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Fractional Firefighting in the Two Dimensional Grid

by

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ABSTRACT

We consider a generalization of the firefighter problem where the number of firefighters available per time step t is not a constant. We show that if the number of firefighters available is periodic in t and the average number per time step exceeds $\frac{3}{2}$, then a fire starting at a finite number of vertices in the two dimensional infinite grid graph can be contained.

1 Introduction

A dynamic problem introduced by Hartnell[5], commonly known as the *firefighter problem* can be described as follows. Given a connected rooted graph (G, r), r is initally set on fire at time 0. At the beginning of each subsequent time period $t \geq 1$, $f(t)$ firefighters are positioned at $f(t)$ different vertices that are currently not on fire nor already have a firefighter positioned. These firefighters remain on their assigned vertices and thus prevent the fire from spreading to that vertex. At the end of each time period, all vertices that are not defended and are adjacent to at least one vertex on fire will catch the fire and be burned. Once a vertex is burned or defended, it remains that way permanently.

If G is a finite graph, the process ends when one of the following occurs:

(a) The fire is *contained*, meaning the fire is unable to spread any further, and there are still vertices in G without a firefighter.

(b) The fire spreads until every vertex in G are either burned or defended.

If G is infinite, then (a) could still happen but the second possibility is replaced by:

(b') The fire cannot be contained, meaning the fire spreads indefinitely.

Most of the exisiting literature considers $f(t)$ to be a constant (usually $f(t) = 1$) independent of t. This means that at every time step, the number of firefighters available for deployment is fixed. Under this condition, the firefighter problem can be stated formally as:

FIREFIGHTER
INSTANCE: A rooted graph (G, r) and an integer $k \geq 1$.
QUESTION: Is there a finite sequence $d_1, d_2, ..., d_t$ of vertices of G that can be defended such that at most k vertices are burned at the end of time t?

The firefighter problem was considered on infinite grids $\mathbb{L}_d$ where d is the dimension by Hartke[9], Wang and Moeller[12] and Fogarty[2]. Fogarty[2] and Hartke[9] also considered a modified firefighter problem where fire starts at more than one vertex at time 0. The firefighter problem was also considered on finite grids of dimensions 2 and 3. In particular, MacGillivray and Wang[10] and Wang and Moeller[12] studied

$$\begin{aligned} & MVS(G, v) \\ = \; & \text{maximum number of vertices that can be saved in } G \text{ if fire starts at } v. \end{aligned}$$

for $G = P_n \times P_n$ while Wang and Moeller[12] also considered $G = P_l \times P_m \times P_n$. Another value

$$R(G, v) = \frac{\text{number of vertices that can be saved}}{\text{number of vertices in } G}$$

where the fire starts at v was also studied by Wang and Moeller[12] and Hartke[9]. NP-completeness of the firefighter problem on bipartite graphs was established by MacGillivray and Wang[10]. More recently MacGillivray et.al[11] showed that the firefighter problem is

NP-complete for trees of maximum degree three, but in P for graphs of maximum degree three if v, the vertex where the fire breaks out, is of degree at most two. Hartnell and Li[7] showed that when G is a tree, the greedy algorithm is a 2-approximation algorithm to an optimum algorithm that saves the maximum number of vertices in G. Finbow and Hartnell[1] and Hartnell et.al[8] considered the problem of constructing graphs that minimizes the effect of fire spreading. Other related publications include [3, 4, 6].

For a two dimensional infinite grid $\mathbb{L}_2$, Wang and Moeller[12] showed that $f(t) = 1$ for all $t \geq 1$ (one firefighter per time step) is insufficient to prevent the fire from spreading indefinitely while $f(t) = 2$ for all t suffices. It was further proven by Hartke[9] that when $f(t) = 2$ for all t, a minimum of 8 time steps are required to successfully contain the fire, resulting in a minimum of 18 vertices being burned. In this paper, we propose a generalization of the firefighter problem, called the *fractional firefighter problem* where $f(t)$ is no longer a constant. We will consider the two dimensional infinite grid $\mathbb{L}_2$ defined by

$$V = \mathbb{Z} \times \mathbb{Z}$$

$$E = \{((m,n),(m',n'))||m - m'| + |n - n'| = 1\}$$

Our function $f(t)$ is *periodic* with period p and for each $t \geq 1$, $f(t) \in \mathbb{N}$. This allows us to define the following *firefighter ratio*

$$R(f) = \frac{\sum_{i=1}^{p} f(i)}{p}$$

which is simply the average number of firefighters we have for deployment at each time step. For any given (G, r), a general question that can be asked is if there is a real number $R(G, r)$ such that any function $f(t)$ with ratio $R(G, r)$ cannot contain the fire, yet any function $g(t)$ with ratio greater than $R(G, r)$ can.

For the case when $G = \mathbb{L}_2$, we assume without loss of generality that $r = (0, 0)$. In this paper, we shall show that any function $f(t)$ with ratio greater than $\frac{3}{2}$ is sufficient to contain the fire in $\mathbb{L}_2$. Thus, if $R(\mathbb{L}_2, r)$ exists, then $1 \leq R(\mathbb{L}_2, r) \leq \frac{3}{2}$.

2 Terminology, Notation and Assumptions

Throughout this paper, we let $G = \mathbb{L}_2$ and assume that the fire starts at $r = (0, 0)$. For $m \geq 1$, let D_m represent the set of vertices in G that are at distance m from r. The distance between two vertices in G is defined to be the length of a shortest path between them. A vertex of $\mathbb{L}_2$ that is on fire and has at least one adjacent vertex not on fire and without a firefighter positioned there is called an *active* vertex.

All functions $f(t)$, $t \geq 1$, are assumed to be periodic and we write f instead of $f(t)$ if there is no confusion. We identify f with a sequence of its period. Therefore, [2,1,2,2] corresponds to a function f such that for $k \geq 0$,

$$\begin{aligned} f(4k+1) &= 2 \\ f(4k+2) &= 1 \\ f(4k+3) &= 2 \\ f(4k) &= 2 \end{aligned}$$

We see that the firefighter ratio for this example is 1.75. There is a *partial order* associated with these functions, defined by

$$f \preceq g \iff \forall n \in \mathbb{N} \left(\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} g(i) \right)$$

If f and g have periods p_f and p_g respectively, then we see that

$$f \preceq g \iff \forall n, 1 \leq n \leq lcm(p_f, p_g) \left(\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} g(i) \right)$$

For each $n \in \mathbb{N}$, we define f_n to be a function of period $2n+1$ as follows:

$$f_n(i) = \begin{cases} 1 & \text{if } 1 \leq i \pmod{2n+1} \leq n; \\ 2 & \text{otherwise.} \end{cases}$$

For example, $f_1 = [1, 2, 2]$ and $f_2 = [1, 1, 2, 2, 2]$. The firefighter ratio for f_n is

$$\frac{n + 2(n+1)}{2n+1} = \frac{3n+2}{2n+1} = 1 + \frac{n+1}{2n+1}$$

which we see is always greater than 1.5, and that the ratio approaches 1.5 as n approaches infinity.

In the next section, we will show that for each $n \in \mathbb{N}$, the sequence of firefighters represented by the function f_n is always sufficient to contain a fire that starts at $(0, 0)$. In fact, a simple corollary will show that even if the fire starts at any finite number of vertices, f_n is still sufficient to contain the fire. Our main goal is to provide a strategy detailing the placement of the firefighters at each time step.

3 The strategy

The complete strategy is divided into *phases*, described below:

(i) **Phase 1 (P1)** The fire starts at $(0, 0)$. During phase 1, at each (odd) time $t = 2k+1$, $k \geq 0$, one firefighter must be positioned at $(k, -k-1)$. These firefighters are called *retreat firefighters* as they make sure that the fire does not wrap around the 'firewall' (a contiguous line of firefighters) from below. All other firefighters positioned during this phase are called *advance firefighters*. Advance firefighters' role is to be aggressive and gain as much ground as they can on the fire. The positions of advance firefighters must satisfy the following:

(a) At time $t = 2$, one advance firefighter must be positioned at $(-1, -1)$.

(b) If there is one advance firefighter at (x_1, y_1), then there must be one at (x_2, y_2) such that $x_2 = x_1 - 1$ and $|y_2 - y_1| \leq 1$.

Phase 1 is completed when one advance firefighter is positioned at $(-C_1, 0)$ for some $C_1 > 0$.

(ii) **Phase 2 (P2)** In phase 2, retreat firefighters continue to be positioned in the same way as in phase 1. Advance firefighters now attempt to 'overtake' the progress of the fire at some point directly above the root. Specifically, the positions of advance firefighters placed during this phase must satisfy the following:

(c) Continuing on from the position $(-C_1, 0)$ in phase 1, if there is one advance firefighter at (x_1, y_1), then there must be one at (x_2, y_2) such that $y_2 = y_1 + 1$ and $|x_2 - x_1| \leq 1$.

Phase 2 is completed when one advance firefighter is positioned at $(0, C_2)$ for some $C_2 > 0$.

(iii) **Phase 3 (P3)** In phase 3, retreat firefighters continue to be positioned in the same way. Advance firefighters, after having overtaken the fire at the top, now starts its move towards the diagonal line of retreat firefighters. This commences the 'closing up' stage. Specifically, the positions of advance firefighters placed during this phase satisfies the following:

(d) Continuing on from the position $(C_2, 0)$ in phase 2, if there is one advance firefighter at $(x_1, y_1$, then there must be one at (x_2, y_2) such that $x_2 = x_1 + 1$ and $|y_2 - y_1| \leq 1$.

Phase 3 is completed when one advance firefighter is positioned at $(C_3, 0)$ for some $C_3 > 0$.

(iv) **Phase 4 (P4)** In this final phase, while the retreat firefighters continue to prevent the fire from wrapping around the firewall, advance firefighters continue its 'descend' to meet the retreat firefighters. This 'closes up' the boundary of firefighters and prevents any further spread of the fire.

Suppose for some $n \in \mathbb{N}$, the sequence of firefighters available for deployment is f_n. We now proceed to prove that each of the 4 phases P_i, can be completed at some finite time t_i, $i = 1, 2, 3, 4$.

Proposition 3.1 Phase 1 can be completed after $t_1 = 2n$ time steps.

Proof: If $n = 1$, phase 1 can be completed by positioning firefighters at (0,-1) when $t = 1$, (-1,-1) and (-2,0) when $t = 2$. Let us first consider the case when $n \geq 2$ is even:

During the first n time steps, we have one firefighter to deploy per time step. For $t = 2k+1, 0 \le k \le \frac{n}{2} - 1$, position one firefighter at $(k, -k-1)$, these are the retreat firefighters. For $t = 2k, 1 \le k \le \frac{n}{2}$, position one firefighter at $(-k, -k)$. Note that all these positions for the firefighters are valid, (that is, the positions are not on fire at the time a firefighter is positioned there) as each position chosen at time t is precisely at distance t from $(0, 0)$.

For $t = n+1, n+2, ..., 2n$, we have two firefighters to deploy per time step. We continue to deploy one firefighter at $(k, -k-1)$ at times $t = 2k+1, \frac{n}{2} \le k \le n-1$. We are left with one firefighter for deployment at times $t = 2k+1, \frac{n}{2} \le k \le n-1$ and two firefighters for deployment at times $t = n+2, n+4, ..., 2n$. We write these times as $n+m, m = 1, 2, ..., n$. For $m = 1, 3, ..., n-1$, we have one firefighter and we deploy it at $(-\frac{n}{2} - \frac{3m-1}{2}, -\frac{n}{2} + \frac{m-1}{2})$. For $m = 2, 4, ..., n$, we have two firefighters and we deploy them at $(-\frac{n}{2} - \frac{3m-2}{2}, -\frac{n}{2} + \frac{m}{2} - 1)$ and $(-\frac{n}{2} - \frac{3m}{2}, -\frac{n}{2} + \frac{m}{2})$. It is easy to verify that the positions of these firefighters are again valid, as the position of a firefighter deployed at time t is again at distance t from $(0, 0)$. With the described deployment, we see that when $m = n$, $t = 2n$ and the last firefighter deployed would be at $(-2n, 0)$ (so $C_1 = 2n$). For illustrative purposes, we show the positions of the firefighters deployed for the case when $n = 6$ in Figure 1. The filled circle represents the position of (0,0) while the number inside each empty circle represents the time step when the firefighter is positioned there.

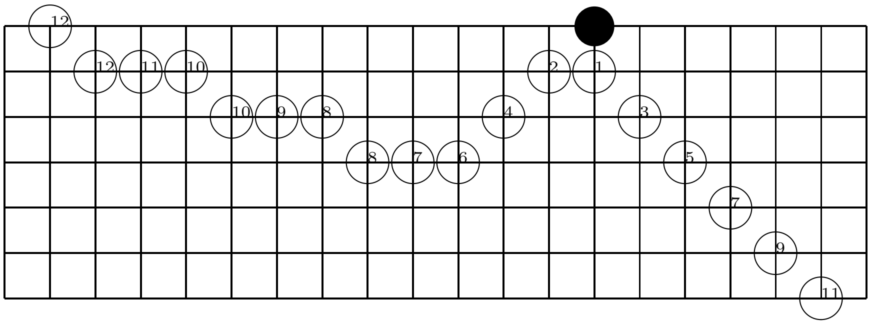


Figure 1

The case when n is odd is very similar. For the first n time steps, we have one firefighter to deploy per time step. For $t = 2k+1, 0 \le k \le \frac{n-1}{2}$, position one firefighter at $(k, -k-1)$ (retreat firefighters). For $t = 2k, 1 \le k \le \frac{n-1}{2}$, position one firefighter at $(-k, -k)$. For $t = n+1, n+2, ..., 2n$, we have two firefighters to deploy per time step. We continue to deploy one firefighter at $(k, -k-1)$ at times $t = 2k+1, \frac{n+1}{2} \le k \le n-1$. We are left with one firefighter for deployment at times $t = 2k+1, \frac{n+1}{2} \le k \le n-1$ and two firefighters for deployment at times $t = n+1, n+3, ..., 2n$. We write these times as $n+m, m = 1, 2, ..., n$. For $m = 2, 4, ..., n-1$, we have one firefighter and we deploy it at $(-\frac{n}{2} - \frac{3m-1}{2}, -\frac{n}{2} + \frac{m-1}{2})$. For $m = 1, 3, ..., n$, we have two firefighters and we deploy them at $(-\frac{n}{2} - \frac{3m-2}{2}, -\frac{n}{2} + \frac{m}{2} - 1)$ and $(-\frac{n}{2} - \frac{3m}{2}, -\frac{n}{2} + \frac{m}{2})$. Note again that all these positions for the firefighters are valid. With the described deployment, we see that when $m = n$, $t = 2n$ and the last firefighter deployed would be at $(-2n, 0)$. For illustrative purposes, we show the positions of the firefighters deployed for the case when $n = 7$ in Figure 2.

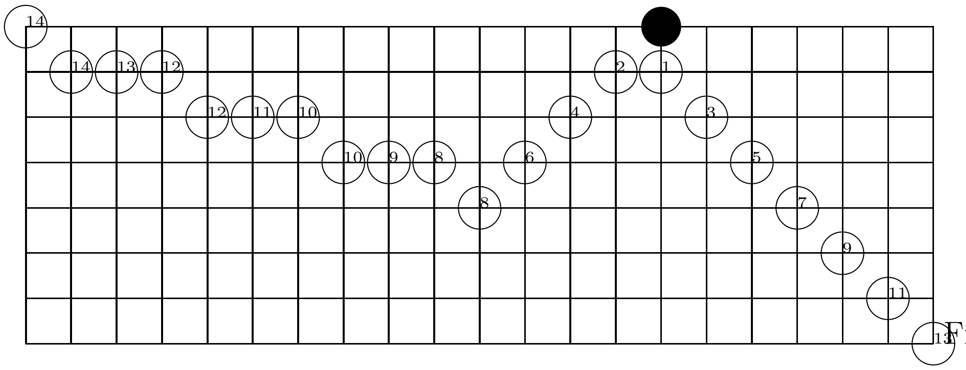


Figure 2

This proves Proposition 3.1.

□

Before we proceed to prove that phase 2 can be completed at some finite time t_2, we first consider a slightly modified version of the firfighter problem. Supposed the fire still starts at (0,0), but the arrival of the firefighters is *delayed* in the sense that the deployment of the firefighters does not commence until some time $d > 1$. This means that by the time the sequence of firefighters corresponding to f_n is available for deployment, all vertices in D_i, $i = 1, ..., d-1$ are already on fire. This is equivalent to considering the firefighter problem where we can start deploying firefighters at $t = 1$, but all the vertices in $\cup_{i=1}^{d} D_i$ are initially set on fire.

Proposition 3.2 Suppose all vertices in $\cup_{i=1}^{d} D_i$ are initially set on fire and f_n corresponds to the firefighter sequence. The firefighters can be postioned in such a way that at time $t = d(4n+2)$, the following conditions are satisfied:

(i) At each (odd) time $t = 2k+1$, $k \geq 0$, one retreat firefighter must be positioned at $(k, -d-k-1)$.

(ii) At time $t = 2$, one advance firefighter is positioned at $(-1, -d-1)$.

(iii) If there is one advance firefighter at (x_1, y_1), then there must be one at (x_2, y_2) such that $x_2 = x_1 - 1$ and $|y_2 - y_1| \leq 1$.

(iv) At time $t = d(4n+2)$, there is one advance firefighter at $(-d(4n+3), 0)$.

Proof: Due to the periodic nature of the number of firefighters available for deployment we will only provide a strategy for deployment for the first $4n+2$ time steps. The way the firefighters are deployed beyond that follows a similar pattern. The strategy for deployment is very similar to that given in Proposition 3.1, the only difference is that the positions of the firefighters are somewhat "shifted down" due to the vertices in $\cup_{i=1}^{d} D_i$ being initially set on fire. First consider when n is even:

(a) For $t = 2k+1$, $0 \leq k \leq \frac{n}{2} - 1$, (that is, $t = 1, 3, 5, ..., n-1$) we have one firefighter to deploy. Position the firefighter at $(k, -d-k-1)$.

(b) For $t = 2k$, $1 \leq k \leq \frac{n}{2}$, (that is, $t = 2, 4, 6, ..., n$) we have one firefighter to deploy. Position the firefighter at $(-k, -d-k)$.

(c) For $t = n+m$, where m is odd and $1 \leq m \leq n+1$, (that is, $t = n+1, n+3, ..., 2n+1$) we have two firefighters to deploy. At time $t = n+m = 2k+1$, $1 \leq m \leq n+1$, $\frac{n}{2} \leq k \leq n$, position one firefighter each at $(k, -d-k-1)$ and $(-\frac{n}{2} - \frac{3m-1}{2}, -d - \frac{n}{2} + \frac{m-1}{2})$.

(d) For $t = n+m$, where m is even and $2 \leq m \leq n$, (that is, $t = n+2, n+4, ..., 2n$) we have two firefighters to deploy. We deploy them at $(-\frac{n}{2} - \frac{3m-2}{2}, -d - \frac{n}{2} + \frac{m}{2} - 1)$ and $(-\frac{n}{2} - \frac{3m}{2}, -d - \frac{n}{2} + \frac{m}{2})$.

(e) For $t = 2(n+k)+1$, $1 \leq k \leq \frac{n}{2}$, (that is, $t = 2n+3, 2n+5, ..., 3n+1$) we have one firefighter to deploy. Position the firefighter at $(n+k, -d-n-k-1)$.

(f) For $t = 2(n+k)$, $1 \leq k \leq \frac{n}{2}$, (that is, $t = 2n+2, 2n+4, ..., 3n$) we have one firefighter to deploy. Position the firefighter at $(-2n-k-1, -d-k+1)$.

(g) For $t = 3n+m$, where m is odd and $3 \leq m \leq n+1$, (that is, $t = 3n+3, 3n+5, ..., 4n+1$) we have two firefighters to deploy. At time $t = 3n+m = 2(n+k)+1$, $3 \leq m \leq n+1$, $\frac{n}{2}+1 \leq k \leq n$, position one firefighter each at $(n+k, -d-n-k-1)$ and $(-2n - \frac{n}{2} - \frac{3m-1}{2}, -d - \frac{n}{2} + \frac{m-1}{2})$.

(h) For $t = 3n+m$, where m is even and $2 \leq m \leq n+2$, (that is, $t = 3n+2, 3n+4, ..., 4n+2$) we have two firefighters to deploy. We deploy them at $(-2n - \frac{n}{2} - \frac{3m-2}{2}, -d - \frac{n}{2} + \frac{m}{2} - 1)$ and $(-2n - \frac{n}{2} - \frac{3m}{2}, -d - \frac{n}{2} + \frac{m}{2})$.

Observe now that at the end of time $t = 4n+2$, by putting $m = n+2$ in (h) above, we would have a firefighter at $(-4n-3, -d+1)$. This proves the propostion when $d = 1$ and for $d > 1$, we just repeat the strategy above for another $(d-1)(4n+2)$ time steps.

The case when n is odd is again very similar. We again provide a strategy for deployment of the firefighters for the first $4n+2$ time steps.

(a) For $t = 2k+1$, $0 \leq k \leq \frac{n-1}{2}$, (that is, $t = 1, 3, 5, ..., n$) we have one firefighter to deploy. Position the firefighter at $(k, -d-k-1)$.

(b) For $t = 2k$, $1 \leq k \leq \frac{n-1}{2}$, (that is, $t = 2, 4, 6, ..., n-1$) we have one firefighter to deploy. Position the firefighter at $(-k, d-k)$.

(c) For $t = n+m$, where m is odd and $1 \leq m \leq n$, (that is $t = n+1, n+3, ..., 2n$) we have two firefighters to deploy. We deploy them at $(-\frac{n-1}{2} - \frac{3m-1}{2}, -d - \frac{n-1}{2} + \frac{m-1}{2} - 1)$ and $(-\frac{n-1}{2} - \frac{3m-1}{2} - 1, -d - \frac{n-1}{2} + \frac{m-1}{2})$.

(d) For $t = n+m$, where m is even and $2 \leq m \leq n+1$, (that is, $t = n+2, n+4, ..., 2n+1$) we have two firefighters to deploy. At time $t = n+m = 2k+1$, $2 \leq m \leq n+1$, $\frac{n+1}{2} \leq k \leq n$, position one firefighter each at $(k, -d-k-1)$ and $(-\frac{n-1}{2} - \frac{3m}{2}, -d - \frac{n-1}{2} + \frac{m}{2} - 1)$.

(e) For $t = 2(n+k)$, $1 \le k \le \frac{n+1}{2}$, (that is, $t = 2n+2, 2n+4, ..., 3n+1$) we have one firefighter to deploy. Position the firefighter at $(-2n-k-1, -d+k+1)$.

(f) For $t = 2(n+k)+1$, $1 \le k \le \frac{n-1}{2}$, (that is, $t = 2n+3, 2n+5, ..., 3n$) we have one firefighter to deploy. Position the firefighter at $(n+k, -d-n-k-1)$.

(g) For $t = 3n+m$, where m is even and $2 \le m \le n+1$, (that is, $t = 3n+2, 3n+4, ...4n+1$) we have two firefighters to deploy. At time $t = 3n+m = 2(n+k)+1$, $2 \le m \le n+1$, $\frac{n+1}{2} \le k \le n$, position one firefighter each at $(n+k, -d-n-k-1)$ and $(-2n - \frac{n+1}{2} - \frac{3m}{2} + 1, -d - \frac{n+1}{2} + \frac{m}{2})$.

(h) For $t = 3n+m$, where m is odd and $3 \le m \le n+2$, (that is, $t = 3n+3, 3n+5, ..., 4n+2$) we have two firefighters to deploy. We deploy them at $(-2n - \frac{n+1}{2} - \frac{3(m-1)}{2}, -d - \frac{n+1}{2} + \frac{m-1}{2})$ and $(-2n - \frac{n+1}{2} - \frac{3(m-1)}{2} - 1, -d - \frac{n+1}{2} + \frac{m-1}{2} + 1)$.

Observe now that at the end of time $t = 4n+2$, by putting $m = n+2$ in (h) above, we would have a firefighter at $(-4n-3, -d+1)$. This proves the proposition when $d = 1$ and for $d > 1$, we just repeat the strategy above for another $(d-1)(4n+2)$ time steps.

This completes the proof of the Proposition 3.2.

$\square$

Proposition 3.3 Phases 1 and 2 can be completed in $t_2 = (2n)(4n+2)$ time steps.

Proof: Figure 3 shows $\mathbb{L}_2$ after the completion of phase 1. The firefighter at $(-2n, 0)$ has just been positioned at time $t_1 = 2n$ and the area enclosed by the bold lines indictates the area where vertices are burned after time $t = 2n$.

Figure 3

At time $t = 2n+1$, we have two firefighters to deploy. As usual, deploy one as a retreat firefighter at $(n, -n-1)$ and deploy the other at $(-2n, 1)$. Figure 4 is essentially the same as Figure 3, only that the two firefighters at time $t = 2n+1$ have been deployed and we rotated Figure 3 by 90^o in the counter-clockwise direction.

Notice that to complete phase 2, we need to progress the sequence of advance firefighters 'upwards' (by looking at Figure 4) so that it reaches the same 'horizontal' (again with respect to Figure 4) level as $(0, 0)$, while at the same time, continuing to deploy one retreat firefighter at every odd time t. This is similar to the scenario in Proposition 3.2 where the advance firefighters have $2n$ levels to move upwards before being able to position one at $(0, C_2)$.

We could calculate explicitly C_2 and the time t_2 by using Proposition 3.2. The number of firefighters available at $t = 2n + 2$ is 1, as we are at the beginning of a new period. The number of time periods needed to move up 1 level is one less than what was calculated in Proposition 3.2 because of the firefighter already at $(-2n, 1)$ saves us 1 move. Due to the periodic nature of f_n, we see that phase 2 can be completed at $t_2 = 2n$ (for phase 1) $+$ $2n(4n+1)$ (for phase 2) $= 2n(4n+2)$. Thus, $C_2 = 2n(4n+2)$ and the proof of Proposition 3.3 is complete.

Figure 4

□

Proposition 3.4 Phases 1, 2 and 3 can be completed in $t_3 = 2n(4n + 2)(4n + 3)$ time steps.

Proof: Figure 5 shows $\mathbb{L}_2$ after the completion of phases 1 and 2. The firefighter at $(0, 2n(4n + 2))$ has just been positioned at time $t_2 = 2n(4n + 2)$ and the area enclosed by the bold lines indicates the area where vertices are burned after $t = 2n(4n + 2)$.

Note that at the next time step $t = 2n(4n+2)+1$, we are at the beginning of the period and have 1 firefighter for deployment. Figure 6 is the same as Figure 5, but rotated by 180^o. Similar to the proof of Proposition 3.3, we see that to complete phase 3, we need to progress the sequence of advance firefighters ‘upwards’ (by looking at Figure 6) so that it reaches the same 'horizontal' (again with respect to Figure 6) level as $(0,0)$. By Proposition 3.2, we know that this can be accomplished in $d(4n+2)$, where $d = 2n(4n+2)$, time steps.

Thus, phases 1, 2 and 3 can be completed at $t_3 = 2n(4n + 2)$ (for phases 1 and 2) $+ 2n(4n+2)^2$ (for phase 3) $= 2n(4n+2)(4n+3)$. We have $C_3 = 2n(4n+2)(4n+3)$ and the proof of Proposition 3.4 is complete.

Figure 5

Figure 6

$\square$

Proposition 3.5 Phases 1, 2, 3 and 4 can be completed in $t_4 = 64n^3 + 64n^2 + 20n$ time steps.

Proof: Throughout this proof, we shall denote $2n(4n + 2)(4n + 3)$ by C_3. At the *end* of time $t_3 = C_3$, we have positioned an advance firefighter at $(C_3, 0)$, the retreat firefighter furthest from r is at $(\frac{C_3}{2} - 1, -\frac{C_3}{2})$. Furthermore, the set of all active vertices are $\{(\frac{C_3}{2} + i, -\frac{C_3}{2} + i) | i = 0, 1, ..., \frac{C_3}{2} - 1\} \subseteq D_{C_3}$.

Figure 7 shows the positions of the following:

(a) The retreat firefighter furthest from r, positioned during time $t = C_3 - 1$ (marked 1);

(b) The advance firefighter at $(C_3, 0)$, positioned during time $t = C_3$ (marked 2);

(c) The active vertices, at the end of time $t = C_3$, forming a diagonal from $(\frac{C_3}{2}, -\frac{C_3}{2})$ (marked 3) to $(C_3 - 1, -1)$ (marked 4). Note that there are exactly $\frac{C_3}{2}$ active vertices.

Figure 7

We first show that if for $t \geq C_3+1$, we only have one firefighter per turn, then the number of active vertices at the end of each turn can be kept a constant. To see this, consider Figure 8, which shows only the retreat firefighter furthest from r at time $t = C_3$, the advance firefighter at $(C_3, 0)$ and for simplicity, only 3 active vertices.

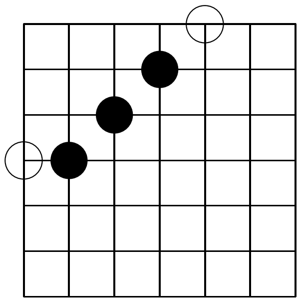


Figure 8

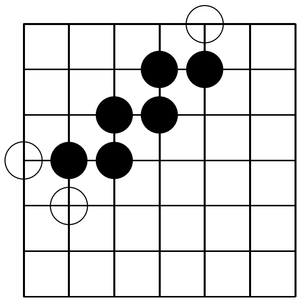


Figure 9

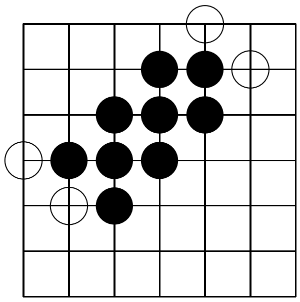
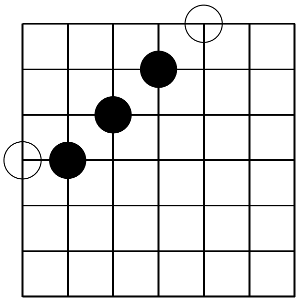


Figure 10

At time $t = C_3 + 1$, we position one retreat firefighter at $(\frac{C_3}{2}, -\frac{C_3}{2} - 1)$, the fire spreads and we still have 3 active vertices (Figure 9). At time $t = C_3 + 2$, we position one advance firefighter at $(C_3 + 1, -1)$, the fire spreads and again we have 3 active vertices (Figure 10). We are back at the same situation as in Figure 8. Thus if we have only one firefighter per time step from $t \geq C_3 + 1$, the number of active vertices can be kept at a constant of $\frac{C_3}{2}$.

We now proceed to show that if the number of firefighters corresponds to the function f_n, phases 1,2,3 and 4 can be completed in $t_4 = 64n^3 + 64n^2 + 20n$ time steps. Note that at

$t = C_3 + 1$, we are at the beginning of the period again. As discussed above, at each time t where there is only one firefighter for deployment does not reduce the number of active vertices. On the other hand, at each t where there are two firefighters for deployment, the number of active vertices is reduced by exactly one. This is regardless of whether t is even or odd. Figures 11 and 12 illustrates this observation.



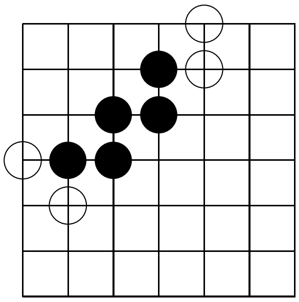
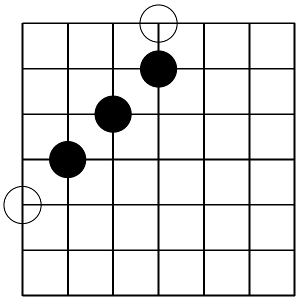


Figure 11(a) (t odd) - beginning of t Figure 11(b) (t odd) - end of t



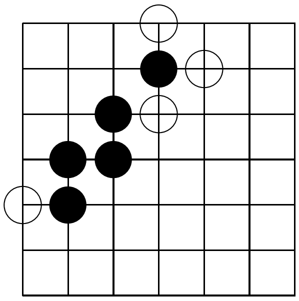


Figure 12(a) (t even) - beginning of t Figure 12(b) (t even) - end of t

Since there are $\frac{C_3}{2}$ active vertices at the beginning of time $t = C_3 + 1$, and in each period, there are $n + 1$ time steps where we have two firefighters for deployment, it can be easily verified that a total of $(16n^2 + 4n + 1)$ periods and an additional $2n - 1$ turns in the next period is required. Thus,

$$
\begin{aligned}
t_4 &= t_3 + (2n+1)(16n^2 + 4n + 1) + 2n - 1 \\
&= 2n(4n+2)(4n+3) + 32n^3 + 24n^2 + 8n \\
&= 64n^3 + 64n^2 + 20n
\end{aligned}
$$

This completes the proof of Proposition 3.5.

□

Corollary 3.6 Suppose at time $t = 0$, fire breaks out at a finite number of vertices in $\mathbb{L}_2$. For any $n \in \mathbb{N}$, a sequence of firefighters corresponding to f_n is able to contain the fire.

Proof: Let $M = \max\{d((0,0),(x,y)) | (x,y) \text{ is on fire at time } 0\}$. By Proposition 3.2, if we assume all vertices in $\cup_{i=1}^{M} D_i$ are initially set on fire, then f_n allows us to complete phase

1. Phases 2,3 and 4 can also be completed by arguments similar to Propositions 3.3 to 3.5, even though the time it takes for the completion of these phases are different.

□

The next two lemmas are required in order to prove the final theorem in this paper.

Lemma 3.7 Let f and g be two functions corresponding to two periodic sequences with periods p_f and p_g respectively. We assume $f(i), g(i) \geq 1$ for all i. If $f \preceq q$ and the fire can be contained using f, then the fire can be contained using g.

Proof: Let $s = lcm(p_f, p_g)$. If $f(i) \leq g(i)$ for each $1 \leq i \leq s$, then using g ,we simply follow the deployment strategy for f. Additional firefighters at each time step can be positioned arbitrarily, and the fire can be contained. If there exists $k \geq 2$ such that $f(k) > g(k)$, let $k^* = \min\{k | f(k) > g(k)\}$ and $x = f(k^*) - g(k^*)$. Since $f \preceq g$, we have

$$\sum_{i=1}^{k^*-1} g(i) - f(i) \geq x.$$

To contain the fire using g, we follow the deployment strategy for f for time $t = 1, 2, ..., k^*-1$. There are at least x 'extra' firefighters during these time steps and we position them 'in advance' in accordance to the deployment strategy of f at time k^*. So, together with the $g(k^*)$ firefighters, we will be able to follow the deployment strategy of f at time k^*. All other k such that $f(k) > g(k)$ can be treated similarly.

□

Lemma 3.8 If g is a function with firefighter ratio strictly greater than $\frac{3}{2}$, then there is an $n \in \mathbb{N}$ such that $f_n \preceq g$.

Proof: Let p_g be the period of g. If p_g is odd, then $f_{\frac{p_g-1}{2}} \preceq g$. If p_g is even, then $f_{\frac{p_g}{2}-1} \preceq g$.

□

Finally, we have the following theorem.

Theorem 3.9 If $g(i) \geq 1, \forall i$ is a periodic sequence with ratio strictly greater than $\frac{3}{2}$, then $g(i)$ is able to contain the fire that breaks out at $(0, 0)$. Consequently, if $R(\mathbb{L}_2, r)$ exists, then $1 \leq R(\mathbb{L}_2, r) \leq \frac{3}{2}$.

□

Remarks:

(1) Note that even if the fire breaks out at a finite number of vertices in $\mathbb{L}_2$, $g(i)$ is still able to contain the fire.

(2) An algorithm implementing the strategy described in this section was written in C language and tested. All the completion times for the four phases were tested experimentally and found to be correct. A pseudocode for the algorithm can be found in Appendix A.

4 Conclusion

In this paper, we have introduced a generalized firefighter problem where the number of firefighters available for deployment per time step, $f(t)$, does not have to be a constant. We specifically looked at the two dimensional infinite grid graph and attempted to 'reconcile' the results for $f(t) = 1$ for all t (not able to contain fire) and $f(t) = 2$ for all t (able to contain fire). This lead us to consider periodic functions f and the definition of $R(G, r)$ for any rooted graph (G, r). Although it is still unknown as to whether $R(\mathbb{L}_2, r)$ exists, we believe that it does, and we conclude this paper with the following conjecture.

Conjecture If $g(i) \geq 1, \forall i$ is a periodic sequence with ratio less than or equal to $\frac{3}{2}$, then $g(i)$ is unable to contain the fire that breaks out at $(0, 0)$. Thus $R(\mathbb{L}_2, r)$ exists and is equal to $\frac{3}{2}$.

Appendix A

```
Input:  Turn #, phase
Algorithm:
F := f(turn #) [number of fighters]
If (turn # = 2k+1 for some k) then
            place fighter at (k,-(k+1))
            F ← F-1
fi

while F > 0 do
(x,y) = location of the fighter on the edge of the advance line
            if phase = 1 then
                        if(x-1,y) is on fire then
                                    place fighter at (x-1,y-1)
                        else if (x-1,y+1) is on fire then
                                    place fighter at (x-1,y)
                        else
                                    place fighter at (x-1,y+1)
                        fi
                        F ← F-1
                        CheckPhase()
            else if phase = 2 then
                        if(x,y+1) is on fire then
                                    place fighter at (x-1,y+1)
                        else if (x+1,y+1) is on fire then
                                    place fighter at (x,y+1)
                        else
                                    place fighter at (x+1,y+1)
                        fi
                        F ← F-1
                        CheckPhase()
            else if phase = 3 then
                        if(x+1,y) is on fire then
                                    place fighter at (x+1,y+1)
                        else if (x+1,y-1) is on fire then
                                    place fighter at (x+1,y)
                        else
                                    place fighter at (x+1,y-1)
                        fi
                        F ← F-1
                        CheckPhase()
            else
```

```
                        if(x+1,y) is on fire then
                                    place fighter at (x+1,y-1)
                        else if (x+1,y-1) is on fire then
                                    place fighter at (x+1,y)
                        else
                                    place fighter at (x+1,y+1)
                        fi
                        F ← F-1
            fi
od

CheckPhase()
if there is a fighter at (k,0) for k>0 then
            phase = 4
else if there is a fighter at (0,k) for k > 0 then
            phase = 3
else if there is a fighter at (k,0) for k < 0 then
            phase = 2
else
            phase = 1
```

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