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Upper Bound on the Number of Vertices of Polyhedra with $0, 1$-Constraint Matrices

by

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ABSTRACT

In this note we show that the maximum number of vertices in any polyhedron $P = \{x \in \mathbb{R}^d \,:\, Ax \leq b\}$ with $0,1$-constraint matrix A and a real vector b is at most $d!$.

1 Introduction

Let $A \in \{0,1\}^{m\times d}$ be a $0,1$-matrix and $b \in \mathbb{R}^m$ be a given real vector. Consider the polyhedron $P = P(A,b) = \{x \in \mathbb{R}^d \ : \ Ax \le b\}$. Denote by $EXT(P)$ the set of vertices of the polyhedron P. The purpose of this short note is to show the following upper bound on the size of $EXT(P)$.

Theorem 1 *For any $A \in \{0,1\}^{m\times d}$ and any $b \in \mathbb{R}^m$, the number vertices of the polyhedron $P = \{x \in \mathbb{R}^d \ : \ Ax \le b\}$ is at most $d!$.*

The bound of Theorem 1 is tight: consider the polyhedron:

$$P = \Pi_{d-1} + \mathbb{R}_- = \mathrm{conv}\{\sigma \in \mathbb{Z}^d : \ \sigma \text{ is permutation of } [d]\} + \mathrm{cone}\{-\mathbf{e}_1, \ldots, -\mathbf{e}_d\},$$

i.e. the polyhedron with all permutation of $[d]$ as vertices (known as *Permutahedron*) and the negative of the unit vectors as extreme directions (where $\mathrm{conv}(X)$ and $\mathrm{cone}(X)$ denote, respectively, the convex hull and conic hull of a set of vectors X). Then P has $d!$ vertices and it is known [3] that P can have the following linear description with a $0,1$-constraint matrix A:

$$P = \{x \in \mathbb{R}^d \ : \ \sum_{i\in S} x_i \le \sum_{i=1}^{|S|}(d-i+1), \ \text{ for all } \ S \subseteq [d]\}.$$

In the next section we give a few preliminaries that will be needed to prove the Theorem 1. In Section 3, we give the proof of the theorem.

2 Preliminaries

First, we may assume without loss of generality, by translating $P = P(A,b)$ if necessary, that the vector b is strictly positive. To deal with the fact that P is unbounded, we define $\hat{P} = P \cap \{x \in \mathbb{R}^d \ : \ \sum_{i=1}^d x_i \ge -M\}$, for a sufficiently big M. Note that $EXT(\hat{P}) \setminus EXT(P) \subseteq \{x \in \mathbb{R}^d \ : \ \sum_{i=1}^d x_i = -M\}$, and that $0 \in \mathrm{int}(P) \cap \mathrm{int}(\hat{P})$ since $b > 0$, where for $X \subseteq \mathbb{R}^d$, we denote by $\mathrm{int}(P)$ the interior of P. We need the following simple facts, which are well-known and also are not difficult to prove.

Observation 1 *Let Q be a d-dimensional polytope and $z \in \mathrm{int}(Q)$ be a point in the interior of Q. Let $F_1 = \mathrm{conv}\{p_1, \ldots, p_r\}$ and $F_2 = \mathrm{conv}\{q_1, \ldots, q_s\}$ be two facets of Q, where $p_1, \ldots, p_r, q_1, \ldots, q_s \in EXT(Q)$ are vertices of Q. Let $\lambda_1, \ldots, \lambda_r$ and $\mu_1, \ldots, \mu_r$ be positive real numbers. Then*

$$\mathrm{int}(\mathrm{conv}\{z, \lambda_1 p_1, \ldots, \lambda_r p_r\}) \cap \mathrm{int}(\mathrm{conv}\{z, \mu_1 q_1, \ldots, \mu_r q_s\}) = \emptyset.$$

Observation 2 ([1]) *The volume $\mathrm{Vol}^d(Q)$ of any d-dimensional polytope Q with integer vertices is an integer multiple of $\frac{1}{d!}$. In particular $\mathrm{Vol}^d(Q) \ge \frac{1}{d!}$.*

Given a polytope Q which contains $\mathbf{0}$ as an interior point, the *polar* polytope Q° is defined as

$$Q^\circ = \{x \in \mathbb{R}^d \ : \ y^T x \le 1 \ \ \forall y \in Q\},$$

see e.g. [2]. It is known that Q° also contains $\mathbf{0}$ as an interior point, and that the vertices and facets of Q are in one-to-one correspondence with the facets and vertices of Q°, respectively.

3 Proof of Theorem 1

Consider $(\hat{P})^{\circ}$, the polar of the polytope $\hat{P}$ defined by adding a bounding inequality to the polyhedron $P = P(A, b)$ as above. Form polarity, it follows that if $F = \{x \in \mathbb{R}^d \ : \ x \in P, \ \ a_i^T x = b_i\}$ is a facet of P, where a_i^T is the ith row of A and b_i is the ith component of b, then the corresponding polar vertex v_i in $EXT((\hat{P})^{\circ})$ lies on the ray connecting the origin to a_i, at a distance $d(a_i, \mathbf{0}) = \frac{1}{b_i}$ from the origin.

Note that $(\hat{P})^{\circ}$ contains exactly one negative vertex $u = (-\frac{1}{M}, \ldots, -\frac{1}{M})$ which does not correspond to a facet of P. It follows then that the vertices of P are in one-to-one correspondence with the facets of $(\hat{P})^{\circ}$ that do not contain u. Consider any two such facets F_1 and F_2. Then $F_1 = \text{conv}\{\frac{1}{b_{i_1}} a_{i_1}, \ldots, \frac{1}{b_{i_r}} a_{i_r}\}$ and $F_2 = \text{conv}\{\frac{1}{b_{j_1}} a_{j_1}, \ldots, \frac{1}{b_{j_s}} a_{j_s}\}$, where $i_1, \ldots, i_r$ and $j_1, \ldots, j_s$ are indices from $[m]$. Now, since $\mathbf{0} \in \text{int}(\hat{P}^{\circ})$ and F_1 and F_2 are facets of $\hat{P}^{\circ}$, we note by Observation 1 that

$$\text{int}(\text{conv}\{\mathbf{0}, a_{i_1}, \ldots, a_{i_r}\}) \cap \text{int}(\text{conv}\{\mathbf{0}, a_{j_1}, \ldots, a_{j_s}\}) = \emptyset.$$

By Observation 2, we have that the volume $\text{Vol}^d(S(a_{i_1}, \ldots, a_{i_r})\})$ of the polytope $S(a_{i_1}, \ldots, a_{i_r}) = \text{conv}\{\mathbf{0}, a_{i_1}, \ldots, a_{i_r}\}$ arising by "stretching" the vertices of a face $F = \text{conv}\{\frac{1}{b_{i_1}} a_{i_1}, \ldots, \frac{1}{b_{i_r}} a_{i_r}\}$ of $\hat{P}^{\circ}$ is at least $1/d!$. Since all such polytopes $S(a_{i_1}, \ldots, a_{i_r})$ lie completely inside the d-dimensional hypercube, whose volume is 1, we conclude that their number, which is equal to the number of facets of $\hat{P}^{\circ}$ is at most $d!$. This finishes the proof.

It can be easily seen that the above proof can be extended to matrices $A \in \{-K, \ldots, -1, 0, 1, \ldots, K\}^{m \times d}$, in which case the corresponding upper-bound on the number of vertices will be $(2K+1)^d d!$. In particular, if the number of bits to represent a given integer constraint matrix is L, then the number of vertices of the polyhedron described by A is at most $(2^L + 1)^d d!$.

References

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