DIMACS Technical Report 2006-07
October 2006

Large Deviations for a Point Process of Bounded Variability

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Abstract

We consider a one-dimensional translation invariant point process of density one with uniformly bounded variance of the number N_I of particles in any interval I. Despite this suppression of fluctuations we obtain a large deviation principle with rate function $\mathcal{F}(\rho) \simeq -L^{-1} \log \mathrm{Prob}(\rho)$ for observing a macroscopic density profile $\rho(x)$, $x \in [0,1]$, corresponding to the coarse-grained and rescaled density of the points of the original process in an interval of length L in the limit $L \to \infty$. $\mathcal{F}(\rho)$ is not convex and is discontinuous at $\rho \equiv 1$, the typical profile.

Key words: Large deviations, hyperuniform point processes, controlled variability.

1. Introduction

One dimensional *point processes of bounded variability* (or *controlled variability*) are translation invariant probability measures describing systems of particles in $\mathbb{R}$ for which the variance of the particle number N_I in an interval I is uniformly bounded: for some constant $C < \infty$ which is independent of I,

$$\mathrm{Var}(N_I) \le C. \tag{1.1}$$

In this paper we describe a particular class of such processes, which we call *G processes* and which are a special case of the *self-correcting processes* introduced by Isham and Westcott [1]. We compute some elementary properties of these systems and then show that they satisfy a large deviation principle.

Point processes satisfying (1.1) have been studied in the statistics literature; see [2,3,4,1]. They were also investigated in [5]; the analysis in that paper and in the references given there was motivated by, and presented in terms of, the statistical mechanics of infinite particle systems. The main result of [5] is that the measure μ on the space of locally finite subsets of $\mathbb{R}$ for a process satisfying (1.1) with density ρ is a superposition of mutually singular translates of some periodic measure μ_0 with period ρ^{-1}:

$$\mu = \rho \int_0^{\rho^{-1}} dx\, \mu_x. \tag{1.2}$$

Here μ_x is the translate of μ_0 by x; the measures μ_x are called the *periodic components* of μ. (The actual condition required in [5] is *tightness* of the family of all random variables $N_I - \rho|I|$, which is weaker than (1.1).)

The constant C of (1.1) must satisfy $C \ge 1/4$; in fact, if $E(N_I) = k + \theta$ with k an integer and $0 \le \theta < 1$ then,

$$\mathrm{Var}(N_I) = \langle (N_I - k)^2 \rangle - \theta^2 = \theta(1-\theta) + E\big((N_I - k)(N_I - k - 1)\big) \ge \theta(1-\theta), \tag{1.3}$$

where the last inequality follows from the fact that N_I is an integer (see [6]). The lower bound (1.3) is obtained by taking μ_0 to be the point mass on a lattice with period ρ^{-1}. In general, however, μ_0 need not be so rigid; it can have arbitrarily large fluctuations in particle number as long as these have small probability.

One such example, the (a, b) process discussed by Lewis and Govier [2], is obtained by displacing the point ai in the lattice $a\mathbb{Z}$ of period $\rho^{-1} = a$ by a random amount b_i, where the variables b_i are independent and have a common continuous distribution with finite first moment; see also [4]. This periodic process has bounded variances in the sense of (1.1); denoting by μ_0 the measure describing it one may thus construct a (translation invariant) process of bounded variability as a superposition of translates μ_x of μ_0 by $x \in [0, 1)$, as in (1.2). These translates can be shown to be mutually singular, so that the construction provides an explicit decomposition of the process into its periodic components. There are alternate constructions for which the periodicity is not built in a priori. An example of such a point process given in [5] is that corresponding to the equilibrium state for positive point charges in a uniform negative background, usually referred to as the one-dimensional one-component plasma or Jellium. This system has very long range interactions between the particles, with the pair potential growing like $|x_i - x_j|$. The self-correcting processes of [1] are other examples; the particular class of these which is the main focus of this paper will be described in section 2, and a brief description of the general class is given in Remark 3.4.

One may of course also consider point processes satisfying conditions similar to but less restrictive than (1.1), or satisfying analogues of these conditions in higher dimensions. We define *superhomogeneous* processes [7] to be translation invariant point processes in d-dimensions (on either $\mathbb{R}^d$ or $\mathbb{Z}^d$) for which the variance of the number N_B of particles in a ball B grows more slowly than the volume $|B|$ of B, i.e., for which

$$\lim_{|B|\to\infty} |B|^{-1} \mathrm{Var}(N_B) = 0. \tag{1.4}$$

It was shown by Beck [8] that $\mathrm{Var}(N_B)$ cannot grow more slowly (in the Cesaro mean sense) than the surface area $|\partial B|$ of B; it is thus only in one dimension that one can have translation invariant point processes with bounded variance in a ball, as in (1.1). *Hyperuniform* systems [9] are then defined as ones which realize this bound:

$$\mathrm{Var}(N_B) \le C|\partial B|. \tag{1.5}$$

Thus, hyperuniform systems in one dimension are the controlled variability processes defined earlier.

Examples of hyperuniform systems in dimensions $d > 1$ are the analogues of the (a, b) process, analyzed in [4], and the one-component plasma with $\log|x_i - x_j|$ interactions in $d = 2$ and $|x_i - x_j|^{-(d-2)}$ interactions in $d > 2$ [10,11]. A well known example of a superhomogeneous but not hyperuniform process in $d = 1$ is the Dyson gas with logarithmic interactions, whose measure μ is (after some normalization) the same as the distribution of the eigenvalues of a random Gaussian Hermitian matrix [12]. For this system $\mathrm{Var}(N_I) \sim \log|I|$. For other examples see [5], [13].

Given their definition, it is reasonable to expect that superhomogeneous systems will have a tendency to suppress also large deviations in N_B. This is in fact true for most of the examples discussed above but, as we shall see, there are exceptions, at least in one dimension. We will discuss this question primarily in the context of large deviations of N_I in one-dimensional systems.

To obtain more information about the probability of large fluctuations in N_I for $|I|$ large we may ask, for example, for what increasing functions $\phi(u)$, $u > 0$, it is true that

$$\mathrm{Prob}(|N_I - E(N_I)| \geq \phi(|I|)) \to 0 \qquad \text{as } |I| \to \infty. \tag{1.6}$$

For processes for which the N_I satisfy a standard central limit theorem—for example, the Poisson process—(1.6) holds only if $\phi(u)$ grows faster than $u^{1/2}$. For a process of bounded variability, on the other hand, it follows immediately from (1.1) that (1.6) holds whenever $\phi(u) \to \infty$ as $u \to \infty$.

When (1.6) does hold, we may ask for the rate of decay of the probability in (1.6) as $|I|$ increases. Of particular interest is the case $\phi(u) = \kappa u$, for which we will study the probability of deviations of specific sign; with $L = |I|$ we write

$$p_L(\kappa) \equiv \begin{cases} \mathrm{Prob}\big(N_I - E(N_I) \geq \kappa L\big), & \text{if } \kappa > 0, \\ 1, & \text{if } \kappa = 0 \\ \mathrm{Prob}\big(N_I - E(N_I) \leq \kappa L\big), & \text{if } -\rho \leq \kappa < 0, \end{cases} \tag{1.7}$$

The random variables N_I satisfy a large deviation principle with rate function $\psi(\kappa)$ if $p_L(\kappa)$ decays as $p_L(\kappa) \simeq \exp[-L\psi(\kappa)]$. Many processes—again, the Poisson case is typical—do satisfy such a principle, with $\psi(\kappa) \sim c\kappa^2$ for κ small. What then of processes of bounded variability?

(i) For the (a, b) processes, $-\log p_L(\kappa)$ grows faster than L, so that $\psi(\kappa)$ does not exist. For example, in order to have $N_I = 0$ (corresponding to $\kappa = -\rho$), each of the $L/3$ points in the central third of I must be displaced at least $L/3$ units, so that

$$\mathrm{Prob}(N_I = 0) \leq [\mathrm{Prob}(b_i \geq L/3)]^{L/3}. \tag{1.8}$$

More generally, a fluctuation of size κL in N_I can occur only if at least order L particles are displaced a distance at least of order L; an estimate as in (1.8), supplemented with some simple counting of the ways these displacements can occur, then verifies the result.

(ii) For the one-component plasma it is believed that $p_L(\kappa) \simeq \exp[-c(\kappa)L^3]$, so that again $\psi(\kappa)$ does not exist (although of course $c(\kappa)$ is a rate function on a different scale). In d dimensions the corresponding quantity $p_R(\kappa)$ describing large fluctuations in N_B, with B a ball of radius R, satisfies $p_R(\kappa) \simeq \exp[-c(\kappa)R^{d+2}]$ (see [14] for the heuristic derivation of the result in 2 and 3 dimensions). We expect similar behavior for most superhomogeneous processes, for example with $p_L(\kappa) \simeq \exp[-c(\kappa)L^2]$ for the Dyson gas in $d = 1$ (see [12] for a discussion of the case $\kappa = -\rho$, i.e., $N_I = 0$).

(iii) For the G process, on the other hand, we will establish in section 4 that there does exist a rate function $\psi(\kappa)$, with $\psi(\kappa) \sim c|\kappa|$ when $\kappa \to 0$. This is a consequence of the more general large deviation theorem given in section 3. We believe that certain other self-correcting process also satisfy a large deviation principle, as we discuss in Remark 3.4.

2. Construction and simple properties of the G process

In this section we construct the G processes with density 1; these are parameterized by a real number $\alpha > 1$ (the construction for arbitrary density ρ is similar, and the parameter satisfies $\alpha > \rho$). As a first step we define a real-valued Markov process $Y_\lambda(t)$, for $t \geq 0$, satisfying $Y_\lambda(t) > -1$; here λ is a probability measure on $(-1, \infty)$. $Y_\lambda(t)$ is defined by two conditions: $Y_\lambda(0)$ is distributed according to λ, and $Y_\lambda(t)$ increases at rate 1 as t increases, except at the points of a Poisson process of density α on $\mathbb{R}_+$, at which it jumps down by one unit—unless this jump would violate the condition $Y_\lambda > -1$, in which case no jump occurs. It is easy to verify explicitly (see below) that this process has a unique stationary single-time distribution $\lambda = \lambda_0$. The corresponding translation invariant process (obtained for example by imposing the initial condition λ_0 at time τ and then taking the Cesaro limit as $\tau \to -\infty$) we denote by $Y(t)$. Having defined Y, we take the points of the G process to be those points at which Y jumps; in other words, the G process is the distribution of the jump points of the Y process. We remark that the points of the G process may be viewed as the output of a so-called D/M/1 queue [15].

Let N_I denote the number of points of the G process in the interval I, and write Prob for the probability measure of the Y process (e.g., on path space) and E for expectation with respect to this measure. Then clearly

$$N_{(s,t]} = t - s - (Y(t) - Y(s)), \tag{2.1}$$

and

$$\mathrm{Var}(N_{(s,t]}) = E[((Y(t) - Y(s))^2]. \tag{2.2}$$

Since from (2.3) below $Y(t)$ has finite mean and variance, (2.1) and (2.2) imply that the process has density 1 and is of bounded variability.

In the remainder of this section we compute some simple properties of this process, including the asymptotic value of $\mathrm{Var}(N_{(s,t]})$ for $t - s$ large. We first note that if λ is a probability distribution with density $f_\lambda(y)$ then the density $f_\lambda(y,t)$ for $Y_\lambda(t)$ satisfies $f_\lambda(y,0) = f_\lambda(y)$ and

$$\frac{\partial f_\lambda}{\partial t}(y,t) + \frac{\partial f_\lambda}{\partial y}(y,t) = \begin{cases} \alpha f_\lambda(y+1,t) - \alpha f_\lambda(y,t), & \text{if } y > 0, \\ \alpha f_\lambda(y+1,t), & \text{if } y \leq 0. \end{cases}$$

The stationary probability density for Y is then seen to be

$$f(y) = \begin{cases} 1 - \beta^{1+y}, & \text{if } -1 < y < 0, \\ (1-\beta)\beta^y, & \text{if } y \geq 0, \end{cases} \tag{2.3}$$

where β is the unique root of the equation

$$\beta = e^{-\alpha(1-\beta)} \tag{2.4}$$

lying in the interval $0 < \beta < 1$. From (2.3),

$$E(Y(t)) = \frac{1}{|\log\beta|} - \frac{1}{2}, \tag{2.5}$$

$$E(Y^2(t)) = \frac{1}{3} - \frac{1}{|\log\beta|} + \frac{2}{\log^2\beta}. \tag{2.6}$$

To obtain the decomposition (1.2) of the process into its periodic components we introduce the random variable $U = Y(n) - \lfloor Y(n) \rfloor$, $n \in \mathbb{Z}$, where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x. The value of U is independent of the choice of the integer n and the periodic components of the process are obtained by conditioning on the value u of U, $0 \leq u < 1$. Under this conditioning we have that the possible values of $Y(t)$ are of the form $k + v$, with $k \geq -1$ an integer and $v = u + t - \lfloor u + t \rfloor$, and from (2.3) that

$$\mathrm{Prob}(Y(t) = k + v \mid U = u) = \begin{cases} 1 - \beta^v, & \text{if } k = -1, \\ (1-\beta)\beta^{k+v}, & \text{if } k \geq 0. \end{cases} \tag{2.7}$$

Thus

$$E(Y(t) \mid U = u) = v - 1 + \frac{\beta^v}{1-\beta}. \tag{2.8}$$

Now we evaluate the variance of $N_{(s,t]}$ for $t - s$ large, from (2.2). Note that if we condition on $U = u$ then the process $Y(n)$, $n \in \mathbb{Z}$, is a transitive Markov chain on the values $k + u$, where $k = 0, 1, \ldots$ if $u = 0$ and otherwise $k = -1, 0, \ldots$. Thus $Y(n)$ and $Y(m)$ will be independent in the $n \to \infty$ limit, and so will be $Y(s)$ and $Y(t)$ as $t \to \infty$. Taking for simplicity $s = 0$ and setting $t = n + \theta$ with $n \in \mathbb{Z}$ and $0 \le \theta < 1$, this leads to

$$\lim_{n\to\infty, n\in\mathbb{Z}} E(Y(0)Y(n+\theta) \mid U = u) = E(Y(0) \mid U = u)E(Y(\theta) \mid U = u),$$

with the approach to the product being exponentially fast. It thus follows from (2.8) that

$$\begin{aligned} \lim_{n\to\infty, n\in\mathbb{Z}} E(Y(0)Y(n+\theta)) &= \int_0^1 E(Y(0) \mid U = u)E(Y(\theta) \mid U = u)\, du \\ &= \frac{1}{3} - \frac{\theta(1-\theta)}{2} - \frac{1}{|\log \beta|} + \frac{2}{\log^2 \beta} - \frac{\beta^\theta + \beta^{1-\theta}}{(1-\beta)|\log \beta|}. \end{aligned} \tag{2.9}$$

Finally, from (2.2), (2.6), and (2.9),

$$W(\theta) \equiv \lim_{n\to\infty, n\in\mathbb{Z}} \mathrm{Var}(N_{(0,n+\theta]}) = \theta(1-\theta) + 2\frac{\beta^\theta + \beta^{1-\theta}}{(1-\beta)|\log \beta|}. \tag{2.10}$$

Thus, for large t, $\mathrm{Var}(N_{(0,t]})$ is not constant but is periodic in t with period 1.

The first term in (2.10) is just the variance of the rigid lattice. The second term goes to zero as $\alpha \nearrow \infty$, when the process tends to that of the lattice, and goes to infinity as $\alpha \searrow 1$, when the process becomes Poisson; see Figure 1. The average $\overline{W}$ of $W(\theta)$ over the interval $0 \le \theta \le 1$ is, from (2.4),

$$\overline{W} = \frac{1}{6} + \frac{4}{\log^2 \beta} \simeq \begin{cases} \dfrac{1}{(\alpha-1)^2}, & \text{as } \alpha \searrow 1, \\ \dfrac{1}{6} + \dfrac{4}{\alpha^2}, & \text{as } \alpha \nearrow \infty. \end{cases}$$

In terms of $Y(t)$, the limit $\alpha \searrow 1$ corresponds to an unbinding transition: for any t, $Y(t) \nearrow \infty$ with probability 1 as $\alpha \searrow 1$.

3. Large deviations

In the introduction we discussed briefly the problem of large deviations for the number of points of a point process in an interval. We will return to this question in section 4, but we must first consider the sharper problem of large deviations in the density profile exhibited by the process. Suppose then that we are given a (macroscopic) density profile $\rho(x)$—a nonnegative integrable function—on the unit interval $[0,1]$. We are interested in the event that the coarse-grained and rescaled density for the point process in the region $[0,L]$ (where L is considered to be very large) agrees with ρ, i.e. that, roughly,

$$N_{(0,xL]} \sim L\int_0^x \rho(\xi)\,d\xi \quad \text{for all } x \text{ in } [0,1]. \tag{3.1}$$

We say that the process satisfies a large deviation principle if the probability of the event (3.1) decays with L as $\exp[-L\mathcal{G}(\rho)]$ for some rate function $\mathcal{G}$.

One way to formalize this notion is to demand that

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}\frac{1}{L}\log \mathrm{Prob}\left[\sup_{0\le x\le 1}\left|L^{-1}N_{(0,xL]}-\int_0^x \rho(\xi)\,d\xi\right|<\epsilon\right]=-\mathcal{G}(\rho). \tag{3.2}$$

In Theorem 3.3 below we give a slightly weaker version of (3.2) for the G process, in which the $L\to\infty$ limit is replaced by either a lim sup or a lim inf. For the G process we may, by (2.1), restate (3.2) as

$$\lim_{\epsilon\to 0}\lim_{L\to\infty}\frac{1}{L}\log \mathrm{Prob}\left[\sup_{0\le x\le 1}\left|\frac{1}{L}Y(xL)-\left(\frac{1}{L}Y(0)+\int_0^x (1-\rho(\xi))\,d\xi\right)\right|<\epsilon\right]=-\mathcal{G}(\rho). \tag{3.3}$$

We will approach (3.3) by first obtaining a large deviation principle for the probability that the rescaled Y process $L^{-1}Y(xL)$ be uniformly close to some specified function $y(x)$. It is this problem which we discuss next.

We first describe precisely the probability space on which our processes are most conveniently regarded as defined. Since (3.2) involves $N_{(0,t]}$ only for $t\ge 0$ we may realize the relevant part of the G process as the jumps of the process Y_{λ_0} introduced in section 2; here and below we write $Y_{\lambda_0}(t)\equiv Y(t)$. These jump points are a subset of the points of a Poisson process $\hat N$ with rate $\alpha>1$; this process is (for any $\alpha>0$) defined on the space $\hat\Omega$ of locally finite point configurations in $(0,\infty)$ and is given by a probability measure $\hat\nu$ on $\hat\Omega$, with the distribution of $\hat N_I(\omega)$, the number of points of the configuration $\omega\in\hat\Omega$ in the interval I, satisfying $\hat\nu(\hat N_I=k)=(\alpha|I|)^k\exp(-\alpha|I|)/k!$. Y is then defined on the probability space $\Omega\equiv(-1,\infty)\times\hat\Omega$ with measure $\nu=\lambda_0\times\hat\nu$ by the conditions

(i) $Y(v,\omega)(t)$ is right continuous, with left-hand limits, in t;

(ii) $Y(v,\omega)(0)=v$;

(iii) $(dY(v,\omega)/dt)(t)=1$ unless $t\in\omega$; and

(iv) if $t\in\omega$ then $Y(v,\omega)(t)=Y(v,\omega)(t-)-1$ unless $Y(v,\omega)(t-)\le 0$, in which case $Y(v,\omega)(t)=Y(v,\omega)(t-)$.

In this context we formulate the problem posed at the end of the previous paragraph as follows. Let D denote the space of real-valued cadlag functions—functions which are right continuous with left-hand limits at each point—on $[0,1]$, furnished with the Skorohod topology [16]. Then with $\Phi_L:\Omega\to D$ defined by $\Phi_L(v,\omega)(x)=L^{-1}Y(v,\omega)(Lx)$, we want to determine the $L\to\infty$ asymptotics of the probability $\nu(\Phi_L^{-1}(S))$ for open and closed subsets $S\subset D$. In particular the specific question raised below (3.3), which involves *uniform* neighborhoods of the continuous function $y(x)$, can be answered in terms of Skorohod neighborhoods, since every Skorohod neighborhood of y contains a uniform neighborhood, and vice versa.

We will adopt the standard terminology of the theory of large deviations [17]. Let $\mathcal{X}$ be a topological space. A function $\mathcal{F}:\mathcal{X}\to[0,\infty]$ is a *good rate function* on $\mathcal{X}$ if it is lower semicontinuous (l.s.c.) and the level sets $\mathcal{F}^{-1}([0,a])$, $a\in\mathbb{R}_+$, are compact. A family $\{\mu_L\}_{L\ge0}$ of Borel measures on $\mathcal{X}$ *satisfies the large deviation principle (LDP) with rate function* $\mathcal{F}$ if

$$\liminf_{L\to\infty}\frac{1}{L}\log\mu_L(S)\ge-\inf_{y\in S}\hat{\mathcal{F}}(y), \quad \textit{if } S\subset\mathcal{X} \textit{ is open,} \tag{3.4}$$

$$\limsup_{L\to\infty}\frac{1}{L}\log\mu_L(S)\le-\inf_{y\in S}\hat{\mathcal{F}}(y), \quad \textit{if } S\subset\mathcal{X} \textit{ is closed.} \tag{3.5}$$

The large deviation theory for the Poisson process $\hat{N}$ itself, for any $\alpha > 0$, may be conveniently stated in this framework. For $L > 0$ define $\hat{\Phi}_L : \hat{\Omega} \to D$ by $\hat{\Phi}_L(\omega)(x) = x - L^{-1}\hat{N}_{(0,Lx]}$; note that we consider sample paths which, in parallel with those of the Y process, drift upward at unit velocity and take downward jumps at points of the Poisson process. We write AC for the set of absolutely continuous paths in D, and define the rate function $\hat{\mathcal{F}} : D \to [0,\infty]$ by

$$\hat{\mathcal{F}}(y) = \begin{cases} \displaystyle\int_0^1 g(1 - y'(x))\,dx, & \text{if } y \in AC,\ y(0) = 0, \text{ and } y'(x) \le 1 \text{ a.e.,} \\ \infty, & \text{otherwise,} \end{cases} \tag{3.6}$$

where

$$g(r)(= g_\alpha(r)) = r \log \frac{r}{\alpha} - r + \alpha. \tag{3.7}$$

Theorem 3.1: *For any $\alpha > 0$, $\hat{\mathcal{F}}$ is a good rate function on D and the measures $\hat{\nu} \circ \hat{\Phi}_L^{-1}$ on D satisfy the LDP with rate function $\hat{\mathcal{F}}$.*

Proof: This result is an easy consequence of Exercise 5.2.12 of [17]; in particular, the measures $\hat{\nu} \circ \hat{\Phi}_L^{-1}$, regarded there as measures on $L^\infty([0,1])$, are supported on D and are Borel measures there. ■

Now we formulate the corresponding result for the G process. The rate function $\mathcal{F}(y)$, for which (essentially) $-L\mathcal{F}(y)$ is the log of the probability that the rescaled Y process $\Phi_L(x) = L^{-1}Y(Lx)$ follows the path $y(x)$ in D, is

$$\mathcal{F}(y) = \begin{cases} y(0)|\log\beta| + \displaystyle\int_{\{\,x\in[0,1] | y(x)>0\,\}} g(1 - y'(x))\,dx, & \text{if } y \in AC,\ y \ge 0, \text{ and } y'(x) \le 1 \text{ a.e.,} \\ \infty, & \text{otherwise.} \end{cases} \tag{3.8}$$

This formula may be understood intuitively as follows. First, $\mathcal{F}(y)$ is finite only if $y \ge 0$ because $Y > -1$ and hence $\Phi_L(x) > -1/L$. Second, during any time interval in which $Y(t)$ is strictly positive the Poisson and G processes are the same, and the contribution of such an interval to the rate function for the G process should correspond to (3.6); on the other hand, wherever $Y(t)$ is nonpositive the Poisson process is free to produce points at its natural rate α, so the contribution to the rate function from such a region should vanish. Finally, the term $y(0)|\log\beta|$ arises from (2.3) as the cost of "preparing" the system at time 0 with $L^{-1}Y(0) \sim y(0)$.

More formally, we will establish a result parallel to Theorem 3.1; the proof is given in section 5.

Theorem 3.2: *For any $\alpha > 1$, $\mathcal{F}$ is a good rate function on D, and the measures $\nu \circ \Phi_L^{-1}$ on D satisfy the LDP with rate function $\mathcal{F}$.*

The rate function $\mathcal{G}(\rho)$ for a density profile $\rho \in L^1([0,1])$, $\rho \ge 0$, is now obtained via

$$\mathcal{G}(\rho) \equiv \inf_{y \in J_\rho} \mathcal{F}(y), \tag{3.9}$$

where $J_\rho \equiv \{y \in AC \mid y' = 1 - \rho\}$; from (3.8) it follows that the infimum in (3.9) is achieved for $y = y_\rho$, with

$$y_\rho(x) \equiv \int_0^x \big(1 - \rho(\xi)\big)\,d\xi - \inf_{x'\in[0,1]} \int_0^{x'} \big(1 - \rho(\xi)\big)\,d\xi.$$

We justify (3.9) by proving a version of equation (3.2):

Theorem 3.3: *Suppose that $\rho \in L^1([0,1])$ with $\rho \ge 0$ and that for $\epsilon > 0$, $E_{L,\epsilon}(\rho) \subset \Omega$ is the event that*

$$\sup_{0\le x\le 1} \left| L^{-1}N_{(0,xL]} - \int_0^x \rho(t)\,dt \right| \le \epsilon.$$

Then

$$\lim_{\epsilon\to 0} \liminf_{L\to\infty} \frac{1}{L} \log \nu(E_{L,\epsilon}(\rho)) = \lim_{\epsilon\to 0} \limsup_{L\to\infty} \frac{1}{L} \log \nu(E_{L,\epsilon}(\rho)) = -\mathcal{G}(\rho). \tag{3.10}$$

The proof is given in section 5. In fact, (3.2) can also be proven, but the proof is considerably more complicated than that of Theorem 3.3 and provides no better justification of the interpretation of $\mathcal{G}$.

Remark 3.4: As indicated in the introduction, the G process studied here is a special case of the self-correcting processes of Isham and Westcott [1]. In the more general model the points of the process are again the jump points of an auxiliary process $Y(t)$. For the density 1 case Y again increases at rate 1 except at its jump points, and these occur randomly at rate $\gamma(Y(t))$ for some specified nonnegative function γ. The G process is obtained by taking $\gamma(v) = \alpha$ for $v > 0$ and $\gamma(v) = 0$ for $v \leq 0$. More generally, [1] requires that $\limsup_{v\to-\infty} \gamma(v) < 1$ and $\liminf_{v\to\infty} \gamma(v) > 1$ and that $\gamma(v)$ be strictly bounded away from 0 for $v > 0$. We expect that these processes will satisfy a large deviation principle in D whenever $\alpha_\pm \equiv \lim_{v\to\pm\infty} \gamma(v)$ exist, with rate function similar to that of the G process (see (3.8)): finite only for $y \in AC$ with $y' \leq 1$ and, for such y,

$$\mathcal{F}(y) = |y(0)\log\beta_{\mathrm{sgn}\ y(0)}| + \int_{\{\, y(x)>0\, \}} g_{\alpha_+}(1-y'(x))\,dx + \int_{\{\, y(x)<0\, \}} g_{\alpha_-}(1-y'(x))\,dx, \tag{3.11}$$

where g_α is defined in (3.7) and $\beta_\pm$ is the non-unit root of $\beta_\pm = \exp[-\alpha_\pm(1-\beta_\pm)]$.

4. Applications of the large deviation principle

In this section we consider the question, originally posed in the introduction, of a large deviation principle for the number N_I of points of the G process in an interval I. We want to evaluate $\lim_{L\to\infty} L^{-1}\log p_L(\kappa)$, where (see (1.7))

$$p_L(\kappa) \equiv \begin{cases} \nu\big(N_{(0,L]} \geq (1+\kappa)L\big), & \text{if } \kappa > 0, \\ 1, & \text{if } \kappa = 0 \\ \nu\big(N_{(0,L]} \leq (1+\kappa)L\big), & \text{if } -1 \leq \kappa < 0, \end{cases} \tag{4.1}$$

Consider first $\kappa \geq 0$. By (2.1), $N_{(0,L]} \geq (1+\kappa)L$ if and only if $Y(L) \leq Y(0) - \kappa L$; setting $S_0 = D$ and $S_\kappa = \{y \in D \mid y(1) \leq y(0) - \kappa\}$ for $\kappa > 0$ we see that $p_L(\kappa) = \nu(\Phi_L^{-1}(S_\kappa))$. Let $\psi(\kappa) \equiv \inf_{y\in S_\kappa} \mathcal{F}(y)$; we will compute $\psi(\kappa)$ below and show that it is continuous for $\kappa \geq 0$. Since for any $\epsilon > 0$, $S_{\kappa+\epsilon}$ is contained in the interior of S_κ, Theorem 3.2 implies that

$$-\psi(\kappa) \geq \limsup_{L\to\infty} \log p_L(\kappa) \geq \liminf_{L\to\infty} \frac{1}{L}\log p_L(\kappa) \geq -\psi(\kappa+\epsilon).$$

Since ϵ is arbitrary,

$$\lim_{L\to\infty} \frac{1}{L}\log p_L(\kappa) = -\psi(\kappa). \tag{4.2}$$

A similar argument shows that (4.2) holds also for $-1 \leq \kappa < 0$, with $\psi(\kappa) \equiv \inf_{y\in S_\kappa} \mathcal{F}(y)$, where $S_\kappa = \{y \in D \mid y(1) \geq y(0) - \kappa\}$.

We next compute $\psi(\kappa) = \inf_{S_\kappa} \mathcal{F}(y)$. Certainly $\psi(0) = \inf_D \mathcal{F} = 0$; the infimum is obtained by the function $y \equiv 0$. In general, since if $y \in D$ is nonnegative, $\mathcal{F}(y) \geq \mathcal{F}(y - \inf y)$, and since if $y \in S_\kappa$ then also $y - \inf y \in S_\kappa$, in computing $\inf_{S_\kappa} \mathcal{F}$ we need only consider nonnegative functions y in S_κ which vanish at some point.

We now take $\kappa > 0$ and determine the function $y \in S_\kappa$ which realizes $\inf_{S_\kappa} \mathcal{F}$. Let $a = \inf\{x \in [0,1] \mid y(x) = 0\}$. From (3.8), y must vanish on $[a,1]$, and by the convexity of g and Jensen's inequality, y must be affine on $[0,a]$; thus, setting $y(0) = b$,

$$\psi(\kappa) = \inf_{\substack{0\leq a\leq 1,\\ \kappa\leq b}} \left[b|\log\beta| + a g\left(\frac{a+b}{a}\right)\right]. \tag{4.3}$$

Since $g \geq 0$, $\psi(\kappa) \geq \kappa|\log\beta|$, and since $g(\alpha) = 0$, this minimum is achieved when $\kappa \leq \alpha - 1$ by taking $b = \kappa$ and $(a+b)/a = \alpha$, i.e., $a = \kappa/(\alpha-1)$. When $\kappa > \alpha - 1$ the minimum for fixed a is achieved at $b = \kappa$, since g is increasing on $[\alpha, \infty)$, and the global minimum $\psi(\kappa) = \kappa|\log\beta| + g(1+\kappa)$ is then at $a = 1$. The minimizing function y is shown in Figure 2.

The analysis for $-1 \le \kappa < 0$ begins in a similar way. The minimizing y in S_κ must vanish on some interval $[0, a]$ and be affine, with $y' \le 1$, on $[a, 1]$, and $b = y(1)$ must satisfy $b \ge |\kappa|$, so that

$$\psi(\kappa) = \inf_{\substack{0 \le a \le 1-|\kappa| \\ |\kappa| \le b \le 1-a}} \left[(1-a) g\left(\frac{1-a-b}{1-a} \right) \right]. \tag{4.4}$$

Since g is decreasing on $[0, \alpha]$ the minimum for fixed $a \le 1 - |\kappa|$ is at $b = |\kappa|$. For $|\kappa| \le 1 - \alpha\beta$ the global minimum $\psi(\kappa) = |\kappa||\log\beta|$ is achieved at $a = (1 - \alpha\beta - |\kappa|)/(1 - \alpha\beta)$; for $|\kappa| > 1 - \alpha\beta$ the global minimum $\psi(\kappa) = g(1+\kappa)$ is achieved at $a = 0$; see Figure 3. Thus

$$\psi(\kappa) = \begin{cases} g(1+\kappa), & \text{if } -1 \le \kappa \le \alpha\beta - 1, \\ |\kappa||\log\beta|, & \text{if } \alpha\beta - 1 \le \kappa \le \alpha - 1, \\ \kappa|\log\beta| + g(1+\kappa), & \text{if } \alpha - 1 < \kappa. \end{cases} \tag{4.5}$$

Remark 4.1: Formula (4.5) for $\kappa < 0$ can perhaps be better understood by considering the time reversal of the Y process with respect to the stationary measure with density given in (2.3). This process involves unit jumps upward at the rate $\alpha\beta$, yielding

$$\psi(\kappa) = \begin{cases} |\kappa||\log\beta| + g_{\alpha\beta}(1+\kappa), & \text{if } -1 \le \kappa < \alpha\beta - 1. \\ |\kappa||\log\beta|, & \text{if } \alpha\beta - 1 \le \kappa \le 0, \end{cases} \tag{4.6}$$

which agrees with (4.5); $g_{\alpha\beta}$ was defined in (3.7). (In more detail: the reversed process drifts downward with unit velocity and takes unit jumps upward, at rate $\alpha\beta$ when $y > 0$ and at rate $\alpha(1-\beta)\beta^{y+1}/(1-\beta^{y+1})$ when $-1 < y \le 0$.)

We may compare (4.5) with the corresponding result for the Poisson process of density 1. Defining $p^*_\kappa(L)$ as in (4.1) but with $N_{(0,L]}$ replaced by $N^*_{(0,L]}$, the number of points of this process in $(0, L]$, we have

$$\lim_{L\to\infty} \frac{1}{L} \log p^*_\kappa(L) = \psi^*(\kappa) \equiv g_1(1+\kappa),$$

where g_1 is defined in (3.7). This follows from a simple computation with Stirling's formula; it may also be obtained from Theorem 3.1, and we then find that the minimizing $y^* \in D$ is $y^*(x) = -\kappa x$, corresponding to constant density $1+\kappa$ for the points of the Poisson process. Figure 4 compares ψ and ψ^*.

One may also ask a related question, for either the Poisson or the G process: what is the rate function for the probability of a constant density profile $\rho(x) = 1+\kappa$ (which of course gives rise to $(1+\kappa)L$ points in the interval $(0, L]$; compare(4.1))? For the Poisson process of rate 1 this is just $g_1(1+\kappa)$, since, as remarked above, the constant profile is the typical one when we condition on the total number of particles in the interval. For the G process, however, the rate function is $\psi_0(\kappa) \equiv \mathcal{G}(\rho)$ as obtained from (3.9), which yields

$$\psi_0(\kappa) = \begin{cases} g(1+\kappa), & \text{if } -1 \le \kappa < 0; \\ 0, & \text{if } \kappa = 0, \\ \kappa|\log\beta| + g(1+\kappa)(= g_{\alpha\beta}(1+\kappa)), & \text{if } 0 < \kappa. \end{cases} \tag{4.7}$$

Since $g(1) = \alpha - 1 - \log\alpha > 0$, $\psi_0(\kappa)$ is nonconvex and discontinuous at $\kappa = 0$. See Figure 5.

5. Proofs of the main results

We will prove Theorem 3.2 via the *contraction principle* [17]: if $\mathcal{X}$ and $\mathcal{Y}$ are Hausdorff spaces, $\{\mu_L\}$ is a family of Borel measures on $\mathcal{X}$ satisfying the LDP with good rate function $\mathcal{F}$, and $f : \mathcal{X} \to \mathcal{Y}$ is continuous at each point $x \in \mathcal{X}$ for which $\mathcal{F}(x) < \infty$, then the family $\{\mu_L \circ f^{-1}\}$ satisfies the LDP on $\mathcal{Y}$ with good rate function $\bar{\mathcal{F}}(y) = \inf_{x\in f^{-1}(y)} \mathcal{F}(x)$.

To proceed we first derive an LDP for a family of measures on D obtained from those of the Poisson process. Define $\Phi_{0L} : \mathbb{R} \to \mathbb{R}$ by $\Phi_{0L}(v) = v/L$; it is easy to see that if $\mathcal{F}_0 : \mathbb{R} \to [0,\infty]$ is defined by

$$\mathcal{F}_0(a) = \begin{cases} a|\log\beta|, & \text{if } a \geq 0, \\ \infty, & \text{if } a < 0, \end{cases}$$

then the measures $\lambda_0 \circ \Phi_{0L}^{-1}$ on $\mathbb{R}$ satisfy the LDP with rate function $\mathcal{F}_0$. Now define $\tilde{\Phi}_L : \Omega \to D$ by $\tilde{\Phi}_L(v,\omega)(x) = \Phi_{0L}(v) + \hat{\Phi}_L(\omega)(x)$ and $\tilde{\mathcal{F}} : D \to [0,\infty]$ by $\tilde{\mathcal{F}}(y) = \mathcal{F}_0(y(0)) + \hat{\mathcal{F}}(y - y(0))$. Finally, let $P \subset D$ be defined by $P = \bigcup_{L\geq 1} \tilde{\Phi}_L(\Omega)$ and let $\bar{P}$ be the closure of P in D.

Remark 5.1: We note that (i) $\nu(\tilde{\Phi}_L^{-1}(P)) = \nu(\tilde{\Phi}_L^{-1}(\bar{P})) = 1$ for all $L \geq 1$. Clearly, any function $z \in \bar{P}$ satisfies

$$z(x_2) - z(x_1) \leq x_2 - x_1 \quad \text{whenever } 0 \leq x_1 \leq x_2 \leq 1; \tag{5.1}$$

conversely, (ii) any continuous function z on $[0,1]$ which satisfies (5.1) may be uniformly approximated, to accuracy $1/L$, by functions in $\tilde{\Phi}_L(\Omega)$, and hence belongs to $\bar{P}$, since one may apply to $\tilde{z}$, where $\tilde{z}(t) = z(t) - t$, the simple fact that any continuous nonincreasing function can be uniformly approximated, to accuracy $1/L$, by a piecewise constant function whose jumps are down and of size $1/L$.

Lemma 5.2: *The measures $\nu \circ \tilde{\Phi}_L^{-1}$ on D, and on $\bar{P}$, satisfy the LDP with good rate function $\tilde{\mathcal{F}}$.*

Proof: The product measures $(\lambda_0 \circ \Phi_{0L}^{-1}) \times (\hat{\nu} \circ \hat{\Phi}_L^{-1})$ satisfy the large deviation principle on $\mathbb{R} \times D$ with good rate function $\mathcal{F}^*(a,y) = \mathcal{F}_0(a) + \hat{\mathcal{F}}(y)$ (see [17]; the proof uses the fact that $\mathbb{R}$ and D are Polish spaces). The large deviation principle for $\nu \circ \tilde{\Phi}_L^{-1}$ on D is obtained from an application of the contraction principle to the map $f : \mathbb{R} \times D \to D$ defined by $f(a,y) = a + y$.

The LDP on $\bar{P}$ follows at once from Remark 5.1(i) and the observation that $\bar{P}$ contains $\{\, \tilde{y} \in D \mid \tilde{\mathcal{F}}(\tilde{y}) < \infty \,\}$, an immediate consequence Remark 5.1(ii). ■

Now for each $L \geq 1$ we have defined two maps, Φ_L and $\tilde{\Phi}_L$, of Ω into D, obtained from the G process and the Poisson process, respectively. The next lemma relates them by a map to which we can apply the contraction principle.

Lemma 5.3: *There exists a map $f : \bar{P} \to D$ such that*

(a) for every $L \geq 1$, $\Phi_L = f \circ \tilde{\Phi}_L$;

(b) if $\tilde{y} \in \bar{P}$ and $\tilde{\mathcal{F}}(\tilde{y}) < \infty$ then f is continuous at $\tilde{y}$;

(c) if $y \in D$ then $\mathcal{F}(y) = \inf_{\tilde{y}\in f^{-1}(y)} \tilde{\mathcal{F}}(\tilde{y})$.

Proof of Theorem 3.2: The theorem follows immediately from Lemma 5.2, Lemma 5.3, and the contraction principle. ■

Before proving Lemma 5.3 we establish two technical lemmas, for which we need the following definitions. First, if $y \in D$ is lower semicontinuous (which means that $y(x-) \geq y(x)$ for all $x \in (0,1]$) then we define $K_y \subset D$ to be the set of all $\tilde{y}$ such that $\tilde{y}(0) = y(0)$ and such that

$$\phi(x) \equiv y(x) - \tilde{y}(x) \tag{5.2}$$

is nondecreasing on $[0,1]$ and is locally constant on the (open) set $\{\, x \in [0,1] \mid y(x) > 0 \,\}$. Second, we let H be the space of strictly increasing, continuous, and onto functions $h : [0,1] \to [0,1]$, for $h \in H$ set

$$\|h\| = \sup_{x_1 \neq x_2} \left| \log\left(\frac{h(x_1) - h(x_2)}{x_1 - x_2} \right) \right|,$$

and for $y, z \in D$ define

$$d(y,z) = \inf_{h\in H} \max\{\|h\|, \|y - z\circ h\|_\infty\}.$$

Then d is a metric for D and in this metric D is complete [16].

Lemma 5.4: *Suppose that $y_1, y_2 \in D$ are l.s.c., that $\tilde y_i \in K_{y_i}$ for $i = 1, 2$, and that for some $\epsilon > 0$, (i) $y_1, y_2 > -\epsilon$ and (ii) $d(\tilde y_1, \tilde y_2) < \epsilon$. Then $d(y_1, y_2) < 3\epsilon$.*

Proof: By (ii) there is a $h \in H$ with $\|h\| < \epsilon$ and $\|\tilde y_1 - \tilde y_2 \circ h\|_\infty < \epsilon$. We will show below that that $y_1(x) - y_2(h(x)) < 3\epsilon$ for all $x \in [0, 1]$; interchanging the roles of y_1 and y_2 and replacing h by h^{-1} we then have that also $y_2(x) - y_1(h^{-1}(x)) < 3\epsilon$ for all x, so that $y_2(h(x)) - y_1(x) < 3\epsilon$ for all x and hence $\|y_1 - y_2 \circ h\|_\infty < 3\epsilon$.

Now we fix $x \in [0, 1]$ and show that $y_1(x)-y_2(h(x)) < 3\epsilon$. If $y_1 > 0$ on $[0, x]$ then set $x^* = 0$. Otherwise, define $x^* = \sup\{\, t \in [0, x] \mid y_1(t) \le 0\}$; since y_1 is l.s.c. and cadlag, $y_1(x^*) = 0$ in this case. Then from (5.2) and hypotheses (i) and (ii) we have, with $\phi_i = y_i - \tilde y_i$, $i = 1, 2$, as in (5.2),

$$\begin{aligned} y_1(x) - y_2(h(x)) &= \big(y_1(x^*) - y_2(h(x^*))\big) + \big(\tilde y_1(x) - \tilde y_2(h(x))\big) - \big(\tilde y_1(x^*) - \tilde y_2(h(x^*))\big) \\ &\quad + \big(\phi_1(x) - \phi_1(x^*)\big) - \big(\phi_2(h(x)) - \phi_2(h(x^*))\big) \\ &< \epsilon + \epsilon + \epsilon + 0 + 0. \end{aligned} \tag{5.3}$$

Note that the first term in (5.3) is bounded by ϵ from hypothesis (ii) if $x^* = 0$ and from hypothesis (i) otherwise, and that $\phi_1(x) = \phi_1(x^*)$ because if $x > x^*$ then y_1 is strictly positive on $(x^*, x]$ and ϕ_1 is right continuous. ■

Lemma 5.5: *If $y \in D$ is nonnegative and l.s.c., and some $\tilde y \in K_y$ is absolutely continuous, then y is also absolutely continuous.*

Proof: For any $\epsilon > 0$ there is a $\delta > 0$ such that if $\{\, [u_i, v_i] \mid i = 1, \dots, n\,\}$ is a collection of nonoverlapping intervals of total length less than δ then

$$\sum_i |\tilde y(v_i) - \tilde y(u_i)| < \epsilon. \tag{5.4}$$

But then also

$$\sum_i |y(v_i) - y(u_i)| < \epsilon. \tag{5.5}$$

To see this, note that if y does not vanish on $[u_i, v_i]$ then $y(v_i) - y(u_i) = \tilde y(v_i) - \tilde y(u_i)$, and if it does then

$$|y(v_i) - y(u_i)| \le |y(v_i) - y(v_i')| + |y(u_i') - y(u_i)|,$$

where $u_i' = \inf\big(\{y = 0\} \cap [u_i, v_i]\big)$ and $v_i' = \sup\big(\{y = 0\} \cap [u_i, v_i]\big)$. But since $|y(v_i) - y(v_i')| = |\tilde y(v_i) - \tilde y(v_i')|$ and $|y(u_i') - y(u_i)| \le |\tilde y(u_i') - \tilde y(u_i)|$, we may then bound (5.5) by a sum of the form (5.4) for the collection of intervals in which $[u_i, v_i]$ is sometimes replaced by $[u_i, u_i']$ and $[v_i', v_i]$. ■

Proof of Lemma 5.3: The function f is unambiguously defined on P by condition (a). For each $\tilde y \in P$ which has at least one discontinuity is of the form $\tilde\Phi_L(v, \omega)$ for a unique L and unique $v = \tilde y(0)L$, and $\omega \cap (0, L]$ is uniquely determined by $\tilde y$. We may then define $f(\tilde y) = \Phi_L(v, \omega)$, since this is determined by $\omega \cap (0, L]$. The exceptional (continuous) $\tilde y \in P$ are those of the form $\tilde y = z_a$ for some $a \in (-1, \infty)$, where $z_a(x) \equiv a + x$; note that $z_a = \tilde\Phi_L(aL, \omega_L^0)$ for any L such that $aL > -1$, with ω_L^0 any point configuration satisfying $\omega_L^0 \cap (0, L] = \emptyset$. In this case, however, $\Phi_L(v, \omega_L^0) = \tilde y$ for all such L, so that we may define $f(\tilde y) = \tilde y$ and satisfy (a). We remark that it is easy to see that $\tilde y \in K_{f(\tilde y)}$ for all $\tilde y \in P$.

To extend f to $\bar P$ we suppose that $\tilde y \in \bar P \setminus P$ and that $\tilde y = \lim_{n\to\infty} \tilde y_n$ for $\tilde y_n \in \tilde\Phi_{L_n}(\Omega)$. Then necessarily $\lim_{n\to\infty} L_n = \infty$, for otherwise there would exist $L^*, M < \infty$ such that a subsequence of $(\tilde y_n)$ would belong to

$$\bigcup_{1\le L\le L^*} \tilde\Phi_L(\Omega) \cap \{z \in D \mid \|z\|_\infty \le M\} \tag{5.6}.$$

But (5.6) is easily seen to be compact, so that we would have $\tilde{y} \in P$, a contradiction. From this and Lemma 5.4 it follows that the sequence $f(\tilde{y}_n)$ is Cauchy in D and we may thus define $f(\tilde{y})$ by $f(\tilde{y}) = \lim_{n\to\infty} f(\tilde{y}_n)$; it also follows easily that $f(\tilde{y})$ is nonnegative and that f is continuous at $\tilde{y}$. The fact that $L_n \to \infty$ also implies that $\tilde{y}$ and $f(\tilde{y})$ are continuous; thus the convergence of $\tilde{y}_n$ and $f(\tilde{y}_n)$ is uniform [16] and from this and the fact that $\tilde{y} \in K_{f(\tilde{y})}$ for $\tilde{y} \in P$ the same conclusion follows for $\tilde{y} \in \bar{P}$.

The function f satisfies (a) on $\bar{P}$ since it does so on P. Since we know that f is continuous at all points of $\bar{P} \setminus P$ we may verify (b) by showing that f is continuous at points of P at which $\tilde{\mathcal{F}}$ is finite. These points are precisely the functions z_a with $a \ge 0$, and it suffices to show that if $z_a = \lim_{n\to\infty} \tilde{y}_n$ with $\tilde{y}_n \in \tilde{\Phi}_{L_n}(\Omega)$ then $\lim_{n\to\infty} f(\tilde{y}_n) = f(z_a) = z_a$. In fact it suffices to consider separately the cases $L_n \to \infty$ and L_n bounded; the first of these is covered by the argument given above for $y \in \bar{P} \setminus P$, and the second is possible only if, for sufficiently large n, $\tilde{y}_n = z_{a_n}$ with $a_n \to a$; then $f(\tilde{y}_n) = z_{a_n} \to z_a = f(z_a)$.

We have seen that if $y = f(\tilde{y})$ then $\tilde{y} \in K_y$. Conversely, when y and $\tilde{y}$ are absolutely continuous, and y is nonnegative with $y' \le 1$ a.e., we have that if $\tilde{y} \in K_y$ then $y = f(\tilde{y})$. For given $\epsilon > 0$ we may by (5.2) and Remark 5.1(ii) approximate $\tilde{y}$ to within ϵ by a function $\tilde{y}^* \in \Phi_L(\Omega)$ with $L > 1/\epsilon$, and it follows then from Lemma 5.4 that $d(f(\tilde{y}), f(\tilde{y}^*)) < 3\epsilon$ and $d(f(\tilde{y}^*), y) < 3\epsilon$. Note in particular that $y = f(y)$ in this case.

We finally verify property (c). If $y \in D$ is negative at some point then $\mathcal{F}(y) = \infty$; if also $y = f(\tilde{y})$ then either $y = \tilde{y} = z_a$ with $a < 0$ or $\tilde{y}$ is discontinuous, and in either case $\tilde{\mathcal{F}}(\tilde{y}) = \infty$. If y is nonnegative and not absolutely continuous then $\mathcal{F}(y) = \infty$, and by Lemma 5.5 we cannot have $y = f(\tilde{y})$ with $\tilde{\mathcal{F}}(\tilde{y}) < \infty$. Suppose then that y is nonnegative and absolutely continuous. If $y \ne f(\tilde{y})$ for any $\tilde{y}$ then we cannot have $y' \le 1$ a.e., since otherwise $y = f(y)$; thus $\mathcal{F}(y) = \infty$ in this case. Finally, suppose that $y = f(\tilde{y})$ for some $\tilde{y}$ (which implies that $y' \le 1$ a.e.) Then $\mathcal{F}(y) \le \tilde{\mathcal{F}}(\tilde{y})$, for if $\tilde{\mathcal{F}}(\tilde{y}) = \infty$ this is trivial, while if $\tilde{\mathcal{F}}(\tilde{y}) < \infty$ then since $\tilde{y} \in K_y$,

$$\begin{aligned}\tilde{\mathcal{F}}(\tilde{y}) &= \tilde{y}(0)|\log\beta| + \int_0^1 g(1 - \tilde{y}'(x))\,dx \\ &\ge \tilde{y}(0)|\log\beta| + \int_{\{\,y>0\,\}} g(1 - \tilde{y}'(x))\,dx \\ &= \mathcal{F}(y).\end{aligned} \tag{5.7}$$

Thus $\mathcal{F}(y) \le \inf_{\tilde{y}\in f^{-1}(y)} \tilde{\mathcal{F}}(\tilde{y})$. On the other hand, for this y we may define $\tilde{y} = y - \phi$ with $\phi(0) = 0$ and

$$\phi'(x) = \begin{cases} 0, & \text{if } y(x) > 0, \\ \alpha - 1, & \text{if } y(x) = 0. \end{cases} \tag{5.8}$$

Then $\tilde{\mathcal{F}}(\tilde{y}) = \mathcal{F}(y)$ and $f(\tilde{y}) = y$, since $y' \le 1$ a.e. This completes the verification of (c). ■

Remark 5.6: There is another way to complete the proof of Lemma 5.3, after defining f on P as above. If z is a real-valued function of bounded variation we will write $z = z_+ - z_-$ for the canonical decomposition of z as the difference of two increasing functions, so that $dz = dz_+ - dz_-$ is the Hahn decomposition of the signed measure dz as the difference of two nonnegative measures; note that if $\tilde{y} = \tilde{\Phi}_L(v,\omega) \in P$ then $\tilde{y}$ is of bounded variation, with $d\tilde{y}_+ = dx$, $d\tilde{y}_- = \sum_{\{x|Lx\in\omega\}} L^{-1}\delta_x$. Now it is easy to verify that for $\tilde{y} \in P$, $y \equiv f(\tilde{y})$ satisfies the integral equation

$$y(x) = \tilde{y}(0) + \int_0^x [d\tilde{y}_+ - \chi_y d\tilde{y}_-], \tag{5.9a}$$

where

$$\chi_y(x) = \begin{cases} 1, & \text{if } y(x-) > 0, \\ 0, & \text{if } y(x-) \le 0. \end{cases} \tag{5.9b}$$

But one may also show directly that (5.9) has a unique solution y whenever $\tilde{y}$ is continuous, of bounded variation, and satisfies $\tilde{y}(0) \ge 0$. Taking this solution as the definition of $f(\tilde{y}) = y$ when $\tilde{y} \in \bar{P} \setminus P$, one verifies the properties of Lemma 5.3.

Finally, we give the

Proof of Theorem 3.3: From (2.1),

$$E_{L,\epsilon}(\rho) = \Phi_L^{-1}\bigl(B_\epsilon^\infty(J_\rho)\bigr),$$

where J_ρ was defined below (3.9) and B_ϵ^∞ denotes an ϵ-ball in the uniform metric on D. Because each $y \in J_\rho$ is a translate of the continuous function y_ρ, however, there is a $\delta > 0$ such that $B_\delta(J_\rho) \subset B_\epsilon^\infty(J_\rho)$; then by Theorem 3.2,

$$\liminf_{L\to\infty} \frac{1}{L} \log \nu(E_{L,\epsilon}(\rho)) \geq \liminf_{L\to\infty} \frac{1}{L} \log \nu\bigl(\Phi_L^{-1}(B_\delta(J_\rho))\bigr) \geq -\mathcal{F}(y_\rho) = -\mathcal{G}(\rho),$$

and hence

$$\liminf_{\epsilon\to 0} \liminf_{L\to\infty} \frac{1}{L} \log \nu(E_{L,\epsilon}(\rho)) \geq -\mathcal{G}(\rho). \tag{5.10}$$

Conversely, since $B_\epsilon^\infty(J_\rho) \subset B_\epsilon(J_\rho) \subset \overline{B_\epsilon(J_\rho)}$, $\mathcal{F}$ is l.s.c., and for $y \in \overline{B_\epsilon(y_\rho)}$, $\mathcal{F}(y+c) - \mathcal{F}(y_\rho + c)$ is independent of c for $c > \epsilon$,

$$\begin{aligned}
\limsup_{\epsilon\to 0} \limsup_{L\to\infty} \frac{1}{L} \log \nu(E_{L,\epsilon}(\rho)) &\leq \limsup_{\epsilon\to 0} \limsup_{L\to\infty} \frac{1}{L} \log \nu\bigl(\Phi_L^{-1}(\overline{B_\epsilon(J_\rho)})\bigr) \\
&\leq -\liminf_{\epsilon\to 0} \inf_{y\in\overline{B_\epsilon(J_\rho)}} \mathcal{F}(y) \\
&= -\inf_{y\in J_\rho} \mathcal{F}(y) = -\mathcal{G}(\rho).
\end{aligned} \tag{5.11}$$

Equations (5.10) and (5.11) together imply (3.10). ■

Added Note:

After this work was completed we were informed by Neil O'Connell that our large deviation result for the G process could be deduced from the results in the parer of A. A. Puhalski and Ward White, Functional Large Deviation Principles for Waiting and Departure Processes, *Probability in the Engineering and Informational Sciences* **12**, 479–507, 1998. The purpose of that paper, which is part of a series, is "to establish large deviation principles for waiting and departure processes in single-server queues . . . ". The focus of our paper, the statistical mechanics of point processes with bounded variability, is quite different, and should be of independent interest.

Acknowledgments: We thank Ofer Zeitouni for a helpful suggestion and Neil O'Connell for pointing out the connection of the G process to queueing theory. The work of J.L.L. was supported by NSF Grant DMR 01–279–26 and AFOSR Grant AF 49620–01–1–0154, and DIMACS and its supporting agencies. The work of S.G. was supported in part by NSF Grant DMS 0504504. E.R.S. thanks the Institute for Advanced Study, and all the authors thank the Institut des Hautes Etudes Scientifique, for hospitality while parts of this work were carried out.

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Figure Captions

Figure 1. Sample configurations of the G process at density $\rho = 1$ in an interval of size $L = 140$, for $\alpha = 1.0$ (the Poisson process) and $\alpha = 1 + 10^k$, $k = -2, -1, 0, 1, 2$.

Figure 2. Function $y(x)$ minimizing $\mathcal{F}(y)$ subject to $y(1) \leq y(0) - \kappa$, for $\kappa \geq 0$.

Figure 3. Function $y(x)$ minimizing $\mathcal{F}(y)$ subject to $y(1) \geq y(0) - \kappa$, for $\kappa \leq 0$.

Figure 4. Rate functions for the event $N_{(0,L]} \sim (1+\kappa)L$: $\psi(\kappa)$ for the G process (solid line) and $\psi^*(\kappa)$ for the Poisson process (dashed line)

Figure 5. Rate function $\psi_0(\kappa)$ for the event $\rho(x) \sim (1+\kappa)$, $0 \leq x \leq 1$, in the G process.

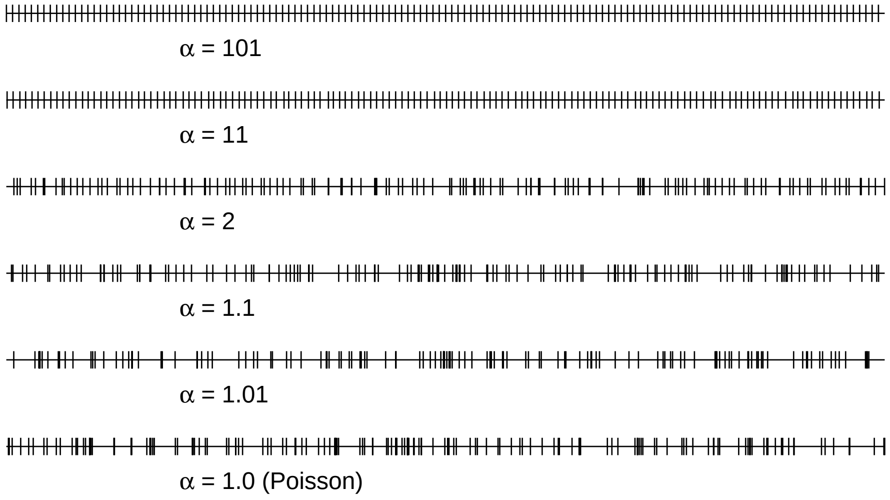
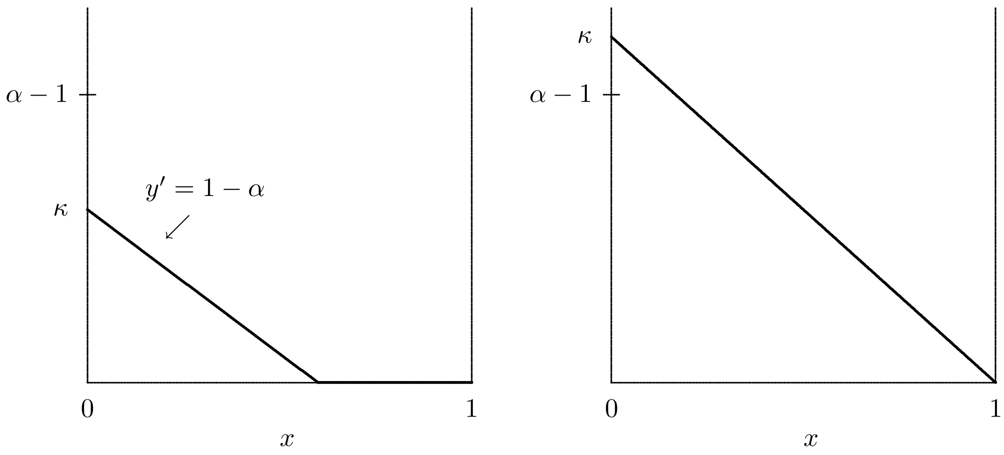
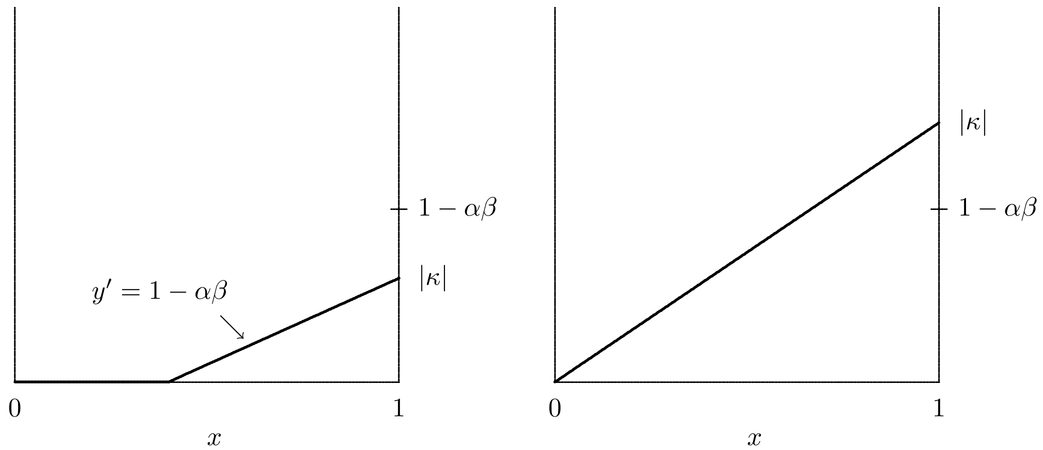


Figure 1



$0 \leq \kappa \leq \alpha - 1$ $\alpha - 1 \leq \kappa$

Figure 2



$1 - \alpha\beta \leq \kappa \leq 0$ $-1 \leq \kappa \leq 1 - \alpha\beta$

Figure 3

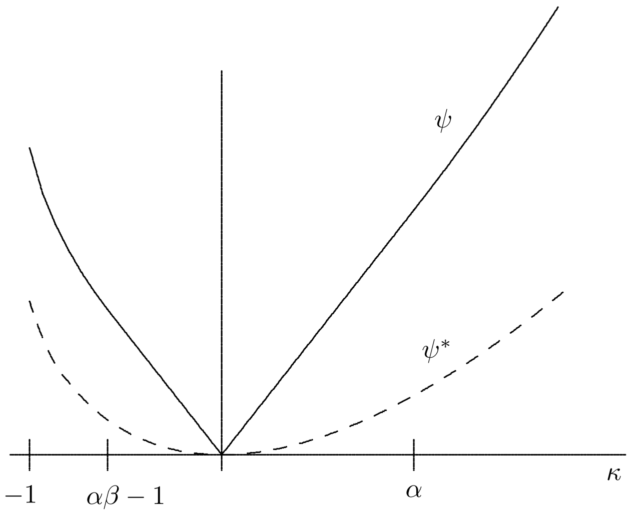


Figure 4

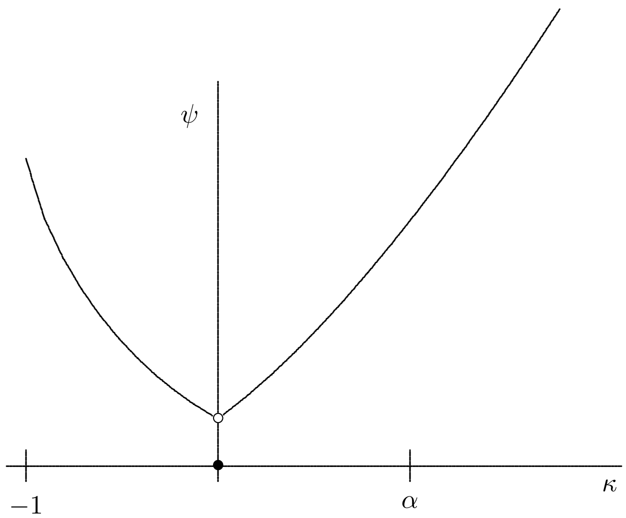


Figure 5