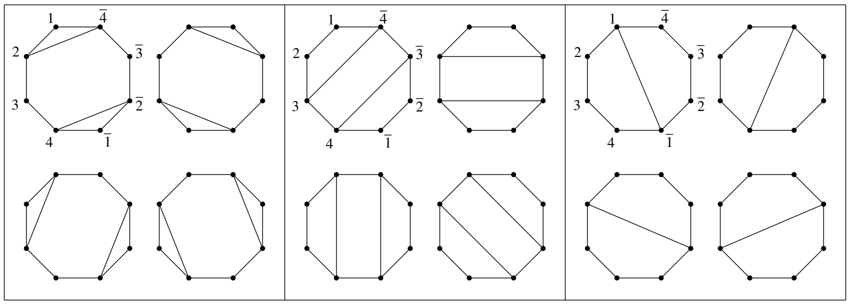

Figure 9: The 1-divisible dissections of an octagon using 1 B-diagonal.

For $t \geq 2$ a divisor of $sn+1$, let ω be a primitive t-th root of unity. Making use of Lemma 3.2 and Corollary 3.3, it is straightforward to prove that

$$\lim_{q\to\omega} \begin{bmatrix} sn+k \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \binom{\frac{sn+1+k}{t}-1}{\frac{k}{t}} \binom{\lfloor\frac{n-1}{t}\rfloor}{\frac{k}{t}} & \text{if } t|k, \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

Note that $\lfloor\frac{n-1}{t}\rfloor = \lfloor\frac{n}{t}\rfloor$ since t divides $sn+1$.

Proposition 4.2 *For $d \geq 2$ a divisor of $2sn+2$, let ω be a primitive d-th root of unity. Then*

$$\lim_{q\to\omega} \begin{bmatrix} sn+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = \begin{cases} \binom{sn+k}{k}\binom{n}{k} & \text{if } d=2, \\ \binom{\frac{sn+1+k}{d}-1}{\frac{k}{d}} \binom{\lfloor\frac{n-1}{d}\rfloor}{\frac{k}{d}} & \text{if } d\geq 3 \text{ odd, } d|k, \\ \binom{\frac{2sn+2+2k}{d}-1}{\frac{2k}{d}} \binom{\lfloor\frac{2(n-1)}{d}\rfloor}{\frac{2k}{d}} & \text{if } d \geq 4 \text{ even, } d|2k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $d=2$, $\lim_{q\to -1}\left[{sn+k \atop k}\right]_{q^2}\left[{n \atop k}\right]_{q^2}=\binom{sn+k}{k}\binom{n}{k}$. For $d\geq 3$, let $t=d$ if d is odd, and $t=\frac{d}{2}$ otherwise. Then t divides $sn+1$, and $\omega':=\omega^2$ is a primitive t-th root of unity. By (12), we have

$$\lim_{q\to\omega}\begin{bmatrix} sn+k \\ k\end{bmatrix}_{q^2}\begin{bmatrix} n \\ k\end{bmatrix}_{q^2}=\lim_{q^2\to\omega'}\begin{bmatrix} sn+k \\ k\end{bmatrix}_{q^2}\begin{bmatrix} n \\ k\end{bmatrix}_{q^2}=\begin{cases}\binom{\frac{sn+1+k}{t}-1}{\frac{k}{t}}\binom{\lfloor\frac{n-1}{t}\rfloor}{\frac{k}{t}} & \text{if } t|k,\\ 0 & \text{otherwise.}\end{cases}$$

The assertion follows. □

For $d\geq 2$ a divisor of $2sn+2$, let C_d be a subgroup of order d of C and let $V(s,n,k,d)\subseteq X$ denote the set of dissections that are invariant under C_d. Note that $V(s,n,k,2)=X$ since all dissections in X are invariant under 2-fold rotation. For $d\geq 3$, we observe that if $V(s,n,k,d)$ is nonempty then each $\pi\in V(s,n,k,d)$ consists of k non-diameter B-diagonals, and hence $k\leq n-1$ and $d|2k$. Moreover, $V(s,n,k,d)=V(s,n,k,2d)$ if d is odd. The following result can be proved by the same argument as the one in the proof of Proposition 3.6 and the proof is omitted.

Proposition 4.3 *For $t\geq 2$ a common divisor of $sn+1$ and k, there is a bijection between $V(s,n,k,2t)$ and the cartesian product of the set of sequences $\{1\leq a_1\leq a_2\leq\cdots\leq a_{\frac{k}{t}}\leq\frac{sn+1}{t}\}$ and the set of sequences $(\epsilon_1,\ldots,\epsilon_m)\in\{0,1\}^m$ with exactly $\frac{k}{t}$ entries equal to 1, where $m=\lfloor\frac{n-1}{t}\rfloor$.*

For illustration, Figure 10(a) is a dissection $\pi\in V(1,9,4,4)$ of a polygon with 20 vertices ($t=2$). The bijection in Proposition 4.3 carries π to the pair of sequences $(a_1,a_2)=(2,5)$ and $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4)=(0,1,1,0)$ as shown in (b)-(d) of Figure 10.

Hence we have the following results.

Corollary 4.4 *For $d\geq 2$ a divisor of $2sn+2$, we have*

$$|V(s,n,k,d)|=\begin{cases}\binom{sn+k}{k}\binom{n}{k} & \text{if } d=2,\\ \binom{\frac{sn+1+k}{d}-1}{\frac{k}{d}}\binom{\lfloor\frac{n-1}{d}\rfloor}{\frac{k}{d}} & \text{if } d\geq 3 \text{ odd, } d|k\\ \binom{\frac{2sn+2+2k}{d}-1}{\frac{2k}{d}}\binom{\lfloor\frac{2(n-1)}{d}\rfloor}{\frac{2k}{d}} & \text{if } d\geq 4 \text{ even, } d|2k\\ 0 & \text{otherwise,}\end{cases}$$

Since the results of Corollary 4.4 agree with that of Proposition 4.2, the proof of Theorem 4.1 is completed.

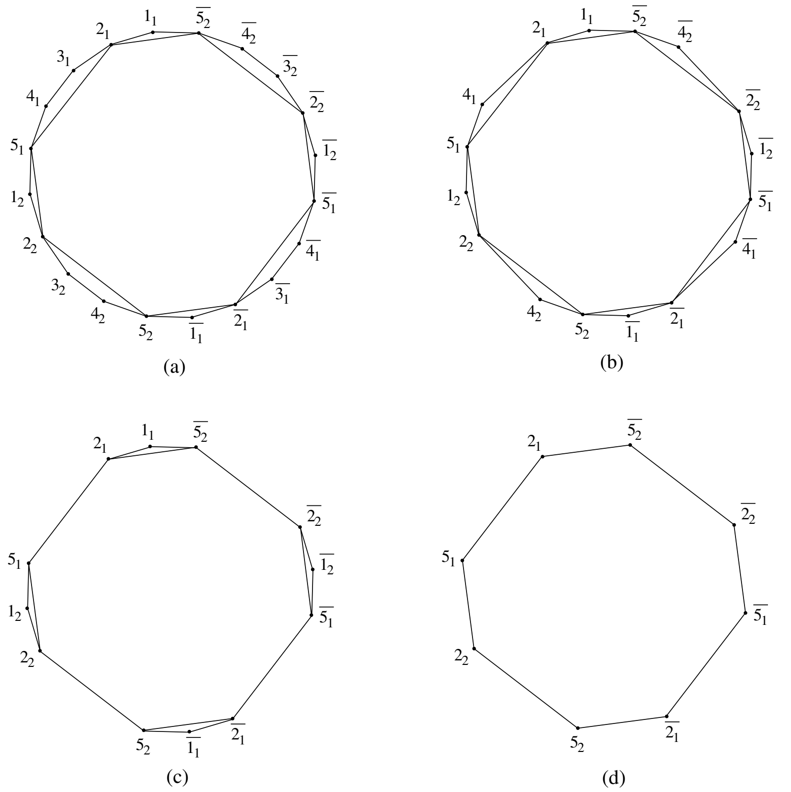

Figure 10: A dissection of a 20-gon carried to the pair $((2,5),(0,1,1,0))$ by the bijection in Proposition 4.3.

5 The cyclic sieving phenomenon for $\Delta^s(D_n)$

Following [3, Section 5.3], the generalized cluster complex $\Delta^s(\Phi)$ for type D_n can be realized as follows. Let P be a regular polygon with $2s(n-1)+2$ vertices. The vertices are labeled by $\{1,2,\ldots,s(n-1)+1,\overline{1},\overline{2},\ldots,\overline{s(n-1)+1}\}$ counterclockwise. There are two copies of each diameter, one colored red and the other colored blue. A D-*diagonal* of P is either (i) a red or a blue diameter, or (ii) a non-diameter B-diagonal. For $1\le i\le s(n-1)+1$, let L_i denote the diameter connecting $i,\overline{i}$, and let $\kappa(L_i)$ denote the color of L_i. The map Γ_s acts on $\Delta^s(D_n)$ by rotating P clockwise, carrying point 2 to point 1, and switching the colors of certain diameters. Specifically, Γ_s carries the ordered pair $(L_j,\kappa(L_j))$ to $(L_{j-1},\kappa(L_{j-1}))$, where $\kappa(L_{j-1})\neq\kappa(L_j)$ if $j=1$ or $j\equiv 2 \pmod s$, and $\kappa(L_{j-1})=\kappa(L_j)$ otherwise. The compatibility among D-diagonals is defined as follows. Two diameters with the same endpoints and different colors are compatible. Two diameters with distinct endpoints are compatible if and only if applying Γ_s repeatedly until either of them is carried to L_1 results in diameters of the same color. In all the other cases, two D-diagonals are compatible if they are noncrossing in the sense of type-B dissections. The faces of $\Delta^s(D_n)$ correspond to dissections of P using compatible D-diagonals. For convenience, the set $\{(sj+2,sj+1) : 0\le j\le n-1\}$ of edges of P are called *color-switchers*, where $s(n-1)+2=\overline{1}$. Figure 11 shows the orbit of a maximal face of $\Delta^2(D_3)$ under the action of Γ_2, along with the color-switchers and their opposite edges, drawn as broken edges, indicating the locations at which diameters change colors.

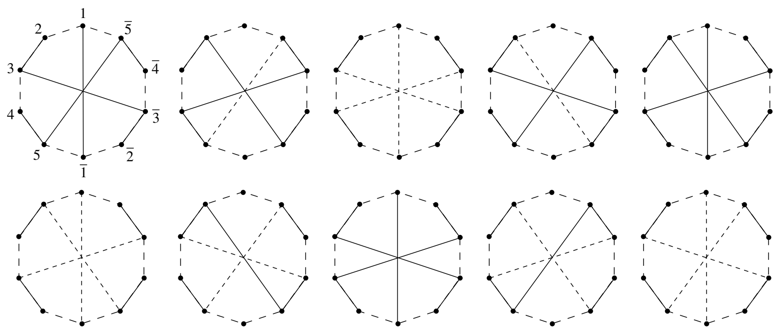

Figure 11: The orbit of a maximal face of $\Delta^2(D_3)$.

To describe the feasible polynomial $X(q)$ that serves our purpose, we rewrite the k-face number of $\Delta^s(D_n)$ (Theorem 2.1(iii)) as the sum of the expressions

$$\binom{s(n-1)+k}{k}\binom{n-1}{k}+\binom{s(n-1)+k}{k}\binom{n-2}{k-1} \tag{13}$$

and

$$\binom{s(n-1)+k-1}{k-1}\binom{n-2}{k-2}+2\binom{s(n-1)+k-1}{k}\binom{n-2}{k-2}. \tag{14}$$

Note that the first term of (13) is $f_k(B_{n-1},s)$, which counts the k-faces of $\Delta^s(D_n)$ with at most one diameter (say colored red). By a result of Tzanaki [11, Corollary 3.4], the second term of (13) is the number of k-faces of $\Delta^s(B_{n-1})$ with one diameter, which counts the k-faces of $\Delta^s(D_n)$ with exactly one diameter (say colored blue). Hence (13) is the number of k-faces of $\Delta^s(D_n)$ with at most one diameter of either color, and the difference of the two terms is equal to the number of k-faces of $\Delta^s(D_n)$ without diameters. On the other hand, the first term of (14) is the number of $(k-1)$-faces of $\Delta^s(B_{n-1})$ with one diameter, which also counts the k-faces of $\Delta^s(D_n)$ with two diameters of different color and the same endpoints. The second term of (14) counts the remaining k-faces of $\Delta^s(D_n)$, i.e., the ones with at least two diameters connecting distinct endpoints. Such faces can be paired off by switching the color of each diameter. For the sake of being consistent with the polynomial $\mathrm{Cat}^{(s)}(D_n,q)$ in (20) when we take q-analogue, (14) is alternatively written as

$$\binom{s(n-1)+k}{k}\binom{n-2}{k-2}+\binom{s(n-1)+k-1}{k}\binom{n-2}{k-2} \tag{15}$$

so that the difference of the two terms is equal to the number of k-faces of $\Delta^s(D_n)$ with two diameters of different color and the same endpoints.

As a q-analogue of the sum of (13) and (15), we define the polynomial $F(s,n,k;q)$ by

$$\begin{aligned} F(s,n,k;q) := & \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot q^n \\ & + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n. \end{aligned} \tag{16}$$

Note that $F(s,n,k;1) = f_k(D_n,s)$. We shall prove that the k-faces of $\Delta^s(D_n)$, along with $F(s,n,k;q)$, exhibit the cyclic sieving phenomenon.

Theorem 5.1 *For positive integers s and n, let X be the set of s-divisible dissections of a $(2s(n-1)+2)$-gon using k compatible D-diagonals. Let C be the cyclic group of order $2s(n-1)+2$ generated by Γ_s that acts on X. Let $X(q) := F(s,n,k;q)$. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

For example, take $s=3$, $n=2$, and $k=2$. $X(q) \equiv 4 + 4q^2 + 4q^4 + 4q^6 \pmod{q^8-1}$. As shown in Figure 12, there are 16 3-divisible dissections of an octagon using 2 compatible D-diagonals. These dissections are partitioned into four orbits under the group action $\langle \Gamma_3 \rangle$, all of which free.

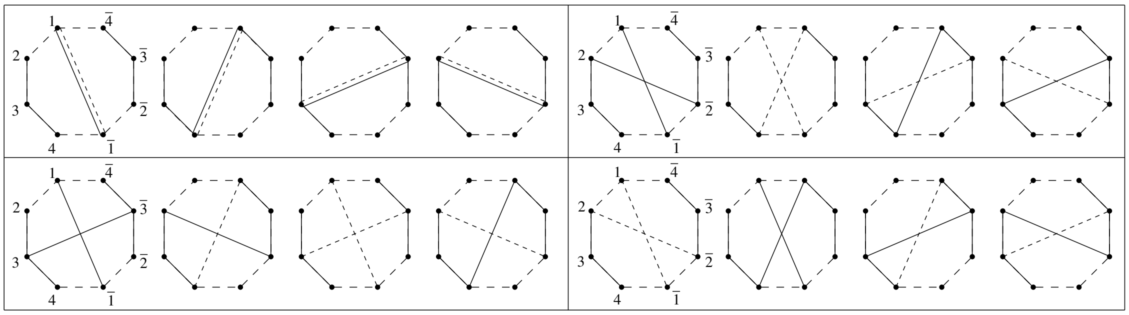

Figure 12: The 3-divisible dissections of an octagon using 2 compatible D-diagonals.

To prove Theorem 5.1, we first evaluate $F(s,n,k,q)$ at a primitive d-th root ω of unity, where d divides $2s(n-1)+2$. For $d=2$, we have an immediate evaluation at $q=-1$.

Proposition 5.2

$$[F(s,n,k,q)]_{q=-1} = \begin{cases} \binom{s(n-1)+k}{k}\binom{n}{k} + \binom{s(n-1)+k-1}{k}\binom{n-2}{k-2} & \textit{if } n \textit{ even}, \\ \binom{s(n-1)+k}{k}\binom{n-2}{k} + \binom{s(n-1)+k-1}{k-1}\binom{n-2}{k-2} & \textit{otherwise}. \end{cases}$$

For $d \geq 3$, by Lemma 3.2 and Corollary 3.3, we the have the following results as an intermediate stage of evaluation.

Lemma 5.3 *For $t \geq 2$ a divisor of $s(n-1)+1$, let ω be a primitive t-th root of unity. Then*

(i)

$$\lim_{q\to\omega} \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \begin{cases} \binom{\frac{s(n-1)+1+k}{t}-1}{\frac{k}{t}} \binom{\lfloor \frac{n-2}{t} \rfloor}{\frac{k}{t}} & \text{if } t|k \\ 0 & \text{otherwise.} \end{cases}$$

(ii)

$$\lim_{q\to\omega} \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q = 0$$

(iii)

$$\lim_{q\to\omega} \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q = \begin{cases} \binom{\frac{s(n-1)+1+k}{t}-1}{\frac{k}{t}} \binom{\frac{n}{t}-1}{\frac{k}{t}-1} & \text{if } t|k,\ t|n \\ 0 & \text{otherwise.} \end{cases}$$

(iv)

$$\lim_{q\to\omega} \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q = \begin{cases} \binom{\frac{s(n-1)+1+k}{t}-1}{\frac{k}{t}} \binom{\frac{n}{t}-1}{\frac{k}{t}-1} & \text{if } t|k,\ t|n \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 5.3, the following results can be proved by an argument similar to the one in the proof of Proposition 4.2.

Proposition 5.4 *For $d \geq 3$ a divisor of $2s(n-1)+2$, let ω be a primitive d-th root of unity. Then*

$$[F(s,n,k;q)]_{q=\omega} =$$

$$\begin{cases} \binom{\frac{s(n-1)+1+k}{d}-1}{\frac{k}{d}} \binom{\frac{n}{d}}{\frac{k}{d}} + \binom{\frac{s(n-1)+1+k}{d}-1}{\frac{k}{d}} \binom{\frac{n}{d}-1}{\frac{k}{d}-1} & \text{if } d \geq 3 \text{ odd, } d|k,\ d|n, \\ \binom{\frac{s(n-1)+1+k}{d}-1}{\frac{k}{d}} \binom{\lfloor \frac{n-2}{d} \rfloor}{\frac{k}{d}} & \text{if } d \geq 3 \text{ odd, } d|k,\ d \nmid n, \\ \binom{\frac{2s(n-1)+2+2k}{d}-1}{\frac{2k}{d}} \binom{\frac{2n}{d}}{\frac{2k}{d}} + \binom{\frac{2s(n-1)+2+2k}{d}-1}{\frac{2k}{d}} \binom{\frac{2n}{d}-1}{\frac{2k}{d}-1} & \text{if } d \geq 4 \text{ even, } d|2k,\ d|n, \\ \binom{\frac{2s(n-1)+2+2k}{d}-1}{\frac{2k}{d}} \binom{\lfloor \frac{2(n-2)}{d} \rfloor}{\frac{2k}{d}} & \text{if } d \geq 4 \text{ even, } d|2k,\ d \nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the set of s-divisible dissections π of P using k compatible D-diagonals, for $0 \le k \le n$. Note that if $\pi \in X$ contains no diameter then $k \le n-2$. Let C be the cyclic group of order $2s(n-1)+2$ generated by Γ_s. For $d \ge 2$ a divisor of $2s(n-1)+2$, let $W(s,n,k,d) \subseteq X$ denote the set of dissections that are invariant under d-fold rotation $C_d \subseteq C$. We observe that if $W(s,n,k,d)$ is nonempty then either $d=2$, or $d \ge 3$ and $d|2k$. These two cases are treated in Propositions 5.5 and 5.10, respectively.

Proposition 5.5 *For positive integers s, n, and k, we have*

$$|W(s,n,k,2)| = \begin{cases} \binom{s(n-1)+k}{k}\binom{n}{k} + \binom{s(n-1)+k-1}{k}\binom{n-2}{k-2} & \text{if } n \text{ even,} \\ \binom{s(n-1)+k}{k}\binom{n-2}{k} + \binom{s(n-1)+k-1}{k-1}\binom{n-2}{k-2} & \text{otherwise.} \end{cases}$$

Proof. Note that, by the difference of the two terms of (13), there are $\binom{s(n-1)+k}{k}\binom{n-2}{k}$ dissections without diameters, all of which are invariant under 2-fold rotation. Given a dissection $\pi \in W(s,n,k,2)$ containing a diameter L, when P is rotated 180 degrees, L passes n color-switchers. Then L remains in the same color if n is even, and switches to the other color otherwise. Hence if n is even, then any dissection that contains diameters is invariant under 2-fold rotation, and hence $|W(s,n,k,2)| = |X|$. Otherwise n is odd and π contains two diameters of different colors with the same endpoints. By the first term of (14), there are $\binom{s(n-1)+k-1}{k-1}\binom{n-2}{k-2}$ such dissections, and hence $|W(s,n,k,2)| = \binom{s(n-1)+k}{k}\binom{n-2}{k} + \binom{s(n-1)+k-1}{k-1}\binom{n-2}{k-2}$. The assertion follows. $\square$

For $d \ge 3$, we have the following necessary condition.

Lemma 5.6 *For $d \ge 3$ a common divisor of $2s(n-1)+2$ and $2k$, if there is a dissection in $W(s,n,k,d)$ containing a diameter then d divides n.*

Proof. Given a $\pi \in W(s,n,k,d)$ containing a diameter M, let $\mathcal{M} = \{M_1, \dots, \mathcal{M}_d\} \subseteq \pi$ be the orbit of diameters containing M under C_d. Note that when d is even $\mathcal{M}$ is a multiset with $M_i = M_{\frac{d}{2}+i}$. If $M_1 = L_1$, then these diameters are of the same color. We apply Γ_s on π repeatedly until M_2 is carried to M_1 (i.e., $M_2 = L_1$). Then each M_i travels through $\frac{2s(n-1)+2}{d}$ edges. Let M_i pass r_i color-switchers. Clearly, $r_1 + \cdots + r_d = 2n$. Since color-switchers are equally s points apart, we have $|r_i - r_j| \le 1$, for $0 \le i, j \le d$. Moreover, the resulting diameters remain in the same color, and this occurs whenever each r_i is even. Consequently, $r_1 = \cdots = r_d = \frac{2n}{d} = 2j$, for some j. Hence d divides n.

If $M_1 \ne L_1$ then apply Γ_s until $M_1 = L_1$ so that the diameters in $\mathcal{M}$ are of the same color, and then apply the argument above. The proof is completed. $\square$

We define the *initial point* of each D-diagonal. Each diameter L_i is oriented from i to $\bar{i}$, for $i \in [s(n-1)+1]$ and i is called the initial point of L_i. Each non-diameter D-diagonal L is oriented in the usual sense, which has two segments with starting points j, $\bar{j}$, for $j \in [s(n-1)+1]$, and j is called the initial point of L. The maximal faces of $\Delta^s(D_n)$ consisting entirely of diameters have the following properties, which are useful in enumerating $W(s,n,k,d)$.

Lemma 5.7 ([3, Lemma 9.1]) *For $n \geq 3$, given n diameters of P, the following conditions are equivalent.*

(i) *There exists a assignment of colors for each diameter such that the colored diameters are pairwise compatible.*

(ii) *If $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq s(n-1)+1$ are the initial points of the diameters, then $a_{j+1} - a_j \leq s$, for $1 \leq j \leq n$ (with the convention $a_{n+1} = a_1 + s(n-1)+1$).*

If these conditions hold, then there are exactly two ways to assign colors to the diameters, which are related by switching the colors of each of the n diameters.

To enumerate $W(s,n,k,d)$, it is partitioned into three subsets $T_0(s,n,k,d)$, $T_1(s,n,k,d)$, and $T_2(s,n,k,d)$, where $T_0(s,n,k,d)$ is the subset of dissections that contain no diameter, and $T_1(s,n,k,d)$ (resp. $T_2(s,n,k,d)$) is the subset of dissections with diameters such that the first diameter is red (resp. blue). Note that, by Lemma 5.6, $T_1(s,n,k,d)$ and $T_2(s,n,k,d)$ are nonempty only if $d|n$ and that there is an immediate bijection between $T_1(s,n,k,d)$ and $T_2(s,n,k,d)$ by switching the colors of the diameters, thus $|T_1(s,n,k,d)| = |T_2(s,n,k,d)|$. In the following, we make use of the same bijective method as the one in the proof of Proposition 4.3 to enumerate $T_1(s,n,k,d)$ and $T_0(s,n,k,d)$. With an analogous argument, a proof is given for completeness.

Proposition 5.8 *For $t \geq 2$ a common divisor of $s(n-1)+1$, k, and n, there is a bijection between $T_1(s,n,k,2t)$ and the cartesian product of the set of sequences $\{1 \leq a_1 \leq a_2 \leq \cdots \leq a_{\frac{k}{t}} \leq \frac{s(n-1)+1}{t}\}$ and the set of sequences $(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$ with exactly $\frac{k}{t}$ entries equal to 1 and $\epsilon_m = 1$, where $m = \frac{n}{t}$.*

Proof. Fixing t, the vertices of the polygon are partitioned into $2t$ sectors of size $b = \frac{s(n-1)+1}{t}$ with a vertex-labeling $\{1_1, 2_1, \ldots, b_1\}, \ldots, \{1_t, 2_t, \ldots, b_t\}$, $\{\overline{1_1}, \overline{2_1}, \ldots, \overline{b_1}\}, \ldots, \{\overline{1_t}, \overline{2_t}, \ldots, \overline{b_t}\}$ in counterclockwise order. For each $\pi \in T_1(s,n,k,2t)$, the D-diagonals of π are partitioned into $\frac{k}{t}$ orbits of size t such that each orbit R contains a unique D-diagonal L the initial point of which lies in the first sector $\{1_1, 2_1, \ldots, b_1\}$. We associate R with the label $i \in [b]$ if the initial point of L is i_1. Let $\mu = (a_1, \ldots, a_{\frac{k}{t}})$ be the sequence of labels associated to the orbits of D-diagonals in π.

The bijection Ψ_m is established by induction on $m = \frac{n}{t}$. Recall that $1 \leq k \leq n$ and t divides k. If $m = 1$ then $\frac{k}{t} = 1$, and Ψ_1 carries π to the pair $((a_1), (\epsilon_1))$, where a_1 is the label of the unique orbit of diameters and $\epsilon_1 = 1$. Assume that $m \geq 2$. If $a_{j+1} - a_j \leq s$, for $1 \leq j \leq \frac{k}{t}$ (with the convention $a_{\frac{k}{t}+1} = a_1 + b$) then by Lemma 5.7 we have $k = n$, so define $\Psi_m(\pi) = (\mu, \nu)$, where $\nu = (1, \ldots, 1)$. Otherwise, locate the first a_j such that $a_j + s + 1 \le a_{j+1}$. We set $\epsilon_1 = 1$ if $L = (a_j(a_j + s + 1), \overline{a_j}(\overline{a_j} + s + 1))$ is a D-diagonal in π, and set $\epsilon_1 = 0$ otherwise. Let R be the D-diagonal-orbit containing L. The remaining $\frac{k}{t} - \epsilon_1$ D-diagonal-orbits form a dissection of a $(2s(n-t-1)+2)$-gon P', obtained from P by removing the vertex-orbit of the set $\{a_j + 1, \ldots, a_j + s, \overline{a_j} + 1, \ldots, \overline{a_j} + s\}$ under C_{2t}. By induction the remaining ϵ_i's are determined.

To find Ψ_m^{-1}, given a pair of sequences $\mu = (a_1, \ldots, a_{\frac{k}{t}})$ and $\nu = (\epsilon_1, \ldots, \epsilon_m)$ with $\frac{k}{t}$ entries equal to 1 and $\epsilon_m = 1$, we retrieve the dissection $\Phi_m^{-1}((\mu, \nu))$ as follows. If $m = 1$ then $\frac{k}{t} = 1$ and $\Psi_1^{-1}(\mu, \nu)$ consists of the unique orbit of diameters with label a_1. Assume that $m \ge 2$. If $a_{j+1} - a_j \le s$, for $1 \le j \le \frac{k}{t}$, then by Lemma 5.7 we have $k = n$. Then $\Psi_m^{-1}(\mu, \nu)$ is defined to be the dissection consisting of the diameter-orbits labeled by the a_i's. Otherwise, locate the first a_j such that $a_j + s + 1 \le a_{j+1}$. Then there is an orbit R containing the D-diagonal $L = (a_j(a_j + s + 1), \overline{a_j}(\overline{a_j} + s + 1))$ if $\epsilon_1 = 1$ and there is not if $\epsilon_1 = 0$. Next, let $\mu' = (a_1, \ldots, a_{\frac{k}{t}})$ if $\epsilon_1 = 0$ and $\mu' = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{\frac{k}{t}})$ otherwise, and $\nu' = (\epsilon_2, \ldots, \epsilon_m)$. Use the pair (μ', ν') to inductively determine the set $\mathcal{R}$ of $\frac{k}{t} - \epsilon_1$ D-diagonal-orbits in the polygon P' described above. Then we define $\Phi_m^{-1}((\mu, \nu))$ to be the dissection determined by $\mathcal{R}$ if $\epsilon_1 = 0$ and by $\mathcal{R} \cup \{R\}$ if $\epsilon_1 = 1$. $\square$

For example, in Figure 13(a), there is a dissection $\pi \in T_1(3, 6, 6, 4)$ of a polygon with 32 vertices ($t = 2$). The bijection Ψ_3 in Proposition 5.8 carries π to the pair of sequences $(a_1, a_2, a_3) = (2, 2, 7)$ and $(\epsilon_1, \epsilon_2, \epsilon_3) = (1, 1, 1)$ as shown in (b) of Figure 13.

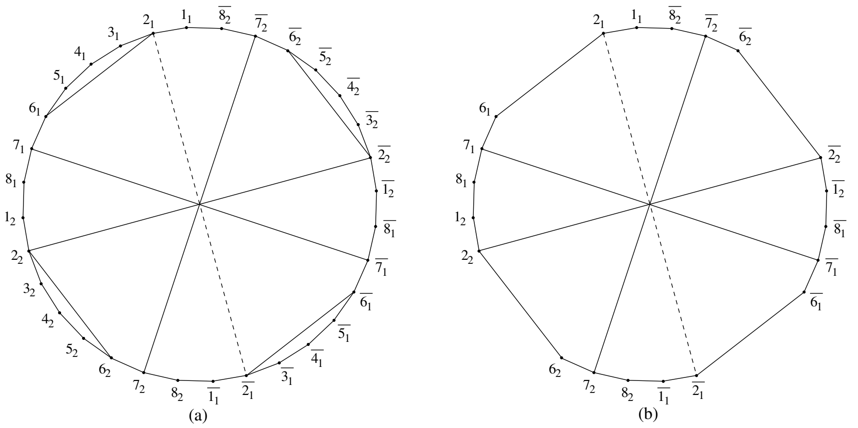

Figure 13: A dissection of a 32-gon carried to the pair $((2, 2, 7), (1, 1, 1))$ by Ψ_3.

On the other hand, we observe that each dissection $\pi \in T_0(s, n, k, 2t)$ consists of k non-diameter D-diagonals, and hence $k \le n - 2$ and $t | k$. As a case of type-B dissection, the following bijective result can be obtained by the same method as the one in the proof of Proposition 3.6 and the proof is omitted.

Proposition 5.9 *For $t \ge 2$ a common divisor of $s(n-1)+1$ and k, there is a bijection between the set $T_0(s, n, k, 2t)$ and the cartesian product of the set of sequences $\{1 \le a_1 \le$*

$a_2 \le \cdots \le a_{\frac{k}{t}} \le \frac{s(n-1)+1}{t}\}$ and the set of sequences $(\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$ with exactly $\frac{k}{t}$ entries equal to 1, where $m = \lfloor \frac{n-2}{t} \rfloor$.

By Propositions 5.8 and 5.9, we have the following results.

Proposition 5.10 *For $d \ge 3$ a divisor of $2s(n-1)+2$, we have*

$$|W(s,n,k,d)| =$$
$$\begin{cases} \binom{\frac{s(n-1)+1+k}{d} - 1}{\frac{k}{d}}\binom{\frac{n}{d}}{\frac{k}{d}} + \binom{\frac{s(n-1)+1+k}{d} - 1}{\frac{k}{d}}\binom{\frac{n}{d}-1}{\frac{k}{d}-1} & \text{if } d \ge 3 \text{ odd, } d|k,\ d|n, \\ \binom{\frac{s(n-1)+1+k}{d} - 1}{\frac{k}{d}}\binom{\lfloor\frac{n-2}{d}\rfloor}{\frac{k}{d}} & \text{if } d \ge 3 \text{ odd, } d|k,\ d\nmid n, \\ \binom{\frac{2s(n-1)+2+2k}{d} - 1}{\frac{2k}{d}}\binom{\frac{2n}{d}}{\frac{2k}{d}} + \binom{\frac{2s(n-1)+2+2k}{d} - 1}{\frac{2k}{d}}\binom{\frac{2n}{d}-1}{\frac{2k}{d}-1} & \text{if } d \ge 4 \text{ even, } d|2k,\ d|n, \\ \binom{\frac{2s(n-1)+2+2k}{d} - 1}{\frac{2k}{d}}\binom{\lfloor\frac{2(n-2)}{d}\rfloor}{\frac{2k}{d}} & \text{if } d \ge 4 \text{ even, } d|2k,\ d\nmid n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $d \ge 3$, let $t = d$ if d is odd and $t = \frac{d}{2}$ otherwise. Note that $W(s,n,k,d) = W(s,n,k,2d)$ if d is odd. If $t|k$ and $d|n$ then by Propositions 5.8 and 5.9, we have

$$\begin{aligned} |W(s,n,k,2t)| &= |T_0(s,n,k,2t)| + |T_1(s,n,k,2t)| + |T_2(s,n,k,2t)| \\ &= \binom{\frac{s(n-1)+1+k}{t} - 1}{\frac{k}{t}}\binom{\frac{n}{t}-1}{\frac{k}{t}} + 2\binom{\frac{s(n-1)+1+k}{t} - 1}{\frac{k}{t}}\binom{\frac{n}{t}-1}{\frac{k}{t}-1} \\ &= \binom{\frac{s(n-1)+1+k}{t} - 1}{\frac{k}{t}}\binom{\frac{n}{t}}{\frac{k}{t}} + \binom{\frac{s(n-1)+1+k}{t} - 1}{\frac{k}{t}}\binom{\frac{n}{t}-1}{\frac{k}{t}-1}. \end{aligned}$$

Moreover, if $t|k$ and $t\nmid n$ then by Lemma 5.6 and Proposition 5.9, we have

$$|W(s,n,k,2t)| = |T_0(s,n,k,t)| = \binom{\frac{s(n-1)+1+k}{t} - 1}{\frac{k}{t}}\binom{\lfloor\frac{n-2}{t}\rfloor}{\frac{k}{t}}.$$

Otherwise $t\nmid k$ and $|W(s,n,k,d)| = 0$. The assertion follows. □

Since the results of Propositions 5.5 and 5.10 agree with that of Propositions 5.2 and 5.4, the proof of Theorem 5.1 is completed.

Remarks: Besides (16), there are another feasible options for the polynomial $X(q)$ exhibiting the cyclic sieving phenomenon, which are also consistent with $\mathrm{Cat}^{(s)}(D_n, q)$ in (20) when $k = n$. For example, we may rewrite expression (13) as

$$\binom{s(n-1)+k}{k}\binom{n-2}{k} + 2\binom{s(n-1)+k}{k}\binom{n-2}{k-1}, \tag{17}$$

and then define $X(q)$, as a q-analogue of the sum of (15) and (17), by

$$X(q) = \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{q^2} \cdot (1+q^n)$$
$$+ \begin{bmatrix} s(n-1)+k \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} + \begin{bmatrix} s(n-1)+k-1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_{q^2} \cdot q^n. \tag{18}$$

Readers may see the discussion of the last section.

6 The cyclic sieving phenomenon for $\Delta^s(I_2(a))$

For a root system Φ of type $I_2(a)$, the complex $\Delta^s(I_2(a))$ is an $(s+1)$-regular graph on $sa+2$ vertices, whose edges are facets. As shown in [3, Example 4.4], this graph can be constructed in the plane on a circle of $sa+2$ points labeled from 0 to $sa+1$ clockwise. For a odd, the edge set has $(am+2)$-fold rotational symmetry and connects each vertex v to the $s+1$ vertices $v+\frac{s(a-1)}{2}+j \pmod{sa+2}$, for $j=1,\ldots,s+1$. Figure 14(a) shows this graph for $s=2$ and $a=5$. In this case, the map Γ_s corresponds to a counterclockwise rotation of the graph by $\frac{2\pi}{sa+2}$. For a even, fixing an odd integer i, the edge set has $(\frac{sa+2}{2})$-fold rotational symmetry and connects 0 to the vertices $i, i+2, \ldots, i+2s$. Figure 14(b) is $\Delta^2(I_2(4))$ drawn in this style with $i=1$. In this case, the map Γ_s acts by a counterclockwise rotation of the graph by $\frac{4\pi}{sa+2}$. Along with the q-analogue $X(q)$ of Theorem 2.1(iv), we prove the cyclic sieving phenomenon for the facets of $\Delta^s(\Phi)$.

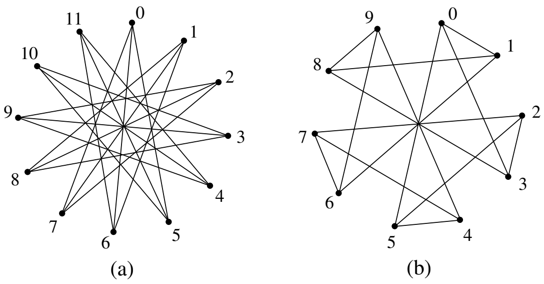

Figure 14: Representation of $\Delta^2(I_2(5))$ and $\Delta^2(I_2(4))$.

Proposition 6.1 *Let X be the edge set of the graph $\Delta^s(I_2(a))$. Define*

$$X(q) := \frac{[sa+2]_q}{[2]_q} \cdot \frac{[sa+a]_q}{[a]_q}. \tag{19}$$

Let $C = \mathbb{Z}_{sa+2}$ be the cyclic group that acts on X by cyclic rotation of the graph. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Proof. For $d \geq 2$ a divisor of $sa+2$, let ω be a primitive d-th root of unity and let $Y(s,a,d) \subseteq X$ be the set of edges that are invariant under a subgroup of order d of C. There are two cases.

For a odd, it is clear that $[X(q)]_{q=\omega} = 0$ for all $d \geq 3$. If $d = 2$ then s is even and $[X(q)]_{q=-1} = \frac{sa+2}{2}$.

On the other hand, an edge of the graph $\Delta^s(I_2(a))$ is invariant under d-fold rotation only if $d = 2$. By the construction of the graph, $Y(s,a,2)$ is nonempty if and only if s is even, in which case $Y(s,a,2)$ consists of the $\frac{sa+2}{2}$ edges connecting antipodal pairs of vertices. This proves that $(X, X(q), C)$ exhibits the cyclic sieving phenomenon for a odd.

For a even, it is clear that $[X(q)]_{q=\omega} = 0$ for all $d \geq 3$ and $[X(q)]_{q=-1} = \frac{(sa+2)(s+1)}{2}$. On the other hand, we remark that under the group action C, odd (resp. even) vertices of $\Delta^s(I_2(a))$ are carried to odd (resp. even) vertices, and each vertex travels two cycles. Since each edge connects an odd vertex and an even vertex, an edge is invariant if and only if it is rotated by a full cycle. Hence $Y(s,a,2) = X$ and $Y(s,a,d) = \emptyset$ for $d \geq 3$. This proves that $(X, X(q), C)$ exhibits the cyclic sieving phenomenon for a even. $\square$

It is straightforward to prove that the vertex set of $\Delta^s(I_2(a))$, along with the polynomial $X(q) = [sa+2]_q$ if a is odd and $X(q) = [sa+2]_{q^2}$ otherwise, exhibits the cyclic sieving phenomenon since all vertices form an orbit if a is odd and two orbits otherwise under C.

7 The cases of exceptional types

In this section, we consider the cluster complex $\Delta(\Phi)$ of exceptional type E_6, E_7, E_8, F_4, H_3, and H_4.

When $k = n$, the polynomial $X(q) = \mathrm{Cat}(\Phi, q)$ defined in (4) is a feasible q-analogue of the number of facets of $\Delta(\Phi)$. We verify Theorem 1.2 affirmatively for $\Delta(\Phi)$ of exceptional type. To see this, with the Coxeter numbers and exponents of Φ listed in Figure 4, the polynomials $X(q)$ are expanded as follows.

(i) In type E_6, $X(q) \equiv 67 + 52q + 67q^2 + 52q^3 + \cdots + 67q^{12} + 52q^{13} \pmod{q^{14}-1}$.

(ii) In type E_7, $X(q) \equiv 416 + 416q^2 + 416q^4 + \cdots + 416q^{18} \pmod{q^{20}-1}$.

(iii) In type E_8, $X(q) \equiv 1574 + 1562q^2 + 1572q^4 + 1562q^6 + \cdots + 1574q^{24} + 1562q^{26} + 1572q^{28} + 1562q^{30} \pmod{q^{32}-1}$.

(iv) In type F_4, $X(q) \equiv 15 + 15q^2 + 15q^4 + \cdots + 15q^{12} \pmod{q^{14}-1}$.

(v) In type H_3, $X(q) \equiv 6 + 5q^2 + 5q^4 + 6q^6 + 5q^8 + 5q^{10} \pmod{q^{12}-1}$.

(vi) In type H_4, $X(q) \equiv 18 + 17q^2 + 18q^4 + 17q^6 + \cdots + 18q^{28} + 17q^{30} \pmod{q^{32}-1}$.

As searched by a computer, the orbit-structures for the k-faces of $\Delta(\Phi)$ under the cyclic group C generated by Γ are shown in Figure 15. We write $a_1(b_1)$, $a_2(b_2)$, ..., $a_t(b_t)$ for the

k	E_6	E_7	E_8	F_4	H_3	H_4
1	14(2), 7(2)	10(7)	16(8)	7(4)	6(3)	16(4)
2	14(26), 7(5)	10(94), 5(1)	16(149), 8(3)	7(19)	6(8)	16(21), 8(1)
3	14(104), 7(13)	10(518)	16(1121)	7(30)	6(5), 2(1)	16(35)
4	14(195), 7(18)	10(1410), 5(1)	16(4211), 8(13), 4(2)	7(15)		16(17), 8(1)
5	14(171), 7(15)	10(2020), 2(1)	16(8778)			
6	14(52), 7(15)	10(1456)	16(10230), 8(22)			
7		10(416)	16(6270)			
8			16(1562), 8(10), 4(2)			

Figure 15: The orbit-structures of the k-faces of $\Delta(\Phi)$ of exceptional types.

orbit-structure of the k-faces that are partitioned into b_i orbits of size a_i, for $1 \le i \le t$, in which case the number of k-faces is equal to $a_1b_1 + \cdots + a_tb_t$.

We observe that the orbit-structures for the facets of $\Delta(\Phi)$ shown in Figure 15 agree with the expansions (i)-(vi) of $X(q)$ (mod $q^{h+2} - 1$). Hence the cyclic sieving phenomenon holds. Consequently, together with the special case $s = 1$ and $k = n$ in Theorems 3.1, 4.1, 5.1, and 6.1, Theorem 1.2 is proved.

8 Discussion and remarks

In this paper, we study the cyclic sieving phenomenon for the faces of the generalized cluster complex $\Delta^s(\Phi)$ associated a root system Φ. As the readers can see, the q-analogue $X(q)$ of face numbers plays an essential role.

First, note that when $k = n$ our polynomials $X(q)$ for the facets of $\Delta^s(\Phi)$ agree with the generalized q-Catalan numbers defined by

$$\mathrm{Cat}^{(s)}(\Phi, q) := \prod_{i=1}^{n} \frac{[sh + e_i + 1]_q}{[e_i + 1]_q}, \tag{20}$$

i.e., $\mathrm{Cat}^{(s)}(A_{n-1}, q) = G(s, n, n)$, $\mathrm{Cat}^{(s)}(B_n, q) = H(s, n, n)$, $\mathrm{Cat}^{(s)}(D_n, q) = F(s, n, n)$, and $\mathrm{Cat}^{(s)}(I_2(a), q)$ is (19). As a result, we prove the following conjecture mentioned by Reiner-Stanton-White, for $\Delta^s(\Phi)$ in types A_n, B_n, D_n, and $I_2(a)$.

Conjecture 8.1 ([9]) *For a positive integer s, let X be the set of facets of the generalized cluster complex $\Delta^s(\Phi)$. Let $X(q) = Cat^{(s)}(\Phi, q)$ be defined in (20). Let the cyclic group C of order $sh + 2$ generated by Γ_s act on X. Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

The cases of exceptional type are not determined for lack of a systematic method, although we believe that the cyclic sieving phenomenon holds. For $k < n$, we would like to point out that the *genuine* q-analogue $X(q)$ of the k-face numbers in the spirit of (20) is quite elusive, as discussed below.

As the readers have seen, what we have (i.e., (6), (11), and (16)) are feasible polynomials $X(q)$ for the purpose of cyclic sieving phenomenon and consistent with (20) when $k = n$. In type A_{n-1} and B_n, we are more or less confident that the polynomials (6) and (11) are the genuine q-analogues of the k-face numbers of $\Delta^s(\Phi)$. However, in type D_n the polynomial (16) seems to be artificially tailored to serve the purpose of cyclic sieving phenomenon, and there is more than one feasible option, e.g., (18). (Note that some obvious q-analogues of the expression in Theorem 2.1(iii) are not feasible.)

Fomin and Reading gave a formula for the k-face numbers of $\Delta^s(\Phi)$

$$f_k(\Phi, s) = c(\Phi, k, s)\binom{n}{k} \prod_{L(e_i)\le k} \frac{sh + e_i + 1}{e_i + 1}, \tag{21}$$

where $c(\Phi, k, s)$ and $L(e_i)$ depend on Φ, see [3, Theorem 8.5] for more information. We remark that except for $k = n$, we can hardly derive any feasible q-analogue of the k-face numbers from (21) even for $s = 1$.

For example, taking $k = 1$ and $s = 1$ in type E_6, we have $f_1(\Phi, 1) = \binom{n}{1}\frac{h+e_1+1}{e_1+1} = \binom{6}{1}\frac{14}{2}$. Taking a natural q-analogue, we have $X(q) = \begin{bmatrix}6\\1\end{bmatrix}_q \frac{[14]_q}{[2]_q} \equiv 3 + 3q + 3q^2 + \cdots + 3q^{13} \pmod{q^{14} - 1}$, which does not agree with the orbit-structure (i.e., with two orbits of size 14 and two orbits of size 7) of the vertex set of $\Delta(E_6)$ under C. What worse is that sometimes the natural q-analogue is not even a polynomial (e.g., taking $s = 1$ and $k = 5$ in type E_6).

In light of these, we are very interested in finding the genuine q-analogue $X(q)$ of face numbers, which involves the Coxeter number and exponents of Φ in the spirit of (20). Such a polynomial $X(q)$ should not only lead to a unified result for the k-faces of $\Delta^s(\Phi)$ as the one in Conjecture 8.1 but also be consistent with other combinatorial structures (e.g. noncrossing partitions) in connection with Coxeter groups. We leave it as an open problem.

Open Problem. What is the genuine q-analogue $X(q)$ of the k-face numbers of $\Delta^s(\Phi)$?

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