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# Blocking sequences of sets in infinite graphs

by

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## ABSTRACT

In this paper, we introduce the graph-theoretic concept of a blocking sequence, and use that concept to prove a conjecture about certain instances of the Firefighter Problem.

More specifically, for any positive integer $d$, let $H^d$ be a $d$-dimensional infinite grid, where at each time-step $(0, 1, 2, \ldots)$, each vertex is in one of three states: *on-fire*, *fireproofed*, or *suseptible*, where a vertex that is suseptible becomes on-fire for the next time step if any of its neighbors are on-fire already, unless it is fireproofed first. Suppose that at time-step 0, some finite set $S_0$ of vertices is on-fire, and that at each time-step $t$, we are allowed to fireproof $f(t)$ vertices of $H^d$ to contain the set of vertices that eventually become on-fire to a finite set. Hartke conjectured in his Ph.D. thesis that $f(t)$ would have to be $\Omega(t^{d-2})$ for us to be successful for all such $S_0$. The main result of this paper is that we prove his conjecture to be correct.

# 1 Introduction

Let $G$ be an arbitrary infinite graph, where each vertex has finite degree. Next, let $U_0, U_1, \ldots$ and $X_1, X_2, \ldots$ be sequences of subsets (with $U_0$ finite) of $V(G)$ such that both

(i) $X_{t+1}$ is disjoint from $U_t$, and

(ii) $U_{t+1} = \mathcal{N}_G(U_t) \setminus (X_1 \cup \ldots \cup X_{t+1})$ for each nonnegative $t$.

(For every graph $G$ and any subset $U \subseteq V(G)$, we define $\mathcal{N}_G(U)$ to be the set of vertices $v$ that satisfy either $v \in U$ or $v$ is adjacent to a vertex in $U$.) We say that $X_1, X_2, \ldots$ is a *blocking sequence for* $U_0$ *in* $G$ there exists a (finite) integer $t_0$ such that $U_t = U_{t_0}$ for each $t \geq t_0$ (and because the degree of each vertex in $G$ is finite, $U_t$ must be a finite set as well).

For example, for any finite subset $U_0$ of $V(G)$, take $X_1$ to be the empty set, and $X_2$ to be the set $\mathcal{N}_G(\mathcal{N}_G(U_0)) \setminus \mathcal{N}_G(U_0)$. Then $U_1$ satisfies $U_1 = \mathcal{N}_G(U_0) \setminus X_1 = \mathcal{N}_G(U_0)$, and $U_2$ satisfies $U_2 = \mathcal{N}_G(U_1) \setminus (X_1 \cup X_2) = U_1$, and finally, for each $t \geq 3$, each $U_t$ satisfies $U_t = U_{t-1} = \ldots = U_2 = U_1$. Furthermore, $X_1, X_2$ is a blocking sequence for $U_0$ in $G$ because $X_1$ is disjoint from $U_0$ and $X_2$ is disjoint from $U_1$, and for each integer $t \geq 1$, $U_t = U_1$.

One motivation for studying blocking sequences comes from computational epidemiology. Suppose that each vertex in $G$ represents a person, and that a contagious infection will spread from every infected person to all of her not-yet-infected neighbors who have not been vaccinated in one time step (two people are considered neighbors if their corresponding vertices are adjacent in $G$), and that our allowed response is to vaccinate a limited number $f(t)$ of people in each time step $t$. Then if our objective is to contain the infection, writing as $U_0$ the set of vertices that correspond to the set of people already infected, we would like to find a blocking sequence $X_1, X_2, \ldots$ of $U_0$, where each $X_t$ satisfies $|X_t| \leq f(t)$. Indeed, then each vertex in $X_t$ corresponds to a person we would vaccinate at time $t$. With this in mind, we first prove Conjecture 1.1, which was posed by S. Hartke in his thesis [1]. We next prove Conjecture 1.2, a generalization of Conjecture 1.1, to be false.

Conjectures 1.1 and 1.2 are stated next.

**Conjecture 1.1** *Let $H$ be the infinite grid on $Z^d$ (where the vertex-set of $H$ is the set of vectors $(a_1, a_2, \ldots, a_d)$ with $d$ coordinates, where the $a_j$'s are integers, and two vertices in $H$ are adjacent if and only if they differ by 1 in exactly one coordinate, and they agree in the remaining $d-1$ coordinates). Next, let $f$ be any function on the set of positive integers such that $\frac{f(t)}{t^{d-2}}$ goes to 0 as $t$ goes to infinity. Then there exists a finite set $S_0$ such that there is no blocking sequence $V_1, V_2, \ldots$ for $S_0$ in $H$, where the $V_t$'s satisfy $|V_t| \leq f(t)$ for all integers $t$.*

**Conjecture 1.2** *Let $G$ be any infinite graph. First, for each positive integer $t$ and each vertex $v \in V(G)$, set $g'_G(t, v)$ to be the number of vertices in $G$ of distance precisely $t$ from*

*$v$. Next, for each such $t$, set $g_G(t) = \min_{v \in V(G)} g'(t, v)$. Then, let $f$ be any positive function on the set of positive integers such that $\sum_{i=1}^{t} \frac{f(i)}{g(t)}$ vanishes for $t$ sufficiently large. Then there exists a finite set $S_0$ such that there is no blocking sequence $V_1, V_2, \ldots$ for $S_0$ in $H$, where the $V_t$'s satisfy $|V_t| \leq f(t)$ for all integers $t$.*

We next state Theorem 1.3, which answers Conjecture 1.1 in the affirmative, and Theorem 1.4, which shows Conjecture 1.2 to be false.

**Theorem 1.3** *Let $H$ be the infinite grid on $Z^d$. Next, let $f$ be any function on the set of positive integers such that $\frac{f(t)}{t^{d-2}}$ goes to 0 as $t$ goes to infinity. Then there exists a finite set $S_0$ such that there is no blocking sequence $V_1, V_2, \ldots$ for $S_0$ in $H$ that satisfy $|V_t| \leq f(t)$ for all integers $t$.*

**Theorem 1.4** *Let $G$ be an infinite $k$-regular graph. Then for any subset $S_0$, there is an infinite blocking sequence $V_1, V_2, \ldots$ of $S_0$ in $G$ such that each $S_t$ has no more than $\frac{(k-1)^t}{t}$ vertices.*

Theorem 1.4 shows Conjecture 1.2 to be false in the following fashion. Suppose that $G$ were an infinite $k$-ary tree. Then $g_G(t)$ grows linearly with $(k-1)^t$, whereas $\sum_{i=1}^{t} \frac{(k-1)^i}{i}$ grows no faster than linearly with $\frac{(k-1)^t}{t}$. To make a connection between Theorem 1.4 and Theorem 1.3 in Epidemiology, a $d$-dimensional grid has maximum degree $2d$. While vaccinating only $o(t^{d-2})$ vertices at each step $t$ is not enough to contain every outbreak (Theorem 1.3), vaccinating $\frac{(4d-1)^t}{t} >> t^{d-2}$ vertices at each step $t$ most assuredly is (Theorem 1.4, although for the special case of infinite $d$-dimensional grids, it actually suffices to vaccinate only $4dt^{d-2}$ vertices at each step $t$).

We prove Theorem 1.3 in the next section, and Theorem 1.4 in §3. In the proofs of both theorems, we use the following notation. Let $G$ be a graph. For any two vertices $v$ and $w$ of $G$, we write as $d_G(v, w)$ the number of edges in the shortest path in $G$ from $v$ to $w$. Furthermore, for any subset $S$ of $V(G)$, we write as $d_G(v, S)$ the number of edges in the shortest path from $v$ to a vertex in $S$. (So if $v$ is in $S$, then $d_G(v, S)$ is 0.) Next, for any positive integer $c$, we denote the set of vertices $v$ such that $d_G(v, S)$ is no larger than $c$, as $\mathcal{N}_G^c(S)$ (so $S \subseteq \mathcal{N}_G^1(S) \subseteq \ldots \subseteq \mathcal{N}_G^c(S)$), and we write as $G^c$ the graph where two distinct vertices $u$ and $v$ are adjacent if $d_G(u, v)$ is less than $c$. (So $\mathcal{N}_{G^c}(U) = \mathcal{N}_G^c(U)$ for every subset $U$ of $G$ and every positive integer $c$.) Finally, for any graph $G$, we define the *extended double-cover* of $G$ to be the bipartite graph with parts $U = \{u_v | v \in V(G)\}$ and $W = \{w_v | v \in V(G)\}$, where $u_v$ and $w_{v'}$ are adjacent if and only if either $v$ and $v'$ are adjacent in $V$, or $v$ and $v'$ are the same vertex.

# 2 Proof of Theorem 1.3

We first prove Claim 2.1, which we use to establish Claim 2.2. We then use Claim 2.2 and 2.3 to finish the proof of Theorem 1.3.

**Claim 2.1** *Let $G$ be an arbitrary infinite graph, where each vertex has finite degree. Next, let $S_0$ be a finite set of vertices of $G$, and let $X_1, X_2, \ldots$ be a blocking sequence for $S_0$ in $G$. Then, for any positive integer $c$, there exists a blocking sequence $X'_1, X'_2, \ldots$ for $S_0$ in $G^c$, where each $X'_{t+1}$ is a subset of $\mathcal{N}_G^c(X_{ct+1} \cup X_{ct+2} \cup \ldots \cup X_{ct+c})$.*

*Proof of Claim 2.1:* First, let $X_1, X_2, \ldots$ and $S_1, S_2, \ldots$ be sequences of sets such that $S_t = \mathcal{N}_G(S_{t-1}) \setminus (X_1 \cup \ldots \cup X_t)$ for each positive $t$. Next, let $X'_1, X'_2, \ldots$ and $S'_1, S'_2, \ldots$ be sequences of sets where $X'_{t+1}$ is the set $X'_{t+1} = \mathcal{N}_{G^c}(X_{ct+1} \cup X_{ct+2} \ldots \cup X_{ct+c}) \setminus S_{ct}$, and where $S'_{t+1} = \mathcal{N}_{G^c}(S'_t) \setminus (X'_1 \cup \ldots \cup X'_{t+1})$.

We claim that $S'_t \subseteq S_{ct}$ for each $t$. To establish this, we use induction on $t$; this is clearly true for $t = 0$. Now suppose that there is a vertex $v \in S'_{t+1} \setminus S'_t$ that is not also in $S_{ct+c}$. Then $v$ is in $\mathcal{N}_G^c(S'_t)$, because $v \in S'_{t+1}$. Therefore, because $S'_t \subseteq S_{ct}$ by our inductive hypothesis, it follows that $v \in \mathcal{N}_{G^c}(S_{ct})$, which implies that $v$ must be in $\mathcal{N}_{G^c}(X_1 \cup X_2 \cup \ldots \cup X_{ct+c})$ (i.e., all paths in $G$ from $v$ to $S_{ct}$ that contain $c$ or fewer edges must contain a vertex of $X_1 \cup \ldots \cup X_{ct+c}$, otherwise $v$ would be in $S_{ct+c}$). Therefore, because $X'_1 \cup \ldots \cup X'_{t+1}$ contains at least every vertex in $\mathcal{N}_{G^c}(X_1 \cup X_2 \cup \ldots \cup X_{ct+c})$ that is not in $S'_t$ (and $v$ is not in $S'_t$), it follows that $v$ must be in $X'_1 \cup \ldots \cup X'_{t+1}$, so $v$ cannot be in $S'_{t+1}$ after all. Therefore, $S'_{t+1}$ must indeed be a subset of $S_{ct+c}$.

That $S'_t$ is a subset of $S_{ct}$ for all positive integers $t$ implies two things. First, the $S'_t$'s converge to a finite set. Indeed, if the $S_t$'s converge to a finite set, then so must the $S_{ct}$'s, and therefore by the above reasoning, so must the $S'_t$'s. Second, $X'_{t+1}$ is disjoint from $S'_t$. Indeed, by definition, $X'_{t+1}$ does not intersect $S_{ct}$, and because $S'_t$ is a subset of $S_{ct}$, it follows that $X'_{t+1}$ does not intersect $S'_t$. So $X'_1, X'_2, \ldots$ is indeed a blocking sequence for $S_0$ in $G^c$, and Claim 2.1 follows. ■

We now use Claim 2.1 to prove Claim 2.2.

**Claim 2.2** *Let $H$ be an infinite $d$-dimensional grid and let $S_0$ be any finite set, and let $f$ be any function on the set of positive integers such that $\frac{f(t)}{t^{d-2}}$ vanishes as $t$ goes to infinity. Suppose that there is a blocking sequence $V_1, V_2, \ldots$ of $S_0$ in $H$, where $|V_t| = f(t)$ for each integer $t$. Then for each positive integer $c \in O_d(1)$, there is a function $f'$ on the set of positive integers that satisfies (a) and (b), stated below.*

*(a)* $\frac{f(t)}{t^{d-2}}$ *vanishes as* $t$ *goes to infinity.*

*(b) There is a blocking sequence* $V_1', V_2', \ldots$ *for* $S_0$ *in* $H^c$ *such that* $|V_t| = f'(t)$ *for each positive integer* $t$*.*

*Proof of Claim 2.2:* This follows from Claim 2.1. Indeed, from Claim 2.1, we may assume that $f'(t+1)$ satisfies $f'(t+1) \leq d^c(f(ct)+f(ct+1)+\ldots+f(ct+c))$. Since both $d$ and $c$ are fixed relative to $t$ and $\frac{f(t)}{t^{d-2}}$ vanishes as $t$ goes to infinity, $t^{-(d-2)} \times d^c(f(ct)+f(ct+1)+\ldots+f(ct+c))$ vanishes as $t$ goes to infinity as well, and Claim 2.2 follows. ■

We are almost ready to use Claim 2.2 to finish the proof of Theorem 1.3. However, we first need a structural result about grids that we will establish in Claim 2.3.

**Claim 2.3** *Let* $H$ *be an infinite* $d$*-dimensional grid, and for each positive integer* $k$*, let* $T_k$ *be the set of vertices* $v$ *where the first coordinate of* $v$ *is* $k$*, and every other coordinate of* $v$ *is nonnegative and is no larger than* $k$*. Then for some* $c \in O_d(1)$*, every subset* $W_k$ *of* $T_k$ *that satisfies* $\frac{|T_k|}{2} \leq |W_k| \leq \frac{3|T_k|}{4}$ *has in* $H^c$ *at least* $|W_k| + |T_{k+1}| - |T_k|$ *neighbors in* $T_{k+1}$*, and every subset* $W_k$ *with more than* $\frac{3|T_k|}{4}$ *vertices has in* $H^c$ *at least* $|W_k|$ *neighbors in* $T_{k+1}$*.*

*Proof of Claim 2.3:* Let $H_k$ be the induced subgraph of $H^2$ between $T_k$ and the set $T_{k+1} \setminus \{v|$ $v$ has another coordinate besides the first equal to $k+1\}$. This graph is isomorphic to the extended double-cover of a $(d-1)$-dimensional grid. With this in mind, $(d-1)$-dimensional grids have the well-known (see e.g., Hartke's thesis) property that every subset $W$ with no more than a $\frac{7}{8}$-fraction of the vertices has at least $\theta(|W|^{\frac{d-2}{d-1}})$ neighbors in $T_{k+1}$ in $H_k$. So it follows that, for some $c \in O_d(1)$, every subset $W$ of vertices of $T_k$ such that $\frac{|T_k|}{2} \leq |W| \leq \frac{3|T_k|}{4}$ will have in $T_{k+1}$ at least $|W|+7d|T_k|^{\frac{d-2}{d-1}}$ neighbors in $(H^2)^{\frac{c}{2}} = H^c$. Since $|T_{k+1}| = (k+1)^{d-1}$ is no larger than $k^{d-1} + 2dk^{d-2} = |T_k| + 2d|T_k|^{\frac{d-2}{d-1}}$, it follows that $|T_{k+1}| - |T_k|$ is no larger than $2d|T_k|$, so $7d|T_k|^{\frac{d-2}{d-1}}$ is larger than $|T_{k+1}| - |T_k|$. Therefore, it follows that $W$ has at least $|W| + |T_{k+1}| - |T_k|$ neighbors in $H^c$ in $T_{k+1}$, and so Claim 2.3 follows. ■

*Remainder of the proof of Theorem 1.3:* Suppose that Theorem 1.3 is not true. We will show that this leads to a contradiction.

First let $c \in O_d(1)$ be as in Claim 2.3. Then by Claim 2.2, there is a function $f'$ that satisfies (a) and (b), stated below.

(a) $\frac{f'(t)}{t^{d-2}}$ vanishes as $t$ goes to infinity.

(b) For every set $S_0$, there is a sequence $V_1', V_2', \ldots$ that is blocking for $S_0$ in $H$, where $|V_t'| = f'(t)$ for each $t$.

So pick a positive integer $k_0$ such that both **(*)** $f'(k_0+t) \leq \frac{(k_0+t)^{d-2}}{100}$ for all positive $t$, and **(**)** $\frac{3(k-1)^d}{4} \geq \frac{2k^d}{3}$ for all $k \geq k_0$. Then pick a positive integer $c \in O(1)$, where $c$ is as Claim 2.3. Then set $S_0$ to be $T_0 \cup T_1 \cup \ldots \cup T_{k_0}$, where the $T_k$'s are as in Claim 2.3. Then, for each positive integer $t$, let $S_t$ be the set $\mathcal{N}_{H^c}(S_{t-1}) \setminus (V_1' \cup \ldots \cup V_t')$, where $V_1', V_2', \ldots$ is a blocking sequence for $S_0$, with each $V_t'$ satisfying $|V_t'| = f'(t)$. We first establish the following inequality:

$$|S_t \cap T_{k_0+t}| \geq \min\left\{|T_{k_0+t}| - (|V_1'| + \ldots + |V_t'|), \frac{|T_{k_0+t}|}{2}\right\} \tag{1}$$

for each positive integer $t$. (Indeed, use induction on $t$. This is trivially true for $t=1$. Now assume that Inequality (1) holds when we replace $t$ with $t-1$. First note that $|V_1'|+\ldots+|V_{t-1}'|$ is no larger than $\frac{(k_0+t-1)^{d-1}}{100} < \frac{(k_0+t-1)^{d-1}}{2} = \frac{|T_{k_0+t-1}|}{2}$ because $f'$ satisfies **(*)**. So this and the inductive hypothesis imply that $|S_{t-1} \cap T_{k_0+t-1}|$ is at least $\frac{|T_{k_0+t-1}|}{2}$. So Claim 2.3 implies that $|S_t \cap T_{k_0+t}|$ is at least $\min\{|T_{k_0+t}| - |T_{k_0+t-1}| + |S_{t-1} \cap T_{k_0+t-1}| - |V_t'|, \frac{3|T_{k_0+t-1}|}{4} - |V_t'|\}$, which equals $\min\{|T_{k_0+t}| - (|V_1'| + \ldots + |V_t'|), \frac{3|T_{k_0+t-1}|}{4} - |V_t'|\}$. However, because $f'$ satisfies **(*)** and $k_0$ satisfies **(**)**, it follows that Inequality (1) indeed holds.)

We now use Inequality (1) to finish the proof of Theorem 1.3. The quantity $|V_1'|+\ldots+|V_t'|$ is no larger than $\frac{(k_0+t)^{d-1}}{100}$, because $f'$ satisfies **(*)**, and $T_{k_0+t}$ has $(k_0+t)^{d-1}$. Therefore, Inequality (1) implies that $|S_t|$ is at least $\frac{(k_0+t)^{d-1}}{2}$ for each such $t$. So the $|S_t|$'s go to infinity as $t$ gets large, and so $V_1', V_2', \ldots$ cannot be a blocking sequence for $S_0$ in $H^c$ after all. Therefore, Theorem 1.3 follows. ■

# 3 Proof of Theorem 1.4

To prove Theorem 1.4, we first make the following observation.

**Claim 3.1** *Let $S_0$ and $U_0$ both be finite subsets of $V(G)$ such that $S_0 \subseteq U_0$. Then if $V_1, V_2, \ldots$ is a blocking sequence of $U_0$, then $V_1, V_2, \ldots$ is also a blocking sequence of $U_0$.*

We now use Claim 3.1. For every finite subset $S_0$ of $G$, there is a vertex $v_0 \in G$ and a (finite) integer $t_0$ such that $S_0 \subseteq \mathcal{N}_G^{t_0}(\{v_0\})$. So Claim 3.2 stated next, follows from this and Claim 3.1.

**Claim 3.2** *Suppose that, for each vertex $v \in G$ and each (finite) positive integer $l$, there exists a blocking sequence $V_1, V_2, \ldots$ of $\mathcal{N}_G^l(\{v\})$ such that each $V_t$ has no more than $\frac{(k-1)^t}{t}$ vertices. Then Theorem 1.4 follows.*

So we devote the rest of the section to proving the following claim, and then Theorem 1.4 will follow.

**Claim 3.3** *For each vertex $v$ and each (finite) positive integer $l$, there exists a blocking sequence $V_1, V_2, \ldots$ of $U_0 = \mathcal{N}_G^l(\{v\})$ such that each $V_t$ has no more than $\frac{(k-1)^t}{t}$ vertices.*

*Proof of Claim 3.3:* First, let $V_1$ be any set of $k-1$ vertices of $N_G(U_0) \setminus U_0$. Then, setting $U_1$ to be the set $\mathcal{N}_G(U_0) \setminus V_1$, let $V_2$ be any set of $\frac{(k-1)^2}{2}$ vertices in $N_G(U_1) \setminus (U_1 \cup V_1)$. Then, setting $U_2$ to be the set $\mathcal{N}_G(U_1) \setminus (V_1 \cup V_2)$, let $V_3$ be any set of $\frac{(k-1)^3}{3}$ vertices in $N_G(U_2) \setminus (U_2 \cup V_1 \cup V_2)$. Continue on in a similar fashion: for each positive integer $t$, setting $U_t$ to be the set $\mathcal{N}_G(U_{t-1}) \setminus (V_1 \cup V_2 \cup V_{t-1})$, let $V_{t+1}$ be any set of $\frac{(k-1)^{t+1}}{t+1}$ vertices in $N_G(U_t) \setminus (U_t \cup V_1 \cup V_2 \cup \ldots \cup V_t)$ (if there is such a set of $\frac{(k-1)^{t+1}}{t+1}$ vertices, otherwise, set $V_{t+1}$ to be the entire (perhaps empty) set $N_G(U_t) \setminus (U_t \cup V_1 \cup V_2 \cup \ldots \cup V_t)$.

We now note that $V_1, V_2, \ldots$ is a blocking sequence of $U_0$ in $G$ if there exists a positive integer $j$ such that $U_{j+1} \setminus U_j$ is empty. This follows from the definition of a blocking sequence. So each $V_t$ has $\frac{(k-1)^t}{t}$ vertices if $V_1, V_2, \ldots V_t$ is not a blocking sequence of $U_0$. We use this next.

Each vertex in each $U_t$ has at most $k-1$ neighbors outside of $U_t$. This is indeed true for $t = 0$ because $U_0$ consists of all vertices of distance $l$ or less from a vertex $v$, and every vertex in $U_0$ of distance $l-1$ or less from $v$ has all neighbors of distance $l$ or less from $v$ (and hence in $U_0$), while every vertex of distance precisely $l$ from $v$ has at least one neighbor of distance $l-1$ from $v$ (and hence in $U_0$). For $t$ positive, each vertex in $U_t \setminus U_{t-1}$ has at least one neighbor in $U_{t-1}$ by definition of $U_t$. From this, it follows that $|U_{t+1} \setminus U_t|$ is no larger than $\{(k-1)|U_t| - |V_{t+1}|$. From this, it follows that $|U_{t+1}|$ is $(k-1)^2|U_{t-1}| - (k-1)|V_t| - |V_{t+1}|$, and so we observe the following equality:

$$|U_{t+1} \setminus U_t| \leq (k-1)^{t+1}|U_0| - (k-1)^t|V_1| - (k-1)^{t-1}|V_2| - \ldots - |V_{t+1}|.$$

Now, as we observed earlier, each $V_j$ has cardinality $\frac{(k-1)^j}{j}$ if $V_1, V_2, \ldots V_t$ is not a blocking sequence of $U_0$, so in this case, the following equality becomes

$$|U_{t+1} \setminus U_t| \leq (k-1)^{t+1} \left( |U_0| - 1 - \frac{1}{2} - \ldots \frac{1}{t} \right),$$

which is strictly less than 0 for all finite $U_0$ and all $t_0$ sufficiently large (relative to $|U_0|$). So $V_1, V_2, \ldots V_t$ must indeed be a blocking sequence of $U_0$, and Claim 3.3 follows. ∎

Theorem 1.4 follows from Claims 3.2 and 3.3. ∎

## References

[1] S. Hartke, "Graph-theoretic models of spread and competition", Ph.D. Thesis for Rutgers University Department of Mathematics, July 2004.