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Monge Property and Bounding Multivariate Probability Distribution Functions with Given Marginals and Covariances

by

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ABSTRACT

Multivariate probability distributions with given marginals are considered, along with linear functionals, to be minimized or maximized, acting on them. The functionals are supposed to satisfy the Monge or inverse Monge or some higher order convexity property and they may be only partially known. Existing results in connection with Monge arrays are reformulated and extended in terms of LP dual feasible bases. Lower and upper bounds are given for the optimum value as well as for unknown coefficients of the objective function based on the knowledge of some dual feasible basis and corresponding objective function coefficients. In the two- and three-dimensional cases dual feasible bases are obtained for the problem, where not only the univariate marginals, but also the covariances of the pairs of random variables are known.

Keywords [1]: Distributions with given marginals, transportation problem, Monge arrays, bounding expectations under partial information.

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1 Introduction

In this paper we consider multivariate discrete probability distributions with given marginals, along with special linear functionals, to be minimized or maximized, acting on them. In other words, we consider transportation problems with special objective functions, where the sum of the marginal values is equal to 1. The latter condition, as not essential, can be dropped in the general theory.

About the objective functions we assume that they enjoy the Monge or inverse Monge or some higher order convexity property.

There is a considerable literature on the Monge property and its use in optimization and other fields of applied mathematics. The papers by Burkard et al. (1995) and Burkard (2004) provide us with an overview about the classical and more recent results. The notion of a discrete higher order convex function was introduced and first studied by the second named author. We elaborate on it in Section 1.1.

The purpose of the paper is the following. First, we reformulate the Monge and inverse Monge properties in terms of dual feasible bases of the transportation problem and obtain further results for them. Secondly, we give lower and upper bounds for the optimum value based on the knowledge of the univariate marginals and the covariances of pairs of bivariate marginals. The results for the latter case concern the two- and three-dimensional transportation problems. Thirdly, we look at partially known objective functions and give lower and upper bounds for entries of the coefficient array. The bounds are based on the knowledge of the univariate marginals in the general, n-dimensional case and on the additional knowledge of the covariances in the two- and three-dimensional cases. In particular, we give lower and upper bounds for the unknown entries of a partially known Monge array.

The linear programming problem:

$$\begin{array}{ll} \min(\max) & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} & \\ & \sum_{j=1}^{n} x_{ij} = a_i, \quad i=1,\dots,m \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j=1,\dots,n \\ & x_{ij} \ge 0, \quad i=1,\dots,m, \quad j=1,\dots,n, \end{array} \tag{1}$$

where

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \quad a_i > 0, \quad i=1,\dots,m, \quad b_j > 0, \quad j=1,\dots,n,$$

and the more general problem:

$$\begin{array}{ll} \min(\max) & \sum_{i_1,\dots,i_d} c(i_1,\dots,i_d)x(i_1,\dots,i_d) \\ \text{subject to} & \\ & \sum_{i_1,\dots,i_d,i_k=i} x(i_1,\dots,i_d) = a_k(i), \\ & \text{for all} \quad i=1,\dots,n_k, \quad k=1,\dots,d \\ & x(i_1,\dots,i_d) \ge 0, \\ & \text{for all} \quad i_k=1,\dots,n_k, \quad k=1,\dots,d, \end{array} \tag{2}$$

where

$$\sum_{i=1}^{n_1} a_1(i) = \cdots = \sum_{i=1}^{n_d} a_d(i), \quad a_k(i) > 0, \quad i = 1, \ldots, n_k, \quad k = 1, \ldots, d$$

are called 2- and d-dimensional transportation problems, respectively.

We can write problems (1) and (2) in the following matrix form:

$$\begin{array}{ll} \min(\max) & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{array} \tag{3}$$

In case of problem (1), we have

$$\begin{aligned} \mathbf{x} &= (x_{11}, \ldots, x_{1n}, \ldots, x_{m1}, \ldots, x_{mn})^T \\ \mathbf{c} &= (c_{11}, \ldots, c_{1n}, \ldots, c_{m1}, \ldots, c_{mn})^T \\ \mathbf{b} &= (a_1, \ldots, a_m, b_1, \ldots, b_n)^T \\ \mathbf{A} &= (\mathbf{a}_{11}, \ldots, \mathbf{a}_{1n}, \ldots, \mathbf{a}_{m1}, \ldots, \mathbf{a}_{mn}), \end{aligned} \tag{4}$$

where

$$\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

and $\mathbf{e}_i$, $\mathbf{e}_{m+j}$ are unit vectors in $\mathbf{E}^{m+n}$ with ones in the i-th and $(m+j)$-th positions, respectively. In case of problem (2),

$$\begin{aligned} \mathbf{x} &= (x(1, \ldots, 1, 1), \ldots, x(1, \ldots, 1, n_d), \ldots, \\ &\quad\ x(n_1, \ldots, n_{d-1}, 1), \ldots, x(n_1, \ldots, n_{d-1}, n_d))^T \\ \mathbf{c} &= (c(1, \ldots, 1, 1), \ldots, c(1, \ldots, 1, n_d), \ldots, \\ &\quad\ c(n_1, \ldots, n_{d-1}, 1), \ldots, c(n_1, \ldots, n_{d-1}, n_d))^T \\ \mathbf{b} &= (a_1(1), \ldots, a_1(n_1), \ldots, a_d(1), \ldots, a_d(n_d))^T \\ \mathbf{A} &= (\mathbf{a}(1, \ldots, 1, 1), \ldots, \mathbf{a}(1, \ldots, 1, n_d), \ldots, \\ &\quad\ \mathbf{a}(n_1, \ldots, n_{d-1}, 1), \ldots, \mathbf{a}(n_1, \ldots, n_{d-1}, n_d)), \end{aligned} \tag{5}$$

where

$$\mathbf{a}(i_1, \ldots, i_d) = \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{n_1+\cdots+n_{d-1}+i_d}, \quad \text{for all} \ \ 1 \leq i_k \leq n_k, \quad 1 \leq k \leq d,$$

and $\mathbf{e}_{i_1}$, $\ldots$, $\mathbf{e}_{n_1+\cdots+n_{d-1}+i_d}$ are unit vectors in $\mathbf{E}^{n_1+\cdots+n_d}$ with ones in the i_1-th, $\ldots$, $(n_1 + \cdots + n_{d-1} + i_d)$-th positions, respectively.

By the definition of $\mathbf{a}_{ij}$ in (4) and $\mathbf{a}(i_1, \ldots, i_d)$ in (5), the constraints in problems (1) and (2) can be written in the following compact forms:

$$\begin{array}{l} \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} \mathbf{a}_{ij} x_{ij} = \mathbf{b} \\ x_{ij} \geq 0, \\ \text{for all} \ \ i = 1, \ldots, m, \quad j = 1, \ldots, n, \end{array} \tag{6}$$

$$\begin{aligned}&\sum_{i_1,\dots,i_d} \mathbf{a}(i_1,\dots,i_d)x(i_1,\dots,i_d) = \mathbf{b}\\&x(i_1,\dots,i_d) \ge 0,\\&\text{for all}\quad i_k = 1,\dots,n_k,\quad k = 1,\dots,d.\end{aligned} \tag{7}$$

The vectors $\mathbf{a}_{ij}$ in problem (6) can be assigned to the cells (i, j) in an $m \times n$ array. Similarly, the vectors $\mathbf{a}_{i_1,\dots,i_d}$ in problem (7) can be assigned to the cells $(i_1, \dots, i_d)$ of an $n_1 \times \cdots \times n_d$ array in the d-space.

If in problem (1) we have the relation $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 1$, then $\{x_{ij}\}$ is a bivariate probability distribution, where its univariate marginals are prescribed to be $\{a_i\}$ and $\{b_j\}$. Problem (1) can then be reformulated in such a way that we minimize or maximize the expectation of $c(X, Y)$, where X, Y are random variables with given distributions, $P(X = i) = a_i$, $i = 1, \dots, m$, $P(Y = j) = b_j$, $j = 1, \dots, n$ and $c(i, j) = c_{ij}$, $i = 1, \dots, m$, $j = 1, \dots, n$ is a given function. Similar is the probabilistic interpretation of problem (2), where there are d random variables $X_1, \dots, X_d$ involved and each has a known probability distribution. Note that we chose the sets $\{1, \dots, m\}$, $\{1, \dots, n\}$ as the supports of X and Y, respectively, but the solution of problem (1) does not depend on the choices of these sets.

In what follows we assume that the sum of each set of marginal values is equal to 1. We are primarily interested in the probabilistic applications. Our results, however, can be generalized in a trivial way for the case, where the sum of the marginal values is not 1.

Sometimes the function $c(i_1, \dots, i_d)$ is known, but the joint distribution of the random variables $X_1, \dots, X_d$ is unknown and our purpose is to give lower and upper bounds for the expectation $E(c(X_1, \dots, X_d))$, i.e., the value of the objective function.

Sometimes, however, some of the $c(i_1, \dots, i_d)$ values are unknown and we want to give lower and upper bounds for them. We provide such bounds based on the knowledge of the univariate marginal distributions and the basic components of the $\mathbf{c}$ function, corresponding to dual feasible bases of the above LP's.

In a linear programming problem of the form (3), where we do not assume that $\mathbf{A}$ has full row rank, a basis $\mathbf{B}$ is called dual feasible in the minimization (maximization) problem if $\mathbf{y}^T\mathbf{A} \le \mathbf{c}^T$ ($\mathbf{y}^T\mathbf{A} \ge \mathbf{c}^T$), where $\mathbf{y}$ is any solution of the equation $\mathbf{y}^T\mathbf{B} = \mathbf{c_B}^T$. The above inequality holds with equality sign if $\mathbf{c}$ is replaced by $\mathbf{c_B}$ and $\mathbf{A}$ is replaced by $\mathbf{A_B}$, i.e., $\mathbf{y}^T\mathbf{A_B} = \mathbf{c_B}^T$, where $\mathbf{c_B}$ and $\mathbf{A_B}$ are those parts of $\mathbf{c}$ and $\mathbf{A}$ which correspond to basic subscripts, respectively. Thus, the knowledge of a dual feasible basis $\mathbf{B}$, together with the corresponding $\mathbf{c_B}$ and $\mathbf{y}$, provide us with lower (upper) bound for the unknown components of the coefficient vector $\mathbf{c}$. At the same time we can obtain lower (upper) bound for the value of the objective function, given that the univariate marginal distributions are known. For a brief introduction to linear programming see Prékopa (1996).

To be able to provide us with the above mentioned bounds some special property has to be assumed in connection with $\mathbf{c}$. Our first assumption is that $\mathbf{c}$ satisfies the Monge property or the inverse Monge property. In the paper we briefly review some of the existing results in connection with Monge arrays, reformulate them in terms of LP dual feasible bases and obtain new results as well. Other problems that we consider in this paper differ from problems (1) and (2) in such a way that, in addition to the constraints that prescribe the univariate marginal distributions, we also prescribe the covariances of pairs of the random variables involved. While in problems (1) and (2) it is unimportant which are the support sets of the random variables, in the problems with covariances these sets play important role. Already the assumption that we impose on the coefficient array of the objective function depends on them. It is a special higher order convexity that we briefly describe in Section 1.1. In addition, one of the constraints of the problem has the elements of the support sets in the coefficients of the variables. We use these problems to give lower and upper bounds for the same values as before, under the modified conditions.

In the further parts of Section 1 we recall some basic notations and facts. Section 2 is devoted to the study of the bivariate and multivariate cases. Existing results in connection with Monge and distribution arrays are reformulated and extended in terms of dual feasibility. Bounds on the expectation and the unknown components of $\mathbf{c}$ are obtained under the condition that the univariate marginals of the random vector are known. In Section 3 the bivariate case is considered, where, in addition to the knowledge of the marginal distributions, we assume the knowledge of the covariance of the two random variables involved. We give bounds for the same values as before. Finally, we present similar results for the three-dimensional case in Sections 4 and 5.

1.1 Multivariate Discrete Higher-order Convexity

Let $f(z), z \in \{z_0, \dots, z_n\}$ be a univariate discrete function, where $z_0, \dots, z_n$ are distinct. Its *divided difference of order* 0, corresponding to z_i, is $f(z_i)$, by definition. The *first-order divided difference* corresponding to z_{i_1}, z_{i_2} is designated and defined by

$$[z_{i_1}, z_{i_2}; f] = \frac{f(z_{i_2}) - f(z_{i_1})}{z_{i_2} - z_{i_1}}, \tag{8}$$

where $z_{i_1} \neq z_{i_2}$. The *kth-order divided difference* is defined recursively (see Popoviciu 1944, Jordan 1965, Prékopa 1998) by

$$[z_{i_1}, z_{i_2}, \dots, z_{i_k}, z_{i_{k+1}}; f] = \frac{[z_{i_2}, \dots, z_{i_{k+1}}; f] - [z_{i_1}, \dots, z_{i_k}; f]}{z_{i_{k+1}} - z_{i_1}}, \tag{9}$$

where $z_{i_1}, \dots, z_{i_{k+1}}$ are pairwise different.

We call the function *kth-order convex* if $z_0 < \cdots < z_n$ and its kth-order divided differences are all nonnegative. First-order convexity means monotonicity; second-order convexity means convexity of the sequence of function values in the traditional sense.

Let $f(z), z \in Z = Z_1 \times \cdots \times Z_d$ be a multivariate discrete function and take the subset

$$Z_{I_1 \dots I_d} = \{z_{1i}, i \in I_1\} \times \cdots \times \{z_{di}, i \in I_d\} = Z_{I_1} \times \cdots \times Z_{I_d}, \tag{10}$$

where $|I_j| = k_j + 1$, $j = 1, \dots, d$. Define the $(k_1, \dots, k_d)$*-order divided difference* of f on the set (10) in an iterative way. First we take the k_1th divided difference with respect to the first variable, then the k_2th divided difference with respect to the second variable etc.. These operations can be executed in any order even in a mixed manner, the result is always the same. Let

$$[z_{1i}, i \in I_1; \dots ; z_{di}, i \in I_d; f] \tag{11}$$

designate the $(k_1, \dots , k_d)$-order divided difference. The sum $k_1 + \cdots + k_d$ is called the *total order* of the divided difference.

We call the function $(k_1, \dots , k_d)$-order convex if all sequences in all $Z_1, \dots, Z_d$ are increasing and all $(k_1, \dots , k_d)$-order divided differences are nonnegative.

1.2 Monge Arrays and Distribution Arrays

An $m \times n$ 2-dimensional array $\mathbf{c} = (c_{ij})$ is called a *Monge array* if it satisfies the *Monge property*:

$$c_{ij} + c_{rs} \le c_{is} + c_{rj} \quad \text{for all} \quad 1 \le i < r \le m, \quad 1 \le j < s \le n. \tag{12}$$

If the inequality in (12) holds strictly, i.e.,

$$c_{ij} + c_{rs} < c_{is} + c_{rj} \quad \text{for all} \quad 1 \le i < r \le m, \quad 1 \le j < s \le n, \tag{13}$$

then it is the *strict Monge property* and $\mathbf{c}$ is called a *strict Monge array.* If the inequalities in (12) and (13) hold in the reverse direction, then it is the *inverse Monge property* and *strict inverse Monge property*, respectively, and $\mathbf{c}$ is called an *inverse Monge array* and *strict inverse Monge array*, respectively. A comprehensive survey on Monge property was presented by Burkard et al.(1996).

If we consider c_{ij} as the value of a function f on (i, j), i.e., $f((i, j)) = c_{ij}$, then the inverse Monge property is equivalent to that the $(1, 1)$-order divided differences are all nonnegative, i.e., f is $(1, 1)$-order convex.

A subclass of Monge arrays can be generated by borrowing an idea from statistics. Let $\mathbf{p}$ be an $m \times n$ array with nonnegative entries p_{lk}, $l = 1, \dots , m$, $k = 1, \dots , n$. Then the array $\mathbf{c} = (c_{ij})$ defined by

$$c_{ij} = \sum_{l=1}^{i} \sum_{k=1}^{j} p_{lk} \quad \text{for all} \quad 1 \le i \le m, \quad 1 \le j \le n \tag{14}$$

is called a *distribution array*, and $\mathbf{p}$ a *density array.* It is easy to check that if $\mathbf{c}$ is a distribution array, then $\mathbf{c}$ is a Monge array and $-\mathbf{c}$ is an inverse Monge array (see Bein et al. 1995). A characterization of 2-dimensional Monge arrays can be given in terms of distribution arrays, first noted by Bein and Pathak. We recall it below.

Theorem 1.1. *(Bein and Pathak 1990) A 2-dimensional array $\mathbf{c} = \{c_{ij}\}$ is a Monge array if and only if there exist a distribution array $\mathbf{d} = \{d_{ij}\}$ and vectors $\mathbf{u} = (u_i)$ and $\mathbf{v} = (v_j)$ such that*

$$c_{ij} = u_i + v_j + d_{ij}.$$

For $d \geq 2$, we define the d-dimensional Monge array due to Aggarwal and Park (1988). An $n_1 \times \cdots \times n_d$ d-dimensional array $\mathbf{c} = \{c(i_1, \ldots, i_d)\}$ has the *Monge property* if for all entries $c(i_1, \ldots, i_d)$ and $c(j_1, \ldots, j_d)$, $1 \leq i_k, j_k \leq n_k$, $1 \leq k \leq d$, we have

$$c(s_1, \ldots, s_d) + c(t_1, \ldots, t_d) \leq c(i_1, \ldots, i_d) + c(j_1, \ldots, j_d), \tag{15}$$

where for all $1 \leq k \leq d$, $s_k = min\{i_k, j_k\}$, $t_k = max\{i_k, j_k\}$. If this inequality holds strictly for all $(s_1, \ldots, s_d) \neq (i_1, \ldots, i_d)$ and $(s_1, \ldots, s_d) \neq (j_1, \ldots, j_d)$, then we say that array $\mathbf{c}$ has the *strict Monge property.* If the inequality in (15) holds in the reverse direction, then it is called the *inverse Monge property* and $\mathbf{c}$ is called an *inverse Monge array.* The definitions of the *strict inverse Monge property* and *strict inverse Monge array* are similar.

1.3 Hoffman's Result

Hoffman's (1963) result for the 2-dimensional minimization transportation problem is stated in terms of Monge sequences, not Monge arrays. Given an $m \times n$ array $\mathbf{c} = (c_{ij})$, an ordering σ of the mn pairs of indices (i, j) of $\mathbf{c}$ is called a *Monge sequence* (with respect to $\mathbf{c}$) if the following conditions hold true:

(i) $1 \leq \sigma(i, j) \leq mn$ for all $1 \leq i \leq m$, $1 \leq j \leq n$,

(ii) $\sigma(i_1, j_1) \neq \sigma(i_2, j_2)$ for any $(i_1, j_1) \neq (i_2, j_2)$,

(iii) whenever $\sigma(i, j) < \sigma(i, s)$ and $\sigma(i, j) < \sigma(r, j)$, for all $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$, we have

$$c_{ij} + c_{rs} \leq c_{is} + c_{rj}. \tag{16}$$

Hoffman proved that the greedy algorithm, or northwest corner rule:

$$\begin{array}{ll} \text{Step 1.} & \text{Set } \; k = 1 \\ \text{Step 2.} & \text{Set } \; x_{ij} = min(a_i, b_j) \; \text{ such that } \; \sigma(i, j) = k \\ \text{Step 3.} & \text{Replace } \; a_i \; \text{ by } \; a_i - x_{ij}. \text{ Replace } \; b_j \; \text{ by } \; b_j - x_{ij} \\ \text{Step 4.} & \text{If } \; k = mn, \; \text{ stop. Otherwise, replace } \; k \; \text{ by } \; k+1 \\ & \text{and go to Step 2} \end{array} \tag{17}$$

solves the 2-dimensional minimization transportation problem if and only if σ is a Monge sequence with respect to the cost array $\mathbf{c}$.

Note that $\mathbf{c}$ is a Monge array if and only if $\sigma(i, j) = (i - 1)n + j$ is a Monge sequence with respect to $\mathbf{c}$.

1.4 Dual Feasible Basis

In what follows, we use the notations and definitions in Prékopa (1996). A *basis* of the columns of a non-zero matrix $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is a collection of linearly independent vectors $\{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r}\}$ such that all column vectors of $\mathbf{A}$ can be represented as linear combinations of these vectors. Assuming $i_1 < \cdots < i_r$, we also call the submatrix $\mathbf{B} = (\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_r})$ a basis and a basis of $\mathbf{A}$. Let I designate the set of subscripts of the vectors in a basis $\mathbf{B}$ of $\mathbf{A}$ and $K = \{1, \ldots, n\} - I$. Then the column vectors $\mathbf{a}_j$, $1 \le j \le n$ can be represented as

$$\mathbf{a}_j = \sum_{i \in I} d_{ij} \mathbf{a}_i.$$

Define $\mathbf{z}_j = \sum_{i \in I} d_{ij} \mathbf{c}_i$. A basis $\mathbf{B}$ is said to be *dual feasible* in the minimization (maximization) problem (3) if $z_j - c_j \le 0$ for all $j \in K$ ($z_j - c_j \ge 0$ for all $j \in K$). The differences $z_j - c_j$ are called relative costs.

1.5 Ordered and Inverse Ordered Sequences

We introduce ordered and inverse ordered subsets for 2-dimensional index sets. Given an $m \times n$ 2-dimensional index set $S = \{(i,j) | 1 \le i \le m, 1 \le j \le n\}$, we call a subset I of S an *ordered subset* if it has the following form:

$$I = \{(1, j_0), \ldots, (1, j_1), (2, j_1), \ldots, (2, j_2), \ldots, (m, j_{m-1}), \ldots, (m, j_m)\}, \tag{18}$$

where $1 = j_0 \le j_1 \le j_2 \le \cdots \le j_{m-1} \le j_m = n$. We call a subset J of S an *inverse ordered subset* if it has the following form:

$$\begin{aligned} J = \{&(1, n-j_0), \ldots, (1, n-j_1), (2, n-j_1), \ldots, (2, n-j_2), \\ &\ldots, (m, n-j_{m-1}), \ldots, (m, n-j_m)\}, \end{aligned} \tag{19}$$

where $0 = j_0 \le j_1 \le j_2 \le \cdots \le j_{m-1} \le j_m = n - 1$.

For an ordered subset I or inverse ordered subset J of S, the collection $\mathbf{B}$ of vectors $\mathbf{a}_{ij}$, $(i,j) \in I$ or $(i,j) \in J$ of the matrix $\mathbf{A}$ in (4) is called an *ordered sequence* or *inverse ordered sequence* of $\mathbf{A}$.

More generally, we define the ordered subset for d-dimensional ($d > 2$) index sets. We call a subset I of the $n_1 \times \cdots \times n_d$ d-dimensional index set $S = \{(i_1, \ldots, i_d) | 1 \le i_k \le n_k, 1 \le k \le d\}$ an *ordered subset* if it has the following form:

$$\begin{aligned} I \;=\; \{&(1, \ldots, 1, 1), \ldots, (1, \ldots, 1, i_{d,1}), \ldots, (1, \ldots, i_{d-1,1}, i_{d,1}), \\ &\ldots, (i_{1,1}, \ldots, i_{d-1,1}, i_{d,1}), \ldots, (i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n-1}), \\ &\ldots, (i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n}), \ldots, (i_{1,n-1}, \ldots, i_{d-1,n}, i_{d,n}), \\ &\ldots, (i_{1,n}, \ldots, i_{d-1,n}, i_{d,n})\}, \end{aligned} \tag{20}$$

where $1 \le i_{k,1} \le \cdots \le i_{k,n-1} \le i_{k,n} = n_k$, for all $1 \le k \le d$. Similarly, for an ordered subset I, the collection $\mathbf{B}$ of vectors $\mathbf{a}(i_1, \ldots, i_d)$, $(i_1, \ldots, i_d) \in I$ of the matrix $\mathbf{A}$ in (5) is called an *ordered sequence* of $\mathbf{A}$.

We have the following theorem for 2-dimensional ordered and inverse ordered sequences and d-dimensional ordered sequence.

Let us assign the cell (i, j), in the $m \times n$ transportation tableau, to the column a_{ij} in the matrix $\mathbf{A}$. Any collection of cells is called a *cell graph.* A cell graph is a *circuit* if its cells can be connected by a closed sequence of lines going alternately horizontally and vertically. A cell graph is a *tree* if there is no circuit in it. A tree that has $m + n - 1$ cells in it is called a *spanning tree.* The following assertions hold true (see, e.g., Hadley, 1963): A collection of vectors a_{ij}, $(i, j) \in I$ is a basis of $\mathbf{A}$ iff I is a spanning tree. Let $I = \{(i_1, j_1), (i_2, j_1), (i_2, j_2), (i_3, j_2), \ldots, (i_{n-1}, j_n), (i_1, j_n)\}$ be a circuit. Then we have the relation

$$a_{i_1 j_1} - a_{i_2 j_1} + a_{i_2 j_2} - a_{i_3 j_2} + \cdots + a_{i_{n-1} j_n} - a_{i_1 j_n} = 0. \tag{21}$$

If I is a spanning tree and $(p, q) \notin I$, then there exists a subset J such that $J \cup \{(p, q)\}$ is a circuit. In view of (21) we can obtain a unique linear combination of a_{pq}, by the use of the vectors of a basis a_{ij}, $(i, j) \in I$, in such a way that we add the vectors corresponding to J along the circuit using alternately $+1$, -1 as coefficients.

Theorem 1.2. *A 2-dimensional ordered or inverse ordered sequence of the matrix* $\mathbf{A}$ *forms a basis of* $\mathbf{A}$.

Proof. Any ordered sequence is a spanning tree hence the assertion follows. □

Theorem 1.3. *A d-dimensional ordered sequence* $\mathbf{B}$ *of the matrix* $\mathbf{A}$ *in (5) forms a basis of* $\mathbf{A}$.

Proof. It is easy to see that the rank of the matrix in (5) is $n_1 + \cdots + n_d - (d - 1)$. The proof goes along the same way as the well-known proof for the case of $d = 2$.

Next, we prove that the vectors in $\mathbf{B}$ are linearly independent. Suppose that for some λ numbers we have the equation:

$$\begin{aligned}
&\lambda(1, \ldots, 1, 1)\mathbf{a}(1, \ldots, 1, 1) + \cdots + \lambda(1, \ldots, 1, i_{d,1})\mathbf{a}(1, \ldots, 1, i_{d,1}) \\
&+ \cdots + \lambda(1, \ldots, i_{d-1,1}, i_{d,1})\mathbf{a}(1, \ldots, i_{d-1,1}, i_{d,1}) \\
&+ \cdots + \lambda(i_{1,1}, \ldots, i_{d-1,1}, i_{d,1})\mathbf{a}(i_{1,1}, \ldots, i_{d-1,1}, i_{d,1}) \\
&+ \cdots + \lambda(i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n-1})\mathbf{a}(i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n-1}) \\
&+ \cdots + \lambda(i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n})\mathbf{a}(i_{1,n-1}, \ldots, i_{d-1,n-1}, i_{d,n}) \\
&+ \cdots + \lambda(i_{1,n-1}, \ldots, i_{d-1,n}, i_{d,n})\mathbf{a}(i_{1,n-1}, \ldots, i_{d-1,n}, i_{d,n}) \\
&+ \cdots + \lambda(i_{1,n}, \ldots, i_{d-1,n}, i_{d,n})\mathbf{a}(i_{1,n}, \ldots, i_{d-1,n}, i_{d,n}) = \mathbf{0}.
\end{aligned}$$

In view of the structure of $\mathbf{a}(i_1, \ldots, i_d)$, the left hand side of the above equation can be written as a linear combination of the unit vectors in $\mathbf{E}^{n_1 + \cdots + n_d}$. Then the coefficient of each unit vector should be equal to 0. If $i_{d,1} > 1$, then since the coefficients of $e_{n_1 + \cdots + n_{d-1} + 1}$, …, $e_{n_1 + \cdots + n_{d-1} + i_{d,1} - 1}$ are $\lambda(1, \ldots, 1, 1)$, …, $\lambda(1, \ldots, 1, i_{d,1} - 1)$, respectively, it follows that

$$\lambda(1, \ldots, 1, 1) = \cdots = \lambda(1, \ldots, 1, i_{d,1} - 1) = 0. \tag{22}$$

Otherwise, $\mathbf{a}(1, \ldots, 1, 1) = \cdots = \mathbf{a}(1, \ldots, 1, i_{d,1})$, which implies (22) too. Similarly, considering the value of $i_{d-1,1}$, we obtain $\lambda(1, \ldots, 1, i_{d,1}) = \cdots = \lambda(1, \ldots, i_{d-1,1} - 1, i_{d,1}) = 0$, and so on. So all λ values are zero, and the vectors in $\mathbf{B}$ are linearly independent.

Thus $\mathbf{B}$ forms a basis of the matrix $\mathbf{A}$. □

2 Monge Property and Dual Feasible Bases

In this section we establish relationship between ordered sequences and dual feasible bases in the 2- and d-dimensional minimization problems (1) and (2), and between the inverse ordered sequences and the dual feasible bases in the 2-dimensional maximization problem (1).

The dual of the 2-dimensional minimization problem (1) is:

$$\begin{array}{rl} \max & \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\ \text{subject to} & \\ & u_i + v_j \leq c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{array} \tag{23}$$

Theorem 2.1. *In the minimization problem (1), any ordered sequence of the matrix* **A** *forms a dual feasible basis if and only if the cost array* **c** *satisfies the Monge property.*

Proof. First we prove that if **c** satisfies the Monge property then any ordered sequence $\mathbf{a}_{ij}, (i,j) \in I$ forms a dual feasible basis of **A**.

Given a cell $(p,q) \notin I$, we form the unique circuit $J \cup \{(p,q)\}$, where $J \subset I$. Assume that the cell (p,q) is above the ordered sequence in the transportation tableau. Then in $J \cup \{(p,q)\} = \{(p,q), (p,j_1), (i_1,j_1), (i_1,j_2), \ldots,$
$(i_h,j_h), (i_h,q)\}$, we have the relations

$$\begin{array}{rl} p > & i_1 > \cdots > i_h \\ & j_1 > \cdots > j_h > q. \end{array} \tag{24}$$

We want to prove that the reduced cost $z_{pq} - c_{pq} \leq 0$. Relations (24) and the Monge property imply:

$$\begin{array}{rcl} c_{pq} - c_{pj_1} + c_{i_1 j_1} - c_{i_1 q} & \geq & 0 \\ c_{i_1 q} - c_{i_1 j_2} + c_{i_2 j_2} - c_{i_2 q} & \geq & 0 \\ & \vdots & \\ c_{i_{h-2} q} - c_{i_{h-2} j_{h-1}} + c_{i_{h-1} j_{h-1}} - c_{i_{h-1} q} & \geq & 0 \\ c_{i_{h-1} q} - c_{i_{h-1} j_h} + c_{i_h j_h} - c_{i_h q} & \geq & 0. \end{array}$$

If we add these relations, then $c_{i_1 q}, \ldots, c_{i_{h-1} q}$ cancel and the sum equals $c_{pq} - z_{pq}$. Thus, $c_{pq} - z_{pq} \geq 0$.

Similar is the proof if (p,q) is below the ordered sequence of the transportation tableau. Since $c_{pq} - z_{pq} \geq 0$ holds for every (p,q) (with equality, if (p,q) is basic), the basis is dual feasible.

To prove the other part of the theorem, suppose that any ordered sequence forms a dual feasible basis. Assume that the cost array **c** does not satisfy the Monge property. Then there exist $1 \leq i < r \leq m$ and $1 \leq j < s \leq n$ such that $c_{ij} + c_{rs} > c_{is} + c_{rj}$. Let B be an ordered sequence which contains $\mathbf{a}_{ij}$ and $\mathbf{a}_{rs}$. By the known condition, B is a dual feasible basis. Let

u_i, $i = 1, \ldots, m$, v_j, $j = 1, \ldots, n$ be the components of the basic solution corresponding to this dual feasible basis. We have the relations:

$$\begin{aligned} c_{ij} &= u_i + v_j, \\ c_{rs} &= u_r + v_s, \\ c_{is} &\geq u_i + v_s, \\ c_{rj} &\geq u_r + v_j. \end{aligned}$$

This implies that

$$c_{is} + c_{rj} \geq u_i + v_s + u_r + v_j = c_{ij} + c_{rs}$$

which contradicts the assumption. □

Since Hoffman's greedy algorithm gives us an ordered sequence, by Theorem 2.1, a dual feasible basis can be obtained by the use of Hoffman's greedy algorithm, i.e., Hoffman's greedy algorithm solves the minimization problem (1) if and only if the cost array is Monge. So, Hoffman's theorem can be seen as a consequence of Theorem 2.1.

Theorem 2.2. *In the minimization problem (1), any dual feasible basis of the matrix* $\mathbf{A}$ *forms an ordered sequence of* $\mathbf{A}$ *if the cost array* $\mathbf{c}$ *satisfies the strict Monge property.*

Proof. Suppose that the cost array $\mathbf{c}$ satisfies the strict Monge property, and $\mathbf{B}$ is a dual feasible basis. Assume that the vectors of $\mathbf{B}$ do not form an ordered sequence of $\mathbf{A}$. Then there must exist vectors $\mathbf{a}_{is}$ and $\mathbf{a}_{rj}$ in $\mathbf{B}$ with $1 \leq i < r \leq m$, $1 \leq j < s \leq n$. Let u_i, $i = 1, \ldots, m$, v_j, $j = 1, \ldots, n$ be the components of the basic solution corresponding to $\mathbf{B}$ in problem (23). This basic solution is feasible, hence

$$\begin{aligned} c_{is} &= u_i + v_s \\ c_{rj} &= u_r + v_j \\ c_{ij} &\geq u_i + v_j \\ c_{rs} &\geq u_r + v_s. \end{aligned}$$

It follows that

$$c_{ij} + c_{rs} \geq u_i + v_j + u_r + v_s = c_{is} + c_{rj}$$

which contradicts the strict Monge property. □

By proofs similar to those of Theorems 2.2 and 2.3, we can obtain the following theorems for the maximization problem (1):

Theorem 2.3. *In the maximization problem (1), any inverse ordered sequence of the matrix* $\mathbf{A}$ *forms a dual feasible basis if and only if the cost array* $\mathbf{c}$ *satisfies the Monge property.*

Theorem 2.4. *In the maximization problem (1), any dual feasible basis of the matrix* $\mathbf{A}$ *forms an inverse ordered sequence if the cost array* $\mathbf{c}$ *satisfies the strict Monge property.*

So, if the cost array of problem (1) satisfies the Monge property, then an ordered sequence and an inverse ordered sequence of the matrix **A** will give us the optimal minimum and maximum values, respectively.

Let $c'_{ij} = c_{m-i+1,j}$, $a'_{ij} = a_{m-i+1,j}$, A cost array $\mathbf{c} = (c_{ij})$ is an inverse Monge array if and only if $\mathbf{c}' = (c'_{ij}) = (c_{(m-i+1)j})$ is a Monge array, a collection of vectors of $\mathbf{A} = (\mathbf{a}_{ij})$ is an inverse ordered sequence of **A** if and only if it is an ordered sequence of $\mathbf{A}' = (\mathbf{a}'_{ij}) = (\mathbf{a}_{(m-i+1)j})$. ¿From these properties and theorems 2.1, 2.2, 2.3, 2.4, we can easily get the following four corollaries for the inverse Monge property:

Corollary 2.1. *In the minimization problem (1), any inverse ordered sequence of the matrix* **A** *forms a dual feasible basis if and only if the cost array satisfies the inverse Monge property.*

Corollary 2.2. *In the minimization problem (1), any dual feasible basis of the matrix* **A** *forms an inverse ordered sequence if the cost array satisfies the strict inverse Monge property.*

Corollary 2.3. *In the maximization problem (1), any ordered sequence of the matrix* **A** *forms a dual feasible basis if and only if the cost array satisfies the inverse Monge property.*

Corollary 2.4. *In the maximization problem (1), any dual feasible basis of the matrix* **A** *forms an ordered sequence if the cost array satisfies the strict inverse Monge property.*

So if the cost array of problem (1) satisfies the inverse Monge property, then an inverse ordered sequence and an ordered sequence of the matrix **A** will give us the optimal minimum and maximum values, respectively.

We summarize all the above results for the minimization (maximization) problem (1) in the following table:

Min (Max)	
Any o.s. (i.o.s.) forms a d.f.b.	*iff* **c** *is* Monge
Any i.o.s. (o.s.) forms a d.f.b.	*iff* **c** *is* inverse Monge
Any d.f.b. forms an o.s. (i.o.s.)	*if* **c** *is* strict Monge
Any d.f.b. forms an i.o.s. (o.s.)	*if* **c** *is* strict inverse Monge

Table 2.1

Here "o.s." means ordered sequence, "i.o.s." means inverse ordered sequence, and "d.f.b." means dual feasible basis.

Next, we present the relationship between ordered sequences and the dual feasible bases in the d-dimensional problem (2) ($d > 2$). So, the vectors **x**, **c**, **b** and the matrix **A**, we use in the rest of this section, are those in (5).

To prove our results we recall three theorems, where the first two are well-known in linear programming (see, Prékopa, 1996, Theorem 5 and 3).

Theorem 2.5. *If problem (3) has a primal feasible solution and a finite optimum, then there exists a primal feasible basis that is also dual feasible.*

Theorem 2.6. *If in problem (3) B is a primal feasible and nondegenerate basis (i.e., the $\mathbf{x_B}$, determined by $\mathbf{Bx_B} = \mathbf{b}$, has all positive components) and $\mathbf{B}$ is optimal, i.e., $(\mathbf{x_B}, \mathbf{x_R})$ is an optimal solution ($\mathbf{x_R} = \mathbf{0}$), then $\mathbf{B}$ is a dual feasible basis.*

Bein et al (1995) extended the algorithm $GREEDY_2$ to the algorithm $GREEDY_d$ and proved the following theorem:.

Theorem 2.7. *(Bein et al. 1995) Given a particular $n_1 \times n_2 \times \cdots \times n_d$ d-dimensional cost array $\mathbf{c}$, the algorithm $GREEDY_d$ solves the corresponding d-dimensional transportation problem for any $\mathbf{b}$ if and only if $\mathbf{c}$ is Monge.*

For d-dimensional minimization problem (2), we prove two theorems.

Theorem 2.8. *In the minimization problem (2), any ordered sequence of the matrix $\mathbf{A}$ forms a dual feasible basis if and only if the cost array $\mathbf{c}$ satisfies the Monge property.*

Proof. For the proof of the "if" direction, assume that $\mathbf{c}$ is Monge. For any given ordered sequence, write positive numbers in the cells of it and what comes out on the r.h.s., let it be the $\mathbf{b}$. For this $\mathbf{b}$, the algorithm $GREEDY_d$ produces the same ordered sequence, by Theorem 2.7, this ordered sequence is a primal non-degenerate, optimal basis to the problem. By Theorem 2.6, this ordered sequence is dual feasible.

For the proof of the "only if" direction, assume that any ordered sequence of matrix $\mathbf{A}$ forms a dual feasible basis. Then for any $\mathbf{b}$ the algorithm $GREEDY_d$ solves the problem optimally, because the algorithm $GREEDY_d$ produces an ordered sequence. Thus, by Theorem 2.7, $\mathbf{c}$ is Monge. $\square$

To prove the next theorem, we need the dual of the minimization problem (2) which is given as follows:

$$
\begin{array}{rl}
\max & \sum_{k=1}^{d} \sum_{i_k=1}^{n_k} a_k(i_k) w_k(i_k) \\
\text{subject to} & \\
& w_1(i_1) + \cdots + w_d(i_d) \le c(i_1, \ldots, i_d) \\
& \text{for all } \; i_k = 1, \ldots, n_k, \quad k = 1, \ldots, d.
\end{array} \tag{25}
$$

Theorem 2.9. *In the minimization problem (2), any dual feasible basis of the matrix $\mathbf{A}$ forms an ordered sequence of $\mathbf{A}$ if the cost array $\mathbf{c}$ satisfies the strict Monge property.*

Proof. Suppose that the cost array satisfies the strict Monge property, and $\mathbf{B}$ is a dual feasible basis of matrix $\mathbf{A}$. Assume that the vectors of $\mathbf{B}$ do not form an ordered sequence, then there must exist vectors $\mathbf{a}(i_1, \ldots, i_d)$ and $\mathbf{a}(j_1, \ldots, j_d)$ in $\mathbf{B}$ such that if $s_k = min\{i_k, j_k\}$,

$t_k = max\{i_k, j_k\}$, then $(s_1, \ldots, s_d) \neq (i_1, \ldots, i_d)$, $(s_1, \ldots, s_d) \neq (j_1, \ldots, j_d)$. Let $w_k(i_k)$, $i_k = 1, \ldots, n_k$, $k = 1, \ldots, d$, be the components of dual vector corresponding to **B**. Then

$$\begin{aligned} c(i_1, \ldots, i_d) &= w_1(i_1) + \cdots + w_d(i_d), \\ c(j_1, \ldots, j_d) &= w_1(j_1) + \cdots + w_d(j_d), \\ c(s_1, \ldots, s_d) &\geq w_1(s_1) + \cdots + w_d(s_d), \\ c(t_1, \ldots, t_d) &\geq w_1(t_1) + \cdots + w_d(t_d). \end{aligned}$$

Since

$$\begin{aligned} & w_1(i_1) + \cdots + w_d(i_d) + w_1(j_1) + \cdots + w_d(j_d) \\ & \qquad = w_1(s_1) + \cdots + w_d(s_d) + w_1(t_1) + \cdots + w_d(t_d), \end{aligned} \tag{26}$$

we have the relation

$$c(s_1, \ldots, s_d) + c(t_1, \ldots, t_d) \geq c(i_1, \ldots, i_d) + c(j_1, \ldots, j_d).$$

It contradicts to the strict Monge property, so any dual feasible basis of the matrix **A** must be an ordered sequence of **A**. □

3 The Use of Covariance in The Two-Dimensional Case

In this section we consider the problem:

$$\begin{array}{ll} \text{min(max)} & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{subject to} & \\ & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \\ & \sum_{i=1}^{m} \sum_{j=1}^{n} y_i z_j x_{ij} = c \\ & x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{array} \tag{27}$$

The first $m + n$ constraints in problem (27) prescribe that the univariate marginals are $\{a_i\}$, $\{b_j\}$. Since the univariate marginals determine the univariate expectations, the last constraint in (27) prescribes that the covariance between the two random variables is also given. Assume that c_{ij} is a function of y_i, z_j: $c_{ij} = g(y_i, z_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Problem (27) can be written in the following matrix form:

$$\begin{array}{ll} \max & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \\ & \overline{\mathbf{A}} \mathbf{x} = \overline{\mathbf{b}} \\ & \mathbf{x} \geq 0, \end{array} \tag{28}$$

where $\overline{\mathbf{A}} = (\overline{\mathbf{a}}_{ij})$, $\overline{\mathbf{a}}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j} + y_i z_j \mathbf{e}_{m+n+1}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $\mathbf{e}_i$, $\mathbf{e}_{m+j}$ and $\mathbf{e}_{m+n+1}$ are unit vectors in $\mathbf{E}^{m+n+1}$ with ones in the i-th, $(m+j)$-th and $(m+n+1)$-th positions, respectively, $\overline{\mathbf{b}} = \begin{pmatrix} \mathbf{b} \\ c \end{pmatrix}$, and $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$ are defined in (4).

Theorem 3.1. *Consider the minimization problem (27), and assume that $\{y_i\}$ is strictly increasing and $\{z_j\}$ is strictly decreasing, and the (1,2)-order and (2,1)-order divided differences of $g(y_i, z_j)$ are nonnegative. Then $B_1 = \{\overline{\mathbf{a}}_{i1}, i = 1, \ldots, m, \overline{\mathbf{a}}_{mj}, j = 2, \ldots, n, \overline{\mathbf{a}}_{1n}\}$ forms a dual feasible basis of $\overline{\mathbf{A}}$.*

Proof. First, let us show that B_1 forms a basis of $\overline{\mathbf{A}}$. It is easy to see that the rank of $\overline{\mathbf{A}}$ is $m+n$, and there are $m+n$ vectors in B_1. To show the linear independence of the $m+n$ vectors in B_1, consider the linear combination of the vectors of B_1:

$$\begin{aligned}
& \sum_{i=1}^{m} \lambda_{i1} \overline{\mathbf{a}}_{i1} + \sum_{j=2}^{n} \lambda_{mj} \overline{\mathbf{a}}_{mj} + \lambda_{1n} \overline{\mathbf{a}}_{1n} \\
= \; & \sum_{i=1}^{m} \lambda_{i1} (\mathbf{e}_i + \mathbf{e}_{m+1} + y_i z_1 \mathbf{e}_{m+n+1}) + \sum_{j=2}^{n} \lambda_{mj} (\mathbf{e}_m + \mathbf{e}_{m+j} + y_m z_j \mathbf{e}_{m+n+1}) \\
& + \lambda_{1n} (\mathbf{e}_1 + \mathbf{e}_{m+n} + y_1 z_n \mathbf{e}_{m+n+1}) \\
= \; & (\lambda_{11} + \lambda_{1n}) \mathbf{e}_1 + \sum_{i=2}^{m-1} \lambda_{i1} \mathbf{e}_i + (\sum_{j=1}^{n} \lambda_{mj}) \mathbf{e}_m + (\sum_{i=1}^{m} \lambda_{i1}) \mathbf{e}_{m+1} + \sum_{j=2}^{n-1} \lambda_{mj} \mathbf{e}_{m+j} \\
& + (\lambda_{mn} + \lambda_{1n}) \mathbf{e}_{m+n} + (\sum_{i=1}^{m} \lambda_{i1} y_i z_1 + \sum_{j=2}^{n} \lambda_{mj} y_m z_j + \lambda_{1n} y_1 z_n) \mathbf{e}_{m+n+1}.
\end{aligned}$$

If it equals 0, then, by the linear independence of the unit vectors, it follows that all λ's must be 0.

Secondly let us show that this basis is dual feasible. For any nonbasic vector $\overline{\mathbf{a}}_{ij}$, $1 \le i < m$, $1 < j \le n$, we have the equations:

$$\begin{gathered}
\overline{\mathbf{a}}_{ij} - \overline{\mathbf{a}}_{i1} + \overline{\mathbf{a}}_{m1} - \overline{\mathbf{a}}_{mj} = (y_i - y_m)(z_j - z_1) \mathbf{e}_{m+n+1} \\
\overline{\mathbf{a}}_{ij} - \overline{\mathbf{a}}_{i1} + \overline{\mathbf{a}}_{11} - \overline{\mathbf{a}}_{1n} + \overline{\mathbf{a}}_{mn} - \overline{\mathbf{a}}_{mj} \\
= [(y_i - y_m)(z_j - z_1) - (y_1 - y_m)(z_n - z_1)] \mathbf{e}_{m+n+1}.
\end{gathered}$$

¿From here we derive the expression of $\overline{\mathbf{a}}_{\mathbf{ij}}$ as the following linear combination of the basic vectors:

$$\begin{aligned}
\overline{\mathbf{a}}_{ij} \; = \; & \frac{(y_1 - y_m)(z_1 - z_n) - (y_i - y_m)(z_1 - z_j)}{(y_1 - y_m)(z_1 - z_n)} (\overline{\mathbf{a}}_{i1} - \overline{\mathbf{a}}_{m1} + \overline{\mathbf{a}}_{mj}) \\
- \; & \frac{(y_i - y_m)(z_1 - z_j)}{(y_1 - y_m)(z_1 - z_n)} (\overline{\mathbf{a}}_{11} - \overline{\mathbf{a}}_{1n} + \overline{\mathbf{a}}_{mn} - \overline{\mathbf{a}}_{mj} - \overline{\mathbf{a}}_{i1}).
\end{aligned}$$

We have to prove that

$$\frac{(y_1 - y_m)(z_1 - z_n) - (y_i - y_m)(z_1 - z_j)}{(y_1 - y_m)(z_1 - z_n)}(c_{i1} - c_{m1} + c_{mj})$$
$$- \quad \frac{(y_i - y_m)(z_1 - z_j)}{(y_1 - y_m)(z_1 - z_n)}(c_{11} - c_{1n} + c_{mn} - c_{mj} - c_{i1}) - c_{ij} \le 0,$$

or, what is the same,

$$(c_{i1} - c_{m1} + c_{mj} - c_{ij}) \le \frac{(y_i - y_m)(z_1 - z_j)}{(y_1 - y_m)(z_1 - z_n)}(c_{11} - c_{m1} + c_{mn} - c_{1n}). \tag{29}$$

Since $(y_i - y_m)(z_1 - z_j) < 0$, the above inequality is equivalent to

$$\frac{c_{i1} - c_{m1} - c_{ij} + c_{mj}}{(y_i - y_m)(z_1 - z_j)} \ge \frac{c_{11} - c_{m1} - c_{1n} + c_{mn}}{(y_1 - y_m)(z_1 - z_n)}.$$

We have assumed that the (1,2)-order and (2,1)-order divided differences of $g(y_i, z_j)$ are nonnegative. The nonnegativity of the (1,2)-order divided difference implies

$$\frac{\frac{c_{11}-c_{m1}-c_{1n}+c_{mn}}{(y_1-y_m)(z_1-z_n)} - \frac{c_{11}-c_{m1}-c_{1j}+c_{mj}}{(y_1-y_m)(z_1-z_j)}}{z_n - z_j} \ge 0.$$

Similarly, the nonnegativity of (2,1)-order divided difference gives

$$\frac{\frac{c_{11}-c_{m1}-c_{1j}+c_{mj}}{(y_1-y_m)(z_1-z_j)} - \frac{c_{i1}-c_{m1}-c_{ij}+c_{mj}}{(y_i-y_m)(z_1-z_j)}}{y_1 - y_i} \ge 0.$$

Since both $z_n - z_j$ and $y_1 - y_i$ are negative, the above two inequalities imply

$$\frac{c_{i1} - c_{m1} - c_{ij} + c_{mj}}{(y_i - y_m)(z_1 - z_j)} \ge \frac{c_{11} - c_{m1} - c_{1j} + c_{mj}}{(y_1 - y_m)(z_1 - z_j)} \ge \frac{c_{11} - c_{m1} - c_{1n} + c_{mn}}{(y_1 - y_m)(z_1 - z_n)}.$$

This completes the proof. □

The following corollaries follow at once from Theorem 3.1.

Corollary 3.1. *Consider the minimization problem (27), and assume that $\{y_i\}$ is strictly decreasing and $\{z_j\}$ is strictly increasing, and the (1,2)-order and (2,1)-order divided differences of $g(y_i, z_j)$ are nonpositive. Then B_1 forms a dual feasible basis of $\overline{\mathbf{A}}$.*

Corollary 3.2. *Consider the minimization problem (27), and assume that both $\{y_i\}$ and $\{z_j\}$ are strictly increasing, the (1,2)-order divided difference of $g(y_i, z_j)$ is nonnegative, and the (2,1)-order divided difference of $g(y_i, z_j)$ is nonpositive. Then B_1 forms a dual feasible basis of $\overline{\mathbf{A}}$.*

Corollary 3.3. *Consider the minimization problem (27), and assume that both* $\{y_i\}$ *and* $\{z_j\}$ *are strictly decreasing, the (1,2)-order divided difference of* $g(y_i, z_j)$ *is nonpositive, and the (2,1)-order divided difference of* $g(y_i, z_j)$ *is nonnegative. Then* B_1 *forms a dual feasible basis of* $\overline{\mathbf{A}}$.

Corollary 3.4. *Consider the maximization problem (27), and assume that* $\{y_i\}$ *is strictly decreasing and* $\{z_j\}$ *is strictly increasing, and the (1,2)-order and (2,1)-order divided differences of* $g(y_i, z_j)$ *are nonnegative. Then* B_1 *forms a dual feasible basis of* $\overline{\mathbf{A}}$.

Corollary 3.5. *Consider the maximization problem (27), and assume that* $\{y_i\}$ *is strictly increasing and* $\{z_j\}$ *is strictly decreasing, and the (1,2)-order and (2,1)-order divided differences of* $g(y_i, z_j)$ *are nonpositive. Then* B_1 *forms a dual feasible basis of* $\overline{\mathbf{A}}$.

Corollary 3.6. *Consider the maximization problem (27), and assume that both* $\{y_i\}$ *and* $\{z_j\}$ *are strictly increasing, the (1,2)-order divided difference of* $g(y_i, z_j)$ *is nonpositive, and the (2,1)-order divided difference of* $g(y_i, z_j)$ *is nonnegative. Then* B_1 *forms a dual feasible basis of* $\overline{\mathbf{A}}$.

Corollary 3.7. *Consider the maximization problem (27), and assume that both* $\{y_i\}$ *and* $\{z_j\}$ *are strictly decreasing, the (1,2)-order divided difference of* $g(y_i, z_j)$ *is nonnegative, and the (2,1)-order divided difference of* $g(y_i, z_j)$ *is nonpositive. Then* B_1 *forms a dual feasible basis of* $\overline{\mathbf{A}}$.

If we use the reasoning in the proof of Theorem 3.1, we can obtain a variety of dual feasible bases for problem (27) under different conditions. Let $B_2 = \{\overline{\mathbf{a}}_{in}, i = 1, \dots, m, \overline{\mathbf{a}}_{1j}, j = 1, \dots, n-1, \overline{\mathbf{a}}_{m1}\}$, $B_3 = \{\overline{\mathbf{a}}_{i1}, i = 1, \dots, m, \overline{\mathbf{a}}_{1j}, j = 2, \dots, n, \overline{\mathbf{a}}_{mn}\}$, $B_4 = \{\overline{\mathbf{a}}_{in}, i = 1, \dots, m, \overline{\mathbf{a}}_{mj}, j = 1, \dots, n-1, \overline{\mathbf{a}}_{11}\}$, the cells corresponding to the vectors in B_1, B_2, B_3 and B_4 are designated by **boldface** points in Figures 3.1, 3.2, 3.3 and 3.4, respectively.

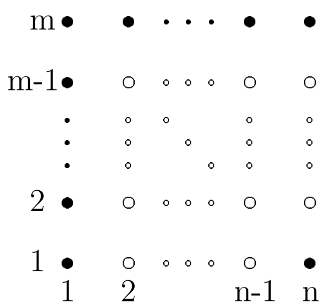

Figure 3.1. (B_1)

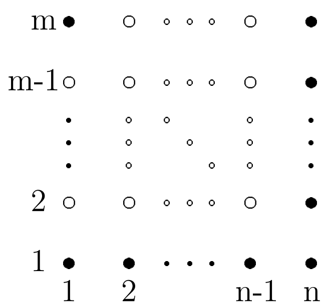

Figure 3.2. (B_2)

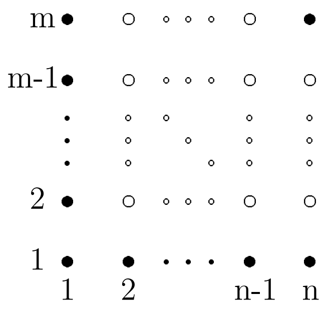

Figure 3.3. (B_3)

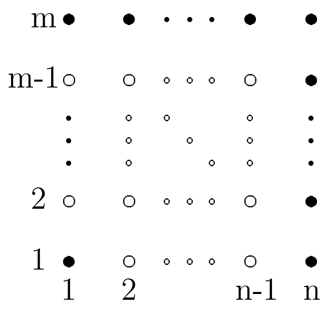

Figure 3.4. (B_4)

We summarize all the results for the minimization (maximization) problem (27) in the following table:

y_i	z_j	(1,2)-order d.d.	(2,1)-order d.d.	d.f.b. of min	d.f.b. of max
$\nearrow$	$\searrow$	≥ 0	≥ 0	B_1	B_2
$\searrow$	$\nearrow$	≤ 0	≤ 0	B_1	B_2
$\nearrow$	$\nearrow$	≥ 0	≤ 0	B_1	B_2
$\searrow$	$\searrow$	≤ 0	≥ 0	B_1	B_2
$\nearrow$	$\searrow$	≤ 0	≤ 0	B_2	B_1
$\searrow$	$\nearrow$	≥ 0	≥ 0	B_2	B_1
$\nearrow$	$\nearrow$	≤ 0	≥ 0	B_2	B_1
$\searrow$	$\searrow$	≥ 0	≤ 0	B_2	B_1
$\nearrow$	$\searrow$	≤ 0	≥ 0	B_3	B_4
$\searrow$	$\nearrow$	≥ 0	≤ 0	B_3	B_4
$\nearrow$	$\nearrow$	≤ 0	≤ 0	B_3	B_4
$\searrow$	$\searrow$	≥ 0	≥ 0	B_3	B_4
$\nearrow$	$\searrow$	≥ 0	≤ 0	B_4	B_3
$\searrow$	$\nearrow$	≤ 0	≥ 0	B_4	B_3
$\nearrow$	$\nearrow$	≥ 0	≥ 0	B_4	B_3
$\searrow$	$\searrow$	≤ 0	≤ 0	B_4	B_3

Table 3.1

The sign $\nearrow$ means strictly increasing, while $\searrow$ means strictly decreasing. "d.d." means divided difference, and "d.f.b." means dual feasible basis.

4 The Three-Dimensional Case

In this section we look at the 3-dimensional transportation problem:

$$
\begin{array}{ll}
\min(\max) & \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} c_{ijk}x_{ijk} \\
\text{subject to} & \\
& \sum_{j=1}^{n_2}\sum_{k=1}^{n_3} x_{ijk} = a_i, \quad i=1,\dots,n_1 \\
& \sum_{i=1}^{n_1}\sum_{k=1}^{n_3} x_{ijk} = b_j, \quad j=1,\dots,n_2 \\
& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} x_{ijk} = c_k, \quad k=1,\dots,n_3 \\
& x_{ijk} \geq 0, \quad i=1,\dots,n_1, \quad j=1,\dots,n_2, \quad k=1,\dots,n_3.
\end{array}
\tag{30}
$$

We can write problem (30) in matrix form if we let

$$
\begin{array}{rcl}
\mathbf{x} & = & (x_{111},\dots,x_{11n_3},\dots,x_{1n_21},\dots,x_{1n_2n_3},\dots, \\
& & x_{n_111},\dots,x_{n_11n_3},\dots,x_{n_1n_21},\dots,x_{n_1n_2n_3})^T \\
\mathbf{c} & = & (c_{111},\dots,c_{11n_3},\dots,c_{1n_21},\dots,c_{1n_2n_3},\dots, \\
& & c_{n_111},\dots,c_{n_11n_3},\dots,c_{n_1n_21},\dots,c_{n_1n_2n_3})^T \\
\mathbf{b} & = & (a_1,\dots,a_{n_1},b_1,\dots,b_{n_2},c_1,\dots,c_{n_3})^T \\
\mathbf{A} & = & (\mathbf{a}_{111},\dots,\mathbf{a}_{11n_3},\dots,\mathbf{a}_{1n_21},\dots,\mathbf{a}_{1n_2n_3},\dots, \\
& & \mathbf{a}_{n_111},\dots,\mathbf{a}_{n_11n_3},\dots,\mathbf{a}_{n_1n_21},\dots,\mathbf{a}_{n_1n_2n_3})^T,
\end{array}
\tag{31}
$$

where $\mathbf{a}_{ijk} = \mathbf{e}_i + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+k}$ and $\mathbf{e}_i$, $\mathbf{e}_{n_1+j}$ and $\mathbf{e}_{n_1+n_2+k}$ are unit vectors in $\mathbf{E}^{n_1+n_2+n_3}$, with ones in the ith, (n_1+j)th and (n_1+n_2+k)th positions, respectively, for all $i=1,\dots,n_1$, $j=1,\dots,n_2$, $k=1,\dots,n_3$. With these definitions problem (30) takes the form (3).

For this problem, we can get the following theorem:

Theorem 4.1. *Consider the minimization problem (30), and assume that the cost array satisfies the Monge property. Then each of the following six sequences of vectors forms a dual feasible basis of the columns of the matrix* $\mathbf{A}$*:*

$$
\begin{array}{ll}
(S_1) & \{\mathbf{a}_{i11}, i=1,\dots,n_1, \mathbf{a}_{n_1j1}, j=2,\dots,n_2, \mathbf{a}_{n_1n_2k}, k=2,\dots,n_3\}, \\
(S_2) & \{\mathbf{a}_{i11}, i=1,\dots,n_1, \mathbf{a}_{n_11k}, k=2,\dots,n_3, \mathbf{a}_{n_1jn_3}, j=2,\dots,n_2\}, \\
(S_3) & \{\mathbf{a}_{1j1}, j=1,\dots,n_2, \mathbf{a}_{in_21}, i=2,\dots,n_1, \mathbf{a}_{n_1n_2k}, k=2,\dots,n_3\}, \\
(S_4) & \{\mathbf{a}_{1j1}, j=1,\dots,n_2, \mathbf{a}_{1n_2k}, k=2,\dots,n_3, \mathbf{a}_{in_2n_3}, i=2,\dots,n_1\}, \\
(S_5) & \{\mathbf{a}_{11k}, k=1,\dots,n_3, \mathbf{a}_{i1n_3}, i=2,\dots,n_1, \mathbf{a}_{n_1jn_3}, j=2,\dots,n_2\}, \\
(S_6) & \{\mathbf{a}_{11k}, k=1,\dots,n_3, \mathbf{a}_{1jn_3}, j=2,\dots,n_2, \mathbf{a}_{in_2n_3}, i=2,\dots,n_1\}.
\end{array}
$$

Proof. Theorem 4.1 can easily be derived by the use of Theorem 2.8. In fact, each of (S_1)-(S_6) is an ordered sequence. A direct proof, however, may be instructive and below we present one.

First let us show that the vectors of S_1 are independent. Consider the linear combination of these vectors:

$$\begin{aligned}
& \sum_{i=1}^{n_1} \lambda_{i11}\mathbf{a}_{i11} + \sum_{j=2}^{n_2} \lambda_{n_1j1}\mathbf{a}_{n_1j1} + \sum_{k=2}^{n_3} \lambda_{n_1n_2k}\mathbf{a}_{n_1n_2k} \\
= & \sum_{i=1}^{n_1} \lambda_{i11}(\mathbf{e}_i + \mathbf{e}_{n_1+1} + \mathbf{e}_{n_1+n_2+1}) + \sum_{j=2}^{n_2} \lambda_{n_1j1}(\mathbf{e}_{n_1} + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+1}) \\
& + \sum_{k=2}^{n_3} \lambda_{n_1n_2k}(\mathbf{e}_{n_1} + \mathbf{e}_{n_1+n_2} + \mathbf{e}_{n_1+n_2+k}) \\
= & \sum_{i=1}^{n_1-1} \lambda_{i11}\mathbf{e}_i + (\sum_{j=1}^{n_2} \lambda_{n_1j1} + \sum_{k=2}^{n_3} \lambda_{n_1n_2k})\mathbf{e}_{n_1} + (\sum_{i=1}^{n_1} \lambda_{i11})\mathbf{e}_{n_1+1} \\
& + \sum_{j=2}^{n_2-1} \lambda_{n_1j1}\mathbf{e}_{n_1+j} + (\sum_{k=1}^{n_3} \lambda_{n_1n_2k})\mathbf{e}_{n_1+n_2} \\
& + (\sum_{i=1}^{n_1} \lambda_{i11} + \sum_{j=2}^{n_2-1} \lambda_{n_1j1})\mathbf{e}_{n_1+n_2+1} + \sum_{k=2}^{n_3} \lambda_{n_1n_2k}\mathbf{e}_{n_1+n_2+k}.
\end{aligned}$$

We can see that if it equals 0, then all λ's must be 0.

Next we show that any vector $\mathbf{a}_{ijk}$ of the matrix $\mathbf{A}$ can be expressed as a linear combination of these vectors, $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, $1 \leq k \leq n_3$. In fact,

$$\begin{aligned}
& \mathbf{a}_{ij1} - \mathbf{a}_{n_1j1} + \mathbf{a}_{n_1jk} \\
= & \mathbf{e}_i + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+1} - (\mathbf{e}_{n_1} + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+1}) \\
& + \mathbf{e}_{n_1} + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+k} \\
= & \mathbf{e}_i + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+k} \\
= & \mathbf{a}_{ijk},
\end{aligned}$$

$$\begin{aligned}
\mathbf{a}_{ij1} &= \mathbf{a}_{i11} - \mathbf{a}_{n_111} + \mathbf{a}_{n_1j1}, \\
\mathbf{a}_{n_1jk} &= \mathbf{a}_{n_1j1} - \mathbf{a}_{n_1n_21} + \mathbf{a}_{n_1n_2k},
\end{aligned}$$

so,

$$\mathbf{a}_{ijk} = \mathbf{a}_{i11} - \mathbf{a}_{n_111} + \mathbf{a}_{n_1j1} - \mathbf{a}_{n_1n_21} + \mathbf{a}_{n_1n_2k}, \tag{32}$$

where $\mathbf{a}_{i11}$, $\mathbf{a}_{n_111}$, $\mathbf{a}_{n_1j1}$, $\mathbf{a}_{n_1n_21}$, $\mathbf{a}_{n_1n_2k} \in S_1$. Thus S_1 forms a basis of matrix $\mathbf{A}$.

Finally, let us show that this basis is dual feasible. According to the definition of dual feasible basis and equation (32), we need to check the nonpositivity of the following value:

$$c_{i11} - c_{n_111} + c_{n_1j1} - c_{n_1n_21} + c_{n_1n_2k} - c_{ijk}. \tag{33}$$

It can be written as

$$
\begin{aligned}
&(c_{i11} - c_{n_111} + c_{n_1j1} - c_{ij1}) + (c_{n_1j1} - c_{n_1n_21} + c_{n_1n_2k} - c_{n_1jk}) \\
&+(c_{ij1} - c_{n_1j1} + c_{n_1jk} - c_{ijk}).
\end{aligned}
\tag{34}
$$

Since the cost array satisfies the Monge property, the values in the parentheses are nonpositive. Thus the sequence of vectors S_1 forms a dual feasible basis of the matrix $\mathbf{A}$. $\square$

The vectors of S_i, $i = 1, \ldots, 6$ are illustrated in Figures 4.1, …, 4.6, respectively. They are represented by the points of the boldface lines.

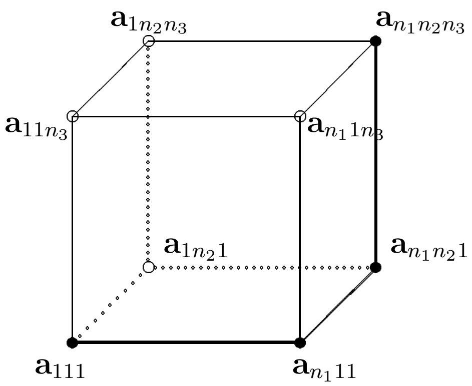

Figure 4.1. (S_1)

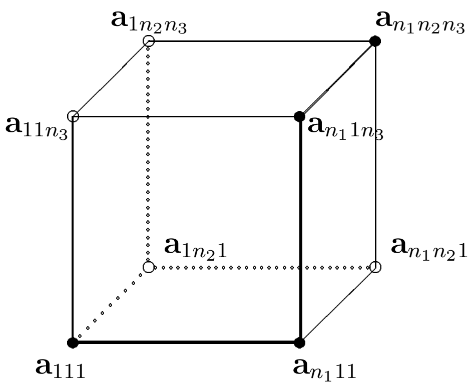

Figure 4.2. (S_2)

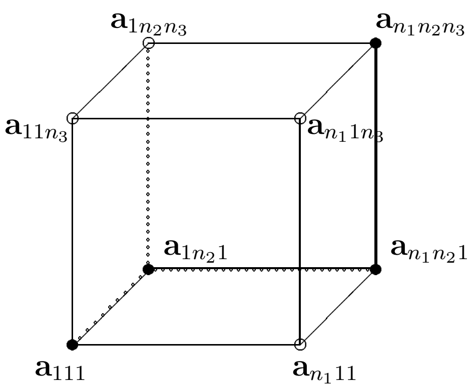

Figure 4.3. (S_3)

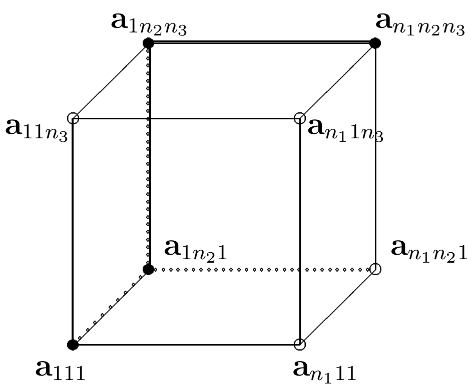

Figure 4.4. (S_4)

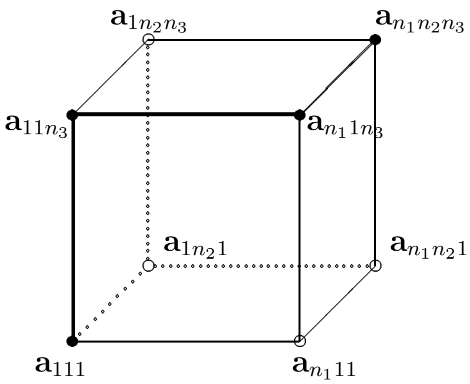

Figure 4.5. (S_5)

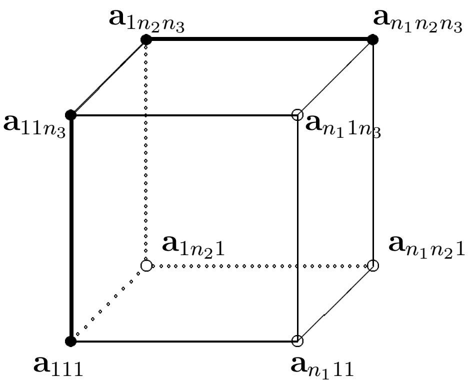

Figure 4.6. (S_6)

Theorem 4.2. *Consider the maximization problem (30). If the cost array satisfies the Monge property, then each of the following sequences of vectors forms a dual feasible basis of the matrix* $\mathbf{A}$*:*

$$\begin{aligned}
&(S_1') \quad \{\mathbf{a}_{i11}, i = 1, \ldots, n_1, \mathbf{a}_{1j1}, j = 2, \ldots, n_2, \mathbf{a}_{11k}, k = 2, \ldots, n_3\},\\
&(S_2') \quad \{\mathbf{a}_{in_2n_3}, i = 1, \ldots, n_1, \mathbf{a}_{n_1jn_3}, j = 1, \ldots, n_2 - 1,\\
&\qquad\quad \mathbf{a}_{n_1n_2k}, k = 1, \ldots, n_3 - 1\}.
\end{aligned}$$

Proof. We prove the assertion for S_1'. The proof for the other one can be carried out similarly.

First we show that these vectors are independent. Consider the linear combination of the vectors of S_1':

$$\begin{aligned}
&\sum_{i=1}^{n_1} \lambda_{i11}\mathbf{a}_{i11} + \sum_{j=2}^{n_2} \lambda_{1j1}\mathbf{a}_{1j1} + \sum_{k=2}^{n_3} \lambda_{11k}\mathbf{a}_{11k}\\
= \;& \sum_{i=1}^{n_1} \lambda_{i11}(\mathbf{e}_i + \mathbf{e}_{n_1+1} + \mathbf{e}_{n_1+n_2+1}) + \sum_{j=2}^{n_2} \lambda_{1j1}(\mathbf{e}_1 + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+1})\\
&+ \sum_{k=2}^{n_3} \lambda_{11k}(\mathbf{e}_1 + \mathbf{e}_{n_1+1} + \mathbf{e}_{n_1+n_2+k})\\
= \;& (\sum_{j=1}^{n_2} \lambda_{1j1} + \sum_{k=2}^{n_3} \lambda_{11k})\mathbf{e}_1 + \sum_{i=2}^{n_1} \lambda_{i11}\mathbf{e}_i + (\sum_{i=1}^{n_1} \lambda_{i11} + \sum_{k=2}^{n_3} \lambda_{11k})\mathbf{e}_{n_1+1}\\
&+ \sum_{j=2}^{n_2} \lambda_{1j1}\mathbf{e}_{n_1+j} + (\sum_{i=1}^{n_1} \lambda_{i11} + \sum_{j=2}^{n_2} \lambda_{1j1})\mathbf{e}_{n_1+n_2+1} + \sum_{k=2}^{n_3} \lambda_{11k}\mathbf{e}_{n_1+n_2+k}.
\end{aligned}$$

We can easily see that if it is 0, then all λ's are zero.

Secondly we show that any vector $\mathbf{a}_{ijk}$ of the matrix $\mathbf{A}$ can be expressed as a linear combination of the vectors in S_1'. In fact, we have the relations:

$$\begin{aligned} & \mathbf{a}_{ij1} - \mathbf{a}_{i11} + \mathbf{a}_{i1k} \\ = \; & \mathbf{e}_i + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+1} - (\mathbf{e}_i + \mathbf{e}_{n_1+1} + \mathbf{e}_{n_1+n_2+1}) \\ & + \mathbf{e}_i + \mathbf{e}_{n_1+1} + \mathbf{e}_{n_1+n_2+k} \\ = \; & \mathbf{e}_i + \mathbf{e}_{n_1+j} + \mathbf{e}_{n_1+n_2+k} \\ = \; & \mathbf{a}_{ijk}, \end{aligned}$$

$$\begin{aligned} \mathbf{a}_{ij1} &= \mathbf{a}_{i11} - \mathbf{a}_{111} + \mathbf{a}_{1j1}, \\ \mathbf{a}_{i1k} &= \mathbf{a}_{i11} - \mathbf{a}_{111} + \mathbf{a}_{11k}. \end{aligned}$$

Then

$$\mathbf{a}_{ijk} = \mathbf{a}_{i11} - \mathbf{a}_{111} + \mathbf{a}_{11k} - \mathbf{a}_{111} + \mathbf{a}_{1j1}, \tag{35}$$

where $\mathbf{a}_{i11}$, $\mathbf{a}_{111}$, $\mathbf{a}_{11k}$, $\mathbf{a}_{1j1} \in S_1'$. So S_1' forms a basis of the matrix $\mathbf{A}$.

Thirdly we show that this basis is dual feasible. According to the definition of the dual feasibility of a basis for the maximization problem and equation (35), we need to prove the nonnegativity of the following value:

$$c_{i11} - c_{111} + c_{11k} - c_{111} + c_{1j1} - c_{ijk}. \tag{36}$$

The value (36) can be written as

$$\begin{aligned} & (c_{i11} - c_{111} + c_{11k} - c_{i1k}) + (c_{i11} - c_{ij1} + c_{1j1} - c_{111}) \\ & + (c_{i1k} - c_{i11} + c_{ij1} - c_{ijk}). \end{aligned} \tag{37}$$

Since the cost array satisfies the Monge property, the values in the three parentheses are nonnegative. Thus the sequence of vectors S_1' forms a dual feasible basis of the matrix $\mathbf{A}$ in the maximization problem (30). □

The vectors of S_1' and S_2' are illustrated in Figures 4.7 and 4.8, respectively. They are represented by the points of the boldface lines.

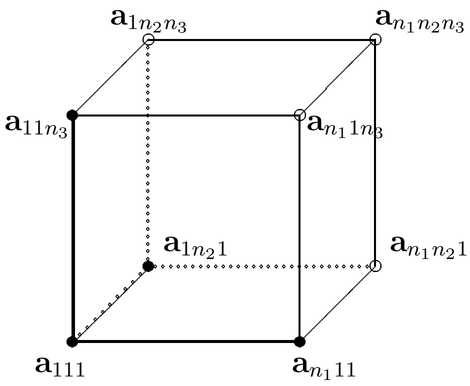

Figure 4.7. (S_1')

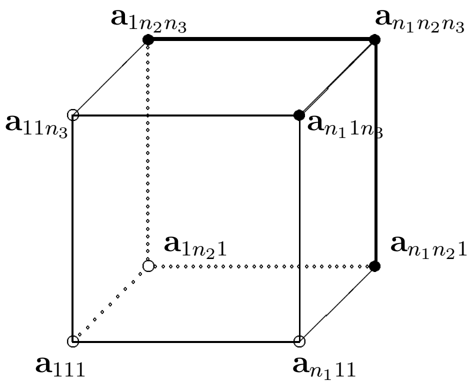

Figure 4.8. (S_2')

5 The Use of Covariances in The Three-Dimensional Case

In this section we supplement the covariance constraints to the constraints of problem (30). The new problem is the following:

$$
\begin{array}{ll}
\min(\max) & \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} c_{ijk}x_{ijk} \\
\text{subject to} & \\
& \sum_{j=1}^{n_2}\sum_{k=1}^{n_3} x_{ijk} = a_i, \quad i = 1,\dots,n_1 \\
& \sum_{i=1}^{n_1}\sum_{k=1}^{n_3} x_{ijk} = b_j, \quad j = 1,\dots,n_2 \\
& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} x_{ijk} = c_k, \quad k = 1,\dots,n_3 \\
& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} r_i s_j x_{ijk} = d_1 \\
& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} s_j t_k x_{ijk} = d_2 \\
& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\sum_{k=1}^{n_3} r_i t_k x_{ijk} = d_3 \\
& x_{ijk} \ge 0, \quad i = 1,\dots,n_1, \quad j = 1,\dots,n_2, \quad k = 1,\dots,n_3.
\end{array}
\tag{38}
$$

Assume that there is a map between r_i, s_j, t_k and c_{ijk}, i.e., $c_{ijk} = g(r_i, s_j, t_k)$, for $i = 1,\dots,n_1$, $j = 1,\dots,n_2$, $k = 1,\dots,n_3$. Problem (38), can be written in the compact form

(28), where

$$\begin{aligned}
\overline{\mathbf{A}} &= (\overline{\mathbf{a}}_{ijk}) = \begin{pmatrix} \mathbf{A} \\ \mathbf{rs} \\ \mathbf{st} \\ \mathbf{rt} \end{pmatrix} \\
\overline{\mathbf{b}} &= \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix} \\
\mathbf{rs} &= (r_1s_1, \dots, r_1s_1, \dots, r_1s_{n_2}, \dots, r_1s_{n_2}, \dots, \\
&\quad\; r_{n_1}s_1, \dots, r_{n_1}s_1, \dots, r_{n_1}s_{n_2}, \dots, r_{n_1}s_{n_2}) \\
\mathbf{st} &= (s_1t_1, \dots, s_1t_1, \dots, s_1t_{n_3}, \dots, s_1t_{n_3}, \dots, \\
&\quad\; s_{n_2}t_1, \dots, s_{n_2}t_1, \dots, s_{n_2}t_{n_3}, \dots, s_{n_2}t_{n_3}) \\
\mathbf{rt} &= (r_1t_1, \dots, r_1t_1, \dots, r_1t_{n_3}, \dots, r_1t_{n_3}, \dots, \\
&\quad\; r_{n_1}t_1, \dots, r_{n_1}t_1, \dots, r_{n_1}t_{n_3}, \dots, r_{n_1}t_{n_3}) \\
\mathbf{d} &= (d_1, d_2, d_3)^T,
\end{aligned} \tag{39}$$

and $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{x}$ are the same as in (31).

Let $\overline{S}_1$ be the collection of those vectors $\overline{\mathbf{a}}_{ijk}$ which have the same subscripts as those in S_1 in Theorem 4.1. We prove

Theorem 5.1. *Consider the minimization problem (38). If r_i, s_j and t_k are strictly increasing, all $(1,2,0)$-order, $(1,0,2)$-order, $(0,1,2)$-order divided differences of $g(r_i,s_j,t_k)$ are nonnegative, and all $(2,1,0)$-order, $(2,0,1)$-order and $(0,2,1)$-order and $(1,1,1)$-order divided differences of $g(r_i,s_j,t_k)$ are nonpositive, then $\overline{S}_1 \cup \{\overline{\mathbf{a}}_{n_11n_3}, \overline{\mathbf{a}}_{1n_21}, \overline{\mathbf{a}}_{11n_3}\}$ forms a dual feasible basis of the matrix $\overline{\mathbf{A}}$.*

Proof. First let us show that vectors of the sequence $\overline{S}_1$ and the vectors $\overline{\mathbf{a}}_{n_11n_3}$, $\overline{\mathbf{a}}_{1n_21}$ and $\overline{\mathbf{a}}_{11n_3}$ form a basis of $\overline{\mathbf{A}}$. By Theorem 4.1 S_1 forms a basis of $\mathbf{A}$, and the basis of $\overline{\mathbf{A}}$ has at most three more vectors than the basis of $\mathbf{A}$. So, we only need to prove that these vectors are linearly independent. It is, however true, because we have the equation (for $n_1+n_2+n_3+3$-dimensional vectors)

$$\begin{gathered}
\textstyle\sum_{i=1}^{n_1} \lambda_{i11}\overline{\mathbf{a}}_{i11} + \sum_{j=2}^{n_2} \lambda_{n_1j1}\overline{\mathbf{a}}_{n_1j1} + \sum_{k=2}^{n_3} \lambda_{n_1n_2k}\overline{\mathbf{a}}_{n_1n_2k} \\
+\lambda_{n_11n_3}\overline{\mathbf{a}}_{n_11n_3} + \lambda_{1n_21}\overline{\mathbf{a}}_{1n_21} + \lambda_{11n_3}\overline{\mathbf{a}}_{11n_3}
\end{gathered}$$

$$
= \begin{pmatrix}
\lambda_{111} + \lambda_{1n_21} + \lambda_{11n_3} \\
\lambda_{211} \\
\vdots \\
\lambda_{(n_1-1)11} \\
\lambda_{n_111} + \cdots + \lambda_{n_1n_21} + \cdots + \lambda_{n_1n_2n_3} + \lambda_{n_11n_3} \\
\lambda_{111} + \cdots + \lambda_{n_111} + \lambda_{n_11n_3} + \lambda_{11n_3} \\
\lambda_{n_121} \\
\vdots \\
\lambda_{n_1(n_2-1)1} \\
\lambda_{n_1n_21} + \cdots + \lambda_{n_1n_2n_3} + \lambda_{1n_21} \\
\lambda_{111} + \cdots + \lambda_{n_111} + \cdots + \lambda_{n_1n_21} + \lambda_{1n_21} \\
\lambda_{n_1n_22} \\
\vdots \\
\lambda_{n_1n_2(n_3-1)} \\
\lambda_{n_1n_2n_3} + \lambda_{n_11n_3} + \lambda_{11n_3} \\
r_1s_1\lambda_{111} + \cdots + r_{n_1}s_1\lambda_{n_111} + \cdots + r_{n_1}s_{n_2}\lambda_{n_1n_21} + \cdots + r_{n_1}s_{n_2}\lambda_{n_1n_2n_3} \\
+r_{n_1}s_1\lambda_{n_11n_3} + r_1s_{n_2}\lambda_{1n_21} + r_1s_1\lambda_{11n_3} \\
s_1t_1\lambda_{111} + \cdots + s_1t_1\lambda_{n_111} + \cdots + s_{n_2}t_1\lambda_{n_1n_21} + \cdots + s_{n_2}t_{n_3}\lambda_{n_1n_2n_3} \\
+s_1t_{n_3}\lambda_{n_11n_3} + s_{n_2}t_1\lambda_{1n_21} + s_1t_{n_3}\lambda_{11n_3} \\
r_1t_1\lambda_{111} + \cdots + r_{n_1}t_1\lambda_{n_111} + \cdots + r_{n_1}t_1\lambda_{n_1n_21} + \cdots + r_{n_1}t_{n_3}\lambda_{n_1n_2n_3} \\
+r_{n_1}t_{n_3}\lambda_{n_11n_3} + r_1t_1\lambda_{1n_21} + r_1t_{n_3}\lambda_{11n_3}
\end{pmatrix}
$$

and if it equals 0, then all λ's must be 0.

Secondly, let us show that this basis is dual feasible. Assume that all $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ sequences are strictly decreasing. The proof for the case where all r_i, s_j and t_k are strictly increasing can be carried out the same way. For any nonbasic vector $\overline{\mathbf{a}}_{ijk}$, $1 \le i \le n_1$, $1 \le j \le n_2$, $1 \le k \le n_3$, we have the following four equations:

$$
\begin{aligned}
&\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{n_111} - \overline{\mathbf{a}}_{n_1j1} + \overline{\mathbf{a}}_{n_1n_21} - \overline{\mathbf{a}}_{n_1n_2k} \\
&= \begin{pmatrix} \mathbf{0} \\ (r_i - r_{n_1})(s_j - s_1) \\ (s_j - s_{n_2})(t_k - t_1) \\ (r_i - r_{n_1})(t_k - t_1) \end{pmatrix},
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{n_111} - \overline{\mathbf{a}}_{n_1n_2k} + \overline{\mathbf{a}}_{n_1n_2n_3} - \overline{\mathbf{a}}_{n_11n_3} + \overline{\mathbf{a}}_{n_111} - \overline{\mathbf{a}}_{n_1j1} \\
&= \begin{pmatrix} \mathbf{0} \\ (r_i - r_{n_1})(s_j - s_1) \\ (s_j - s_{n_2})(t_k - t_1) - (s_1 - s_{n_2})(t_{n_3} - t_1) \\ (r_i - r_{n_1})(t_k - t_1) \end{pmatrix},
\end{aligned} \tag{41}
$$

$$\begin{aligned}
&\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{n_1 j 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 n_2 k} + \overline{\mathbf{a}}_{n_1 1 n_3} - \overline{\mathbf{a}}_{11 n_3} + \overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{i11} \\
= &\begin{pmatrix} \mathbf{0} \\ (r_i - r_{n_1})(s_j - s_1) \\ (s_j - s_{n_2})(t_k - t_1) \\ (r_i - r_{n_1})(t_k - t_1) - (r_1 - r_{n_1})(t_{n_3} - t_1) \end{pmatrix},
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 j 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 n_2 k} \\
= &\begin{pmatrix} \mathbf{0} \\ (r_i - r_{n_1})(s_j - s_1) - (r_1 - r_{n_1})(s_{n_2} - s_1) \\ (s_j - s_{n_2})(t_k - t_1) \\ (r_i - r_{n_1})(t_k - t_1) \end{pmatrix},
\end{aligned} \tag{43}$$

where $\mathbf{0}$ is a zero vector in $\mathbf{R}^{n_1+n_2+n_3}$. For simplicity, let

$$\begin{aligned}
A_1 &= (r_i - r_{n_1})(s_j - s_1), \\
A_2 &= (s_j - s_{n_2})(t_k - t_1), \\
A_3 &= (r_i - r_{n_1})(t_k - t_1), \\
B_1 &= (r_1 - r_{n_1})(s_{n_2} - s_1), \\
B_2 &= (s_1 - s_{n_2})(t_{n_3} - t_1), \\
B_3 &= (r_1 - r_{n_1})(t_{n_3} - t_1),
\end{aligned}$$

$$\mathbf{a}_{13} = \begin{pmatrix} \mathbf{0} \\ A_1 \\ 0 \\ A_3 \end{pmatrix}, \mathbf{a}_{12} = \begin{pmatrix} \mathbf{0} \\ A_1 \\ A_2 \\ 0 \end{pmatrix}, \mathbf{a}_{23} = \begin{pmatrix} \mathbf{0} \\ 0 \\ A_2 \\ A_3 \end{pmatrix}.$$

¿From (40) and (41) we obtain

$$\begin{aligned}
&\frac{A_2 - B_2}{A_2}(\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 j 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 n_2 k} - \mathbf{a}_{13}) \\
= \;&\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 n_2 k} + \overline{\mathbf{a}}_{n_1 n_2 n_3} - \overline{\mathbf{a}}_{n_1 1 n_3} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 j 1} - \mathbf{a}_{13}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\mathbf{a}}_{ijk} &= (\overline{\mathbf{a}}_{i11} - \overline{\mathbf{a}}_{n_1 11} + \overline{\mathbf{a}}_{n_1 j 1} - \overline{\mathbf{a}}_{n_1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 k} + \mathbf{a}_{13}) \\
&\quad - \frac{A_2}{B_2}(\overline{\mathbf{a}}_{n_1 n_2 n_3} - \overline{\mathbf{a}}_{n_1 1 n_3} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 n_2 1}).
\end{aligned} \tag{44}$$

Equations (40) and (42) imply

$$\begin{aligned}
\overline{\mathbf{a}}_{ijk} &= (\overline{\mathbf{a}}_{i11} - \overline{\mathbf{a}}_{n_1 11} + \overline{\mathbf{a}}_{n_1 j 1} - \overline{\mathbf{a}}_{n_1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 k} + \mathbf{a}_{12}) \\
&\quad - \frac{A_3}{B_3}(\overline{\mathbf{a}}_{n_1 1 n_3} - \overline{\mathbf{a}}_{11 n_3} + \overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{n_1 11}).
\end{aligned} \tag{45}$$

Similarly, equations (40) and (43) imply

$$\begin{aligned}\overline{\mathbf{a}}_{ijk} &= (\overline{\mathbf{a}}_{i11} - \overline{\mathbf{a}}_{n_1 11} + \overline{\mathbf{a}}_{n_1 j1} - \overline{\mathbf{a}}_{n_1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 k} + \mathbf{a}_{23}) \\ &\quad -\frac{A_1}{B_1}(\overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 11}). \end{aligned} \tag{46}$$

Finally, from the definition of $\mathbf{a}_{13}$, $\mathbf{a}_{12}$, $\mathbf{a}_{23}$ and (38), we derive the relation:

$$\mathbf{a}_{13} + \mathbf{a}_{12} + \mathbf{a}_{23} = 2(\overline{\mathbf{a}}_{ijk} - \overline{\mathbf{a}}_{i11} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 j1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 n_2 k}).$$

Summing up (41), (42) and (43), we find that

$$\begin{aligned}\overline{\mathbf{a}}_{ijk} &= \overline{\mathbf{a}}_{i11} - \overline{\mathbf{a}}_{n_1 11} + \overline{\mathbf{a}}_{n_1 j1} - \overline{\mathbf{a}}_{n_1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 k} \\ &\quad -\frac{A_2}{B_2}(\overline{\mathbf{a}}_{n_1 n_2 n_3} - \overline{\mathbf{a}}_{n_1 1 n_3} + \overline{\mathbf{a}}_{n_1 11} - \overline{\mathbf{a}}_{n_1 n_2 1}) \\ &\quad -\frac{A_3}{B_3}(\overline{\mathbf{a}}_{n_1 1 n_3} - \overline{\mathbf{a}}_{11 n_3} + \overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{n_1 11}) \\ &\quad -\frac{A_1}{B_1}(\overline{\mathbf{a}}_{111} - \overline{\mathbf{a}}_{1 n_2 1} + \overline{\mathbf{a}}_{n_1 n_2 1} - \overline{\mathbf{a}}_{n_1 11}). \end{aligned} \tag{47}$$

To prove the dual feasibility, we need to prove that

$$\begin{aligned}& c_{i11} - c_{n_1 11} + c_{n_1 j1} - c_{n_1 n_2 1} + c_{n_1 n_2 k} \\ & -\frac{A_2}{B_2}(c_{n_1 n_2 n_3} - c_{n_1 1 n_3} + c_{n_1 11} - c_{n_1 n_2 1}) \\ & -\frac{A_3}{B_3}(c_{n_1 1 n_3} - c_{11 n_3} + c_{111} - c_{n_1 11}) \\ & -\frac{A_1}{B_1}(c_{111} - c_{1 n_2 1} + c_{n_1 n_2 1} - c_{n_1 11}) - c_{ijk} \le 0, \end{aligned}$$

or, what is the same,

$$\begin{aligned}& (c_{i11} - c_{n_1 11} + c_{n_1 j1} - c_{ij1}) \\ & +(c_{ij1} - c_{n_1 j1} + c_{n_1 jk} - c_{ijk}) \\ & +(c_{n_1 j1} - c_{n_1 n_2 1} + c_{n_1 n_2 k} - c_{n_1 jk}) \\ \le\ & \frac{A_2}{B_2}(c_{n_1 n_2 n_3} - c_{n_1 1 n_3} + c_{n_1 11} - c_{n_1 n_2 1}) \\ & +\frac{A_3}{B_3}(c_{n_1 1 n_3} - c_{11 n_3} + c_{111} - c_{n_1 11}) \\ & +\frac{A_1}{B_1}(c_{111} - c_{1 n_2 1} + c_{n_1 n_2 1} - c_{n_1 11}) \end{aligned} \tag{48}$$

If we fix one of the i, j, k subscripts in c_{ijk}, then, in view of Corollary 2.6, (29) will hold for the remaining two subscripts. Thus we can obtain the following inequalities:

$$c_{i11} - c_{n_1 11} + c_{n_1 j1} - c_{ij1} \le \frac{A_1}{B_1}(c_{111} - c_{1 n_2 1} + c_{n_1 n_2 1} - c_{n_1 11}), \tag{49}$$

$$c_{n_1j1} - c_{n_1n_21} + c_{n_1n_2k} - c_{n_1jk} \leq \frac{A_2}{B_2}(c_{n_1n_2n_3} - c_{n_11n_3} + c_{n_111} - c_{n_1n_21}), \tag{50}$$

$$c_{ij1} - c_{n_1j1} + c_{n_1jk} - c_{ijk} \leq \frac{A_3}{B_3}(c_{n_1jn_3} - c_{1jn_3} + c_{1j1} - c_{n_1j1}). \tag{51}$$

Also, from the nonpositivity of the (1,1,1)-order divided difference of $g(r_i, s_j, t_k)$, we obtain

$$\frac{\frac{c_{111}-c_{n_111}-c_{11n_3}+c_{n_11n_3}}{(r_1-r_{n_1})(t_1-t_{n_3})} - \frac{c_{1j1}-c_{n_1j1}-c_{1jn_3}+c_{n_1jn_3}}{(r_1-r_{n_1})(t_1-t_{n_3})}}{(s_1 - s_j)} \leq 0,$$

where $s_1 - s_j < 0$, $r_1 - r_{n_1} < 0$, $t_1 - t_{n_3} < 0$. This implies

$$c_{1j1} - c_{n_1j1} - c_{1jn_3} + c_{n_1jn_3} \leq c_{111} - c_{n_111} - c_{11n_3} + c_{n_11n_3}.$$

By (51) and the inequality $\frac{A_3}{B_3} > 0$, we have

$$c_{ij1} - c_{n_1j1} + c_{n_1jk} - c_{ijk} \leq \frac{A_3}{B_3}(c_{111} - c_{n_111} - c_{11n_3} + c_{n_11n_3}). \tag{52}$$

Summing up (49), (50) and (52), we can obtain (48). Thus the basis is dual feasible.

□

By the similar proof, we can obtain the following theorem

Theorem 5.2. *Under the same conditions with Theorem 5.1 except that* $(1,1,1)$*-order divided differences of* $g(r_i, s_j, t_k)$ *are nonnegative, then* $\overline{S}_1 \cup \{\overline{\mathbf{a}}_{n_11n_3}, \overline{\mathbf{a}}_{1n_21}, \overline{\mathbf{a}}_{1n_2n_3}\}$ *forms a dual feasible basis of the matrix* $\overline{\mathbf{A}}$.

Define $\overline{S}_2$, …, $\overline{S}_6$, $\overline{S'}_1$, $\overline{S'}_2$ the same way as we have defined $\overline{S}_1$. Let

$$
\begin{aligned}
\overline{S}_{11} &= \overline{S}_1 \cup \{\overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{1 1 n_3}\}, \\
\overline{S}_{12} &= \overline{S}_1 \cup \{\overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{1 n_2 n_3}\}, \\
\overline{S}_{21} &= \overline{S}_2 \cup \{\overline{\mathbf{a}}_{1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{1 n_2 1}\}, \\
\overline{S}_{22} &= \overline{S}_2 \cup \{\overline{\mathbf{a}}_{1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{1 n_2 n_3}\}, \\
\overline{S}_{31} &= \overline{S}_3 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 n_3}, \overline{\mathbf{a}}_{1 1 n_3}\}, \\
\overline{S}_{32} &= \overline{S}_3 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 n_3}, \overline{\mathbf{a}}_{n_1 1 n_3}\}, \\
\overline{S}_{41} &= \overline{S}_4 \cup \{\overline{\mathbf{a}}_{1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 1}\}, \\
\overline{S}_{42} &= \overline{S}_4 \cup \{\overline{\mathbf{a}}_{1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 n_3}\}, \\
\overline{S}_{51} &= \overline{S}_5 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 n_3}, \overline{\mathbf{a}}_{1 n_2 1}\}, \\
\overline{S}_{52} &= \overline{S}_5 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}\}, \\
\overline{S}_{61} &= \overline{S}_6 \cup \{\overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{n_1 1 1}\}, \\
\overline{S}_{62} &= \overline{S}_6 \cup \{\overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 1}\}, \\
\overline{S'}_{11} &= \overline{S'}_1 \cup \{\overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{1 n_2 n_3}\}, \\
\overline{S'}_{12} &= \overline{S'}_1 \cup \{\overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{n_1 n_2 n_3}\}, \\
\overline{S'}_{13} &= \overline{S'}_1 \cup \{\overline{\mathbf{a}}_{n_1 n_2 1}, \overline{\mathbf{a}}_{n_1 n_2 n_3}, \overline{\mathbf{a}}_{1 n_2 n_3}\}, \\
\overline{S'}_{14} &= \overline{S'}_1 \cup \{\overline{\mathbf{a}}_{n_1 n_2 n_3}, \overline{\mathbf{a}}_{n_1 1 n_3}, \overline{\mathbf{a}}_{1 n_2 n_3}\}, \\
\overline{S'}_{21} &= \overline{S'}_2 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{1 1 n_3}\}, \\
\overline{S'}_{22} &= \overline{S'}_2 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{1 1 1}\}, \\
\overline{S'}_{23} &= \overline{S'}_2 \cup \{\overline{\mathbf{a}}_{n_1 1 1}, \overline{\mathbf{a}}_{1 1 1}, \overline{\mathbf{a}}_{1 1 n_3}\}, \\
\overline{S'}_{24} &= \overline{S'}_2 \cup \{\overline{\mathbf{a}}_{1 1 1}, \overline{\mathbf{a}}_{1 n_2 1}, \overline{\mathbf{a}}_{1 1 n_3}\}.
\end{aligned}
$$

Then we can derive results for the three-dimensional problem (38), we present them in Tables 5.1-5.8. In the following tables, $\nearrow$ means strictly increasing, $\searrow$ means strictly decreasing, (i, j, k) means (i, j, k)-order divided difference, $0 \leq i, j, k \leq 2$, 'd.f.b. of min' and 'd.f.b. of max' mean 'dual feasible basis of the minimization problem (38)' and 'dual feasible basis of the maximization problem (38)', respectively.

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,1,0)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(1,0,2)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,0,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(0,1,2)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(0,2,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b. of min	$\overline{S}_{11}$							
	$(\overline{S}_{12})$							
d.f.b. of max	$\overline{S}_{62}$							
	$(\overline{S}_{61})$							

Table 5.1

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,1,0)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(1,0,2)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,0,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(0,1,2)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(0,2,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b. of min	$\overline{S}_{61}$							
	$(\overline{S}_{62})$							
d.f.b. of max	$\overline{S}_{12}$							
	$(\overline{S}_{11})$							

Table 5.2

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,1,0)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(1,0,2)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,0,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(0,1,2)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(0,2,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b. of min	$\overline{S}_{21}$							
	$(\overline{S}_{22})$							
d.f.b. of max	$\overline{S}_{42}$							
	$(\overline{S}_{41})$							

Table 5.3

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,1,0)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(1,0,2)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,0,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(0,1,2)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(0,2,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b. of min	$\overline{S}_{41}$							
	$(\overline{S}_{42})$							
d.f.b. of max	$\overline{S}_{22}$							
	$(\overline{S}_{21})$							

Table 5.4

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,1,0)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(1,0,2)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,0,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(0,1,2)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(0,2,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b.	$\overline{S}_{31}$							
of min	$(\overline{S}_{32})$							
d.f.b.	$\overline{S}_{52}$							
of max	$(\overline{S}_{51})$							

Table 5.5

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,1,0)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(1,0,2)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,0,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(0,1,2)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(0,2,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b.	$\overline{S}_{51}$							
of min	$(\overline{S}_{52})$							
d.f.b.	$\overline{S}_{32}$							
of max	$(\overline{S}_{31})$							

Table 5.6

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,1,0)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(1,0,2)$	≤ 0	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0	≥ 0
$(2,0,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(0,1,2)$	≤ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≥ 0
$(0,2,1)$	≤ 0	≥ 0	≤ 0	≤ 0	≥ 0	≥ 0	≤ 0	≥ 0
$(1,1,1)$	≥ 0				≤ 0			
	(≤ 0)				(≥ 0)			
d.f.b.	$\overline{S'}_{11}$							
of min	$(\overline{S'}_{12}$, $\overline{S'}_{13}$, or $\overline{S'}_{14})$							
d.f.b.	$\overline{S'}_{22}$, $\overline{S'}_{23}$, or $\overline{S'}_{24}$							
of max	$(\overline{S'}_{21})$							

Table 5.7

r_i	$\nearrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$
s_j	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$
t_k	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$
$(1,2,0)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0	≤ 0
$(2,1,0)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(1,0,2)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0
$(2,0,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(0,1,2)$	≥ 0	≥ 0	≤ 0	≥ 0	≤ 0	≥ 0	≤ 0	≤ 0
$(0,2,1)$	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	≤ 0	≥ 0	≤ 0
$(1,1,1)$	≤ 0				≥ 0			
	(≥ 0)				(≤ 0)			
d.f.b.	$\overline{S'}_{22}$, $\overline{S'}_{23}$, or $\overline{S'}_{24}$							
of min	$(\overline{S'}_{21})$							
d.f.b.	$\overline{S'}_{11}$							
of max	$(\overline{S'}_{12}$, $\overline{S'}_{13}$, or $\overline{S'}_{14})$							

Table 5.8

6 Applications and Illustrative Examples

Monge and inverse Monge arrays came up in many practical applications. A collection of them is presented in Burkard et al. (1996). In this section we present three more applications which, at the same time, illustrate the ways that we can make use the results of the present paper.

6.1 Bounding Unknown Entries in Partially Known Arrays

As we have mentioned in the Introduction, any dual feasible bases in an LP may serve for bounding and approximation of unknown components of the coefficient vector of the objective function. If $\mathbf{B}_1$ ($\mathbf{B}_2$) is a dual feasible basis in a minimization (maximization) problem such that $\mathbf{c}_{\mathbf{B}_1}$ ($\mathbf{c}_{\mathbf{B}_2}$) is known, then we have the bound for any unknown c_k:

$$\begin{gathered} \mathbf{y}^T\mathbf{a}_k \le c_k \\ (\mathbf{y}^T\mathbf{a}_k \ge c_k), \end{gathered} \tag{53}$$

where $\mathbf{y}$ is any solution of the equation $\mathbf{y}^T\mathbf{B}_1 = \mathbf{c}_{\mathbf{B}_1}^T$ ($\mathbf{y}^T\mathbf{B}_2 = \mathbf{c}_{\mathbf{B}_2}^T$). The best theoretical bound is the largest (smallest) of all these bounds.

In Sections 2 and 4 we have presented dual feasible bases for LP's with Monge arrays in the objective function. We have done the same in Sections 3 and 5 for some higher order convex objective function coefficient arrays. Thus, we have created methods for bounding the entries of the above-mentioned arrays, if they are only partially known. If both the lower and upper bounds can be given for c_k and the bounds are close, then they may be used for the approximation of that value.

The dual feasibility of a basis, however, does not depend on the right hand side values in the equality constraints. Therefore, the bounds presented in Sections 2-5 may not be close enough to the unknown value. There are two possibilities to improve on them.

The first one is to carry out steps according to the rules of the dual method. Any time when a dual step is carried out, first we decide on the vector that leaves the basis $\mathbf{B}$ and the right hand side values in the constraints, to do that. As regards the entering vector, we can find one as long as there is an $\mathbf{a}_k$ such that c_k is known and $\mathbf{a}_k$ can enter the basis according to the rules of the dual method. Whenever no further $\mathbf{a}_k$ can be included into the basis, either because primal feasible basis has been found or the corresponding c_k is unknown, then we take the best bound so far as our final (lower or upper) bound.

The second one is the reduction of the size of the problem. This can be done only in connection with the problems discussed in Sections 2 and 4, unless the available information allows for applying it to problems in Sections 3 and 5.

In the problems of Sections 2 and 4 the right hand side values, i.e., the univariate marginal distributions are fully known, by assumption, so are the univariate marginal conditional distributions, given that the univariate random variables are restricted to some subsets of the support sets. The new problem, written up with the conditional probabilities on the right hand side, becomes smaller and the bounds for the unknown entries of the Monge array in the objective function may be better. This method can be applied for problems described in Sections 3 and 5 only if the conditional covariances are also known.

Any distribution array is a Monge array and the entries of a distribution array are values of a probability distribution function. Hence, the methodology of Sections 2, 4 provides us with a methodology for bounding and approximation of unknown values of a multivariate discrete p.d.f.. The results of Sections 3 and 5 improve on the bounds. In those cases the distribution functions must have the special property: nonnegativity of some divided differences.

Example 6.1 In the problem (1), the known c_{ij} and a_i,b_j values are given in the following table:

$a_4 = 14$	$c_{41} = 22$	$c_{42} = 20$	$c_{43} = 20$	$c_{44} = 16$	$c_{45} = 12$	$c_{46} = 13$
$a_3 = 39$	$c_{31} = 17$	c_{32}	c_{33}	c_{34}	c_{35}	$c_{36} = 12$
$a_2 = 22$	$c_{21} = 17$	c_{22}	c_{23}	c_{24}	c_{25}	$c_{26} = 20$
$a_1 = 18$	$c_{11} = 8$	$c_{12} = 10$	$c_{13} = 13$	$c_{14} = 11$	$c_{15} = 14$	$c_{16} = 23$
	$b_1 = 10$	$b_2 = 11$	$b_3 = 13$	$b_4 = 20$	$b_5 = 24$	$b_6 = 15$

Table 6.1

To find the lower bounds for the unknown values of c_{ij}, we choose

$$\mathbf{L}_1 = \{\mathbf{a}_{11}, \mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}, \mathbf{a}_{15}, \mathbf{a}_{16}, \mathbf{a}_{26}, \mathbf{a}_{36}, \mathbf{a}_{46}\}$$

as the initial dual feasible basis. Carrying out the dual method, no more dual feasible basis can be obtained. Since the solution of the equation $\mathbf{y}_1^T\mathbf{L}_1 = \mathbf{c}_{\mathbf{L}_1}^T$ is $\mathbf{y}_1^T = (23, 20, 12, 13, -15, -13, -10, -12, -$ the lower bounds for the unknown c_{ij} obtained from the dual feasible basis L_1 are as follows:

$$c_{22} \geq 7, \quad c_{23} \geq 10, \quad c_{24} \geq 8, \quad c_{25} \geq 11,$$
$$c_{32} \geq -1, \quad c_{33} \geq 2, \quad c_{34} \geq 0, \quad c_{35} \geq 3.$$

Again we can choose

$$\mathbf{L}_2 = \{\mathbf{a}_{11}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{41}, \mathbf{a}_{42}, \mathbf{a}_{43}, \mathbf{a}_{44}, \mathbf{a}_{45}, \mathbf{a}_{46}\}$$

as the initial dual feasible basis. Carrying out the dual method, no more dual feasible basis can be obtained. Since the solution of the equation $\mathbf{y}_2^T\mathbf{L}_2 = \mathbf{c}_{\mathbf{L}_2}^T$ is $\mathbf{y}_2^T = (-1, 8, 8, 13, 9, 7, 7, 3, -1, 0)$, the lower bounds for the unknown c_{ij} obtained from the dual feasible basis L_2 are as follows:

$$c_{22} \geq 15, \quad c_{23} \geq 15, \quad c_{24} \geq 11, \quad c_{25} \geq 7,$$
$$c_{32} \geq 15, \quad c_{33} \geq 15, \quad c_{34} \geq 11, \quad c_{35} \geq 7.$$

So the final lower bounds for the unknown c_{ij} are as follows:

$$c_{22} \geq 15, \quad c_{23} \geq 15, \quad c_{24} \geq 11, \quad c_{25} \geq 11,$$
$$c_{32} \geq 15, \quad c_{33} \geq 15, \quad c_{34} \geq 11, \quad c_{35} \geq 7.$$

To find the upper bounds for the unknown values of c_{ij}, similarly we choose

$$\mathbf{U}_1 = \{\mathbf{a}_{16}, \mathbf{a}_{15}, \mathbf{a}_{14}, \mathbf{a}_{13}, \mathbf{a}_{12}, \mathbf{a}_{11}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{41}\}$$

or

$$\mathbf{U}_2 = \{\mathbf{a}_{16}, \mathbf{a}_{26}, \mathbf{a}_{36}, \mathbf{a}_{46}, \mathbf{a}_{45}, \mathbf{a}_{44}, \mathbf{a}_{43}, \mathbf{a}_{42}, \mathbf{a}_{41}\}.$$

as the initial dual feasible basis. The final upper bounds for the unknown c_{ij} are as follows:

$$\begin{array}{llll} c_{22} \leq 19, & c_{23} \leq 22, & c_{24} \leq 20, & c_{25} \leq 19, \\ c_{32} \leq 19, & c_{33} \leq 19, & c_{34} \leq 15, & c_{35} \leq 11. \end{array}$$

Example 6.2 In example 6.1, if one more constraint is given with the form of the last constraint in problem (27), where

$$\begin{array}{c} c = 1076, \quad y_1 = 1, \quad y_2 = 2, \quad y_3 = 3, \quad y_4 = 4, \\ z_1 = 16, \quad z_2 = 9, \quad z_3 = 4, \quad z_4 = 3, \quad z_5 = 2, \quad z_6 = 1, \end{array}$$

then we can get a problem with the form of problem (27). It is easy to see that $\{y_i\}$ is strictly increasing and $\{z_j\}$ is strictly decreasing. Assume that $c_{ij} = g(y_i, z_j)$, then both the (1,2)-order and (2,1)-order divided differences are nonpositive. According to the results obtained in Section 3,

$$B_2 = \{\mathbf{a}_{11}, \mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}, \mathbf{a}_{15}, \mathbf{a}_{16}, \mathbf{a}_{26}, \mathbf{a}_{36}, \mathbf{a}_{46}, \mathbf{a}_{41}\},$$

$$B_1 = \{\mathbf{a}_{11}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{41}, \mathbf{a}_{42}, \mathbf{a}_{43}, \mathbf{a}_{44}, \mathbf{a}_{45}, \mathbf{a}_{46}, \mathbf{a}_{16}\}$$

are the dual feasible bases of the minimization and maximization problems respectively. ¿From these two bases, the lower and upper bounds for the unknown c_{ij} can be obtained as follows:

$$\begin{array}{ll} 11.3 \leq c_{22} \leq 22.5, & 11.6 \leq c_{23} \leq 27.8, \\ 9.07 \leq c_{24} \leq 24.9, & 11.5 \leq c_{25} \leq 21.9, \\ 7.53 \leq c_{32} \leq 18.7, & 5.2 \leq c_{33} \leq 21.4, \\ 2.13 \leq c_{34} \leq 17.9, & 4.07 \leq c_{35} \leq 14.5. \end{array}$$

6.2 Bounding Expectations

Let X, Y be two independent random vectors with the same discrete support set. Suppose that the probability distribution of X is fully known but the distribution of Y is only partially known. We know all univariate marginal distributions of its components and, if $d = 2$ or 3, all covariances of the pairs. Then we can give lower and upper bounds for

$$P(X \leq Y). \tag{54}$$

Each lower or upper bound is based on a dual feasible basis. Any dual feasible basis that we presented in Sections 2-5 provides us with a bound. If the basis is both primal and dual feasible, then the bound is sharp, no better bound can be given based on the available information. If we use our dual feasible bases as initial bases and carry out the lexicographic dual method, then the last basis is both primal and dual feasible, and the optimum value of the problem is the sharp bound.

Example 6.3 Suppose that the 2-dimensional random vectors $X_1 = (Y_1, Z_1)$ and $X_2 = (Y_2, Z_2)$ can take on the values (y_i, z_j), $i = 1, \dots, 4$, $j = 1, \dots, 6$, with probabilities $P_{X_1}(y_i, z_j)$ and $P_{X_2}(y_i, z_j)$, respectively, where

$$\begin{aligned} y_1 = 1, y_2 = 3, y_3 = 4, y_4 = 5,\\ z_1 = 18, z_2 = 11, z_3 = 6, z_4 = 3, z_5 = 2, z_6 = 1. \end{aligned} \tag{55}$$

If $P_{X_2}(y_i, z_j)$ are unknown or partially known, but all univariate marginal distributions $P(Y_2 = y_i) = a_i$, $P(Z_2 = z_j) = b_j$ are given as follows

$$\begin{aligned} a_1 = 0.19, a_2 = 0.24, a_3 = 0.42, a_4 = 0.15,\\ b_1 = 0.11, b_2 = 0.12, b_3 = 0.14, b_4 = 0.21, b_5 = 0.26, b_6 = 0.16. \end{aligned} \tag{56}$$

The probability distribution function of X_1 ($F_{X_1}(y_i, z_j)$) denoted as c_{ij} is given in the following table:

$c_{41} = 0.17$	$c_{42} = 0.34$	$c_{43} = 0.53$	$c_{44} = 0.68$	$c_{45} = 0.81$	$c_{46} = 1.00$
$c_{31} = 0.12$	$c_{32} = 0.23$	$c_{33} = 0.36$	$c_{34} = 0.47$	$c_{35} = 0.57$	$c_{36} = 0.72$
$c_{21} = 0.07$	$c_{22} = 0.14$	$c_{23} = 0.23$	$c_{24} = 0.30$	$c_{25} = 0.37$	$c_{26} = 0.49$
$c_{11} = 0.02$	$c_{12} = 0.05$	$c_{13} = 0.09$	$c_{14} = 0.12$	$c_{15} = 0.16$	$c_{16} = 0.22$

Table 6.2

Since

$$P(X_1 \le X_2) = E[P(X_1 \le X_2 | X_2)] = \sum_{i=1}^{4} \sum_{j=1}^{6} c_{ij} P_{X_2}(y_i, z_j),$$

the lower and upper bounds for $P(X_1 \le X_2)$ can be obtained by solving the minimization and maximization problems (1) with the a_i, b_j and c_{ij} values provided in (56) and Table 6.2.

Taking into account that the c_{ij} values satisfy inverse Monge property, we choose

$$\mathbf{L}_1 = \{\mathbf{a}_{11}, \mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}, \mathbf{a}_{15}, \mathbf{a}_{16}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{41}\}$$

as the initial dual feasible basis for the minimization problem. Carrying out the lexicographic dual method, we obtain a sequence of dual feasible bases. The last one is also primal feasible, whose subscripts are $\{33, 34, 24, 25, 15, 16, 32, 42, 41\}$. Thus the optimum value 0.3232 is the lower bound for $P(X_1 \le X_2)$. Similarly, solving the maximization problem, we obtain a sequence of dual feasible bases The last one is also primal feasible, whose subscripts are $\{11, 12, 22, 23, 24, 34, 35, 36, 46\}$. Thus the optimum value 0.4381 is the upper bound for $P(X_1 \le X_2)$. Therefore, the sharp bounds for $P(X_1 \le X_2)$ are as follows:

$$0.3232 \le P(X_1 \le X_2) \le 0.4381.$$

Example 6.4 In Example 6.3, suppose that the covariance of Y_2 and Z_2 is -5.003. Since

$$E[Y_2] = \textstyle\sum_{i=1}^{4} y_i a_i = 3.34,$$
$$E[Z_2] = \textstyle\sum_{j=1}^{6} z_j b_j = 5.45,$$

$$\begin{aligned} E[Y_2 Z_2] &= \sum_{i=1}^{4}\sum_{j=1}^{6} y_i z_j P_{X_2}(y_i, z_j) \\ &= \mathrm{Cov}(Y_2, Z_2) + E[Y_2]E[Z_2] \\ &= -5.003 + (3.34)(5.45) = 13.2. \end{aligned} \tag{57}$$

Furthermore, the bounding problem can be formulated as the minimization and maximization problems (27) with one more constraint described in (57).

Assume that $c_{ij} = g(y_i, z_j)$, then the (1,2)-order divided difference of $g(y_i, z_j)$ is nonnegative and the (2,1)-order divided difference of $g(y_i, z_j)$ is nonpositive. Then

$$B_4 = \{\mathbf{a}_{16}, \mathbf{a}_{26}, \mathbf{a}_{36}, \mathbf{a}_{46}, \mathbf{a}_{41}, \mathbf{a}_{42}, \mathbf{a}_{43}, \mathbf{a}_{44}, \mathbf{a}_{45}, \mathbf{a}_{11}\}$$

and

$$B_3 = \{\mathbf{a}_{11}, \mathbf{a}_{21}, \mathbf{a}_{31}, \mathbf{a}_{41}, \mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{14}, \mathbf{a}_{15}, \mathbf{a}_{16}, \mathbf{a}_{46}\}$$

are the initial dual feasible bases for the minimization and maximization problems, respectively. Carrying out the lexicographic dual method, we obtain a sequence of dual feasible bases for the minimization and maximization problems, respectively. The subscripts of the last ones, which are also primal feasible, are $\{22, 26, 25, 35, 12, 33, 43, 44, 34, 11\}$ and $\{11, 21, 24, 36, 12, 13, 23,$
$34, 35, 46\}$ for the minimization and maximization problems, respectively. Thus the optimum values are also the bounds for $P(X_1 \le X_2)$ as follows:

$$0.4084 \le P(X_1 \le X_2) \le 0.4337.$$

These bounds improve the ones in Example 6.3.

6.3 The Wasserstein Distance of Two Probability Distributions

The Wasserstein distance between the probability distributions μ and ν, defined in R^n, is the value:

$$W(\mu,\nu) = \inf_{\pi\in\Pi} \sqrt{\int\int \frac{1}{2} d(x,y)^2 d\pi(x,y)}, \tag{58}$$

where Π is the set of all probability distributions in $R^n \times R^n$ with marginals μ and ν, respectively, i.e., if $\pi \in \Pi$, then $\pi(\circ \times R^n)) = \mu$, $\pi(R^n \times \circ)) = \nu$ and $d(x,y)$ is the Euclidean distance between x and y.

Let $n = 1$ and μ, ν be the discrete distributions with supports $\{y_1, \ldots, y_m\}$, $\{z_1, \ldots, z_n\}$ and corresponding probabilities $\{a_i\}$, $\{b_j\}$, respectively. Then

$$\begin{array}{rl} W^2(\mu,\nu) = & \min \ \sum_{i=1}^m \sum_{j=1}^n \frac{1}{2}(y_i - z_j)^2 x_{ij} \\ & \text{subject to} \\ & \sum_{j=1}^n x_{ij} = a_i, \quad i = 1, \ldots, m \\ & \sum_{i=1}^m x_{ij} = b_j, \quad j = 1, \ldots, n \\ & x_{ij} \geq 0, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{array} \tag{59}$$

Assume that both $\{y_i\}$ and $\{z_j\}$ are increasing sequences. It is easy to see that the array

$$c_{ij} = \frac{1}{2}(y_i - z_j)^2, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

has the Monge property (its $(1,1)$-order divided difference is constant, equal to -1).

By Hoffman's result, the greedy algorithm, described in Section 1.3, solves optimally problem (59).

By Theorem 2.1 the objective function value, corresponding to any ordered sequence of the transportation min problem, is a lower bound on $W^2(\mu,\nu)$. That ordered sequence which is also primal feasible, provides us with the exact value of $W^2(\mu,\nu)$.

7 Conclusions

We have presented results in connection with multivariate probability distributions with given marginals and given covariances of the pairs of random variables involved. We have also formulated optimization problems, where the coefficient array of the objective function is supposed to have the Monge, or inverse Monge property, or is a special higher order convex function.

We have reformulated basic results, obtained in connection with Monge arrays, in terms of dual feasible bases in the two-, three- and multidimensional transportation problem. We have also obtained general results in connection with the structures of dual feasible bases in the two- and three-dimensional cases, where the constraints of the transportation problems are supplemented by covariance constraints.

Our results allow for creating lower and upper bounds for unknown entries in partially known Monge and inverse Monge arrays. We have also shown how the results can be used to obtain lower and upper bounds for the expectation of a function of a discrete random vector, where the function is a Monge or inverse Monge array and the univariate marginal distributions are known. In the two- and three-dimensional cases we have obtained improved bounds under the condition that the covariances of the pairs of the random variables are also known. In this case the coefficient array of the objective function is supposed to have some special higher order convexity property.

Illustrative numerical results have been presented and we mentioned some among the many possible applications of our results.

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