DIMACS Technical Report 2005-15
May, 2005

Santa Claus' Towers of Hanoi

by

Xiaomin Chen [1,2]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

Bin Tian [3]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

Lei Wang [4]
Dept. of Computer Science
Rutgers University
New Brunswick, New Jersey 08903

[1]xiaomin@paul.rutgers.edu
[2]Supported by (theory of computation) NSF grant 0523866.
[3]tian@paul.rutgers.edu
[4]lewang@pail.rutgers.edu

ABSTRACT

Two new variants of the Towers of Hanoi problem are proposed. In both variations, one is allowed to put a bigger disk directly on the top of a smaller one under some restrictions. We give procedures to solve these two versions, and prove the optimality of our procedures. Our solution also resolves a problem, which is similar to one of our versions, proposed by D. Wood twenty-four years ago.

1 The Towers of Hanoi and Two New Variations

The well known towers of Hanoi, treasured by most mathematicians and many others as their childhood toy for the mental gymnastics, was invented by the French mathematician Edouard Lucas in 1883. In the legend, 64 sacred disks were initially stacked in increasing size on one of three pegs, with the largest at the bottom. A monk has to move the entire tower to another peg. The disks are fragile; only one can be carried at a time. And most importantly, the monk must obey the following rule:

> *The Divine Rule*: A disk may not be placed on top of a smaller disk.

The disks are of size $1, 2, \ldots, n$, usually we denote the disk of size i by $\langle i \rangle$; and we denote the set of disks by $[n] = \{\langle i \rangle : 1 \leq i \leq n\}$. The three pegs are P_1, P_2, and P_3.

The Towers of Hanoi and its many variations are well studied. Here we just mention a few of them: The cyclic towers of Hanoi was first studied by Atkinson [1]. In [6], Klein and Minsker solved the variation where there can be bigger disks above smaller disks in the initial configuration, but the moves still obey the divine rule. The multi-peg Towers of Hanoi problem was proposed by Stewart [7], and remains a big open problem till today. (See related works by Stewart [8], Frame [4], Szegedy [10], Klavžar, Milutinović, and Petr [5], and Chen and Shen [2].) For a good bibliography with more than 200 entries on this subject, see Paul Stockmeyer's manuscript ([9]), which can be found on the Internet.

In this article, we study two new variations of the game. Both variations are on three pegs, and the monk may violate the divine rule slightly in some way.

In the *sinner's mode*, the divine rule may be violated a certain number of times. There is a prescribed number k. And the rule becomes

> *Sinner's Rule with k cheats*: For at most k times a disk may be placed directly on the top of a smaller disk.

In the *Santa Claus'* mode we have, maybe for a better aesthetic reason, another slightly imperfect rule. There is a number d. And the rule is

> *Santa's Rule with discrepancy d*: When a disk $\langle x \rangle$ is put on a pile of other disks, before the move, disks smaller than $\langle x \rangle$ can only occur at the top up to $d - 1$ positions in that pile, but none of them can be of size less than or equal to $x - d$.

We justify the two new versions based on two reasons. In our age, not so many people care about pure divine rules. Besides, Santa Claus likes Christmas trees. It is interesting to point out that when E. Lucas wrote about the towers of Hanoi, he used the name N. Claus.

As we searched through the literature, we believe the sinner's tower has not been studied before. In Section 3 we give a procedure for this problem and prove the optimality of our procedure. In the literature we found a variant that is very similar to Santa's tower in [11]. We reword it here as the *generous Santa Claus' mode*.

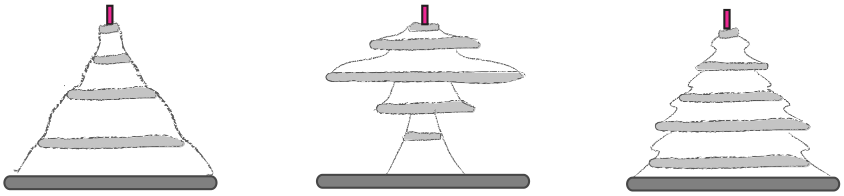

Figure 1: (a) a peg in the classical Hanoi Towers (b) a peg in the Sinner's mode (c) a peg in Santa Claus' mode.

> *Generous Santa's Rule with discrepancy* d: When a disk $\langle x \rangle$ is put on a pile of other disks, smaller disks may occur at any positions in that pile, but none of them can be of size less than or equal to $x - d$.

As we will work out, the solution to the Santa Claus' mode and the generous Santa Claus' mode are almost identical. We give the complete solution to them in Section 4, thus resolve the open question raised in [11].

In the next section, we give notations, and prove some basic facts about Hanoi towers in the canonical version and our versions.

2 The Hanoi Towers

The classical game of moving a set X of disks from peg P to peg Q, where the disks are sorted in increasing order in both initial and final configurations, is denoted by $H(X, P, Q)$. The games under the sinner's rule, Santa's rule, and generous Santa's rule are denoted by $H_S(X, P, Q, k)$, $H_C(X, P, Q, d)$, and $H_G(X, P, Q, d)$, respectively. Often, we denote a move of disk $\langle x \rangle$ from P to Q as $\langle x \rangle(P \to Q)$.

A *state* is a specification of the sets of disks on three pegs, as well as their order on each peg. A *procedure* is a sequence of disk moves. Note that in our games there might be disks on the top of smaller disks in a state, and the moves in a procedure do not necessarily obey the divine rule.

We state the canonical recursive procedure `hanoi` to solve the game $H([n], P_i, P_j)$, where $\{i, j\} \subset \{1, 2, 3\}$, and list some easy facts.

Procedure 2.1 `hanoi([n], a, b)`

```
c := 6-a-b;
hanoi([n-1], a, c);
move disk n from peg a to peg b;
hanoi([n-1], c, b);
```

Fact 2.2 *The procedure* `hanoi([n], a, b)` *obeys the divine rule, takes* $2^n - 1$ *moves, and the i-th biggest disk, i.e.,* $\langle n - i + 1 \rangle$*, is moved* 2^{i-1} *times.*

Fact 2.3 *Any procedure obeying the divine rule needs to take at least $2^n - 1$ moves, if the n disks are in the same peg at the end and the biggest disk is moved at least once.*

Definition 2.4 *Let $\mathcal{A}$ be a procedure with a sequence of m moves, $(\langle x_i\rangle(P_i \to Q_i) : i = 1 \cdots m)$. The* reverse *of $\mathcal{A}$ is defined to be the sequence $(\langle x_{m+1-i}\rangle(Q_{m+1-i} \to P_{m+1-i}) : i = 1 \cdots m)$.*

Lemma 2.5 *Let $\mathcal{A}$ be any procedure that takes a state S_1 to state S_2; let $\mathcal{A}'$ be the reverse of $\mathcal{A}$. Then $\mathcal{A}'$ takes S_2 to S_1. Moreover, if the moves of $\mathcal{A}$ obey the divine rule, yet there are k disks in S_1 such that each one is bigger than the one below it, then $\mathcal{A}'$ violates the divine rule at most k times.*

Proof. It is obvious that $\mathcal{A}'$ takes the state S_2 back to S_1. Suppose there are m moves in $\mathcal{A}$. If $\langle x_{m+1-i}\rangle(Q_{m+1-i} \to P_{m+1-i})$ is a move in $\mathcal{A}'$ which violates the divine rule, then the move $\langle x_i\rangle(P_i \to Q_i)$ in $\mathcal{A}$ takes $\langle x\rangle$ away from a smaller disk below it. Since there are at most k such disks in the initial state of $\mathcal{A}$, and $\mathcal{A}$ never violates the divine rule, the lemma follows. □

Definition 2.6 *Let $\mathcal{A}$ be a procedure playing any game of towers of Hanoi with disk set X; let D be any subset of X; the cost of D in $\mathcal{A}$, $c(\mathcal{A}, D)$, is defined to be the number of steps $\mathcal{A}$ moves disks in D.*

3 The Sinner's Towers

Define $S(n, k)$ to be the minimum number of moves one needs to move n disks obeying the sinner's rule. It is clear that $S(n, k) = 2n - 1$ when $k \geq n - 2$.

Definition 3.1 *We define a function $g(n, k)$ on integers to be*

$$g(n,k) = \begin{cases} 2n - 1 & \textit{if } n \leq k+2, \\ 4n - 2k - 5 & \textit{if } k+2 \leq n \leq 2k+2, \\ 2^{n-2k} + 6k - 1 & \textit{if } 2k+2 \leq n. \end{cases}$$

Note that in the definition the values are consistent where the ranges overlap.

Proposition 3.2 *For all n and k, $S(n, k) \leq g(n, k)$.*

Proof. For $n > 2k + 2$, we do the game $G = H_S([n], P_1, P_2, k)$ in five stages.

1. We simulate the game $H([n - 2k - 1], P_1, P_3)$ to move the top $n - 2k - 1$ disks to the third peg.
2. Now we have $2k+1$ disks left on P_1. We divide the top $2k$ disks into k pairs of adjacent disks. Starting from the top, for each pair $\{\langle i\rangle, \langle i+1\rangle\}$, we move $\langle i\rangle$ to P_2, then move $\langle i+1\rangle$ to P_3 (thus violate the divine rule once), then move $\langle i\rangle$ to P_3.

3. Move $\langle n\rangle$ to P_2.

4. For each pair $\{\langle i\rangle, \langle i+1\rangle\}$ in (2), we move $\langle i\rangle$ to P_1, then move $\langle i+1\rangle$ to P_2, and move $\langle i\rangle$ to P_2.

5. Simulate the game $H([n-2k-1], P_3, P_2)$.

The total number of steps is

$$F(2^{n-2k-1}-1)+3k+1+3k+(2^{n-2k-1}-1)=2^{n-2k}+6k-1.$$

And the divine rule is violated k times in stage (2).

For $2k \geq n > k+2$, we do the game G in five stages. Let $s = n-k-2$.

1. Move $\langle 1\rangle$, $\langle 2\rangle$, …, $\langle n-1-2s\rangle$ one by one to P_3. The divine rule is violated $n-2-2s$ times.

2. Now we have $2s+1$ disks left on P_1. Group the top $2s$ into s pairs, then do the same thing as in the stage (2) in the first case. The divine rule is violated s times here.

3. Move $\langle n\rangle$ to P_2.

4. For the top s pairs on P_3, do the same thing as in stage (4) in the first case.

5. The $n-1-2s$ disks are now in the reversed order on P_3. We move them one by one to P_2.

The divine rule is violated $n-2-2s+s=k$ times, and the total number of steps is

$$(n-1-2s)+3s+1+3s+(n-1-2s)=2n+2s-1=4n-2k-5.$$

Finally, if $n \leq k+2$, we simply move the top $n-1$ disks one by one to P_3, and move $\langle n\rangle$ to P_2, then move all the other disks to P_2. □

The rest of this section is devoted to the proof that $S(n,k)=g(n,k)$.

Proposition 3.3 *Let $n \geq 1$; let S_1 be the state such that $[n]$ is stacked on one peg in increasing order, and the other two pegs are empty; let S_2 be a state where $[n]$ is on one peg (maybe the same as the original one) in any order, and the other two pegs are empty; let $\mathcal{A}$ be any procedure that takes S_1 to S_2 with at most k violations of the divine rule, and assume $\langle n\rangle$ is moved at least once in $\mathcal{A}$. Then there are at least $2^{n-2k}-1$ steps in $\mathcal{A}$.*

Proof. We prove the proposition by induction on n. The base cases $n=1$ and $n \leq 2k$ are trivial. Now assume $n>1$, $n>2k$, and the theorem is true for all $n' < n$. We color the disk $\langle x\rangle$ black if $\langle x\rangle$ is ever moved onto a smaller disk in $\mathcal{A}$. Otherwise we color $\langle x\rangle$ as white.
Case 1. $\langle n\rangle$ is black. Suppose there are s ($1 \leq s \leq k$) consecutive black disks at the bottom of the stack. They are $\langle n\rangle$, …, $\langle n-s+1\rangle$. Disk $\langle n-s\rangle$ is white. There are

$n-s-1>0$ disks above $\langle n-s\rangle$, we call this set of disks T. Notice that, since $\langle n-s\rangle$ is white, right before the first move of $\langle n-s\rangle$, the whole set T is carried to another peg with at most $k-s$ violations of the divine rule. By induction, the procedure already has at least $2^{n-s-1-2(k-s)}-1=2^{n-2k+s-1}-1\geq 2^{n-2k}-1$ steps at this moment.
Case 2. $\langle n\rangle$ is white. Focus on the moment right before the first move of $\langle n\rangle$, the whole set $[n-1]$ is moved from one peg to another peg.
Case 2.1. $\mathcal{A}$ violates the divine rule less than k times before this moment, then it has already taken at least $2^{n-1-2(k-1)}-1>2^{n-2k}-1$ steps.
Case 2.2. Otherwise, all the k violations occur before this moment. It has taken at least $2^{n-1-2k}-1$ steps. Consider the state of $[n-1]$ at this moment, call it S_1'. Call the final state of $[n-1]$ S_2', which is $[n-1]$ in increasing order on a peg, because all the k violations occurred before the state S_1'. Since $\langle n\rangle$ is white, it serves only as ground to $[n-1]$. The procedure $\mathcal{A}$ from this moment, with moves on $\langle n\rangle$ excluded, is a procedure taking S_1' to S_2' without any violation of the divine rule. Notice that, in S_1' there can be at most k disks such that it is bigger than the disk below it. By Lemma 2.5 and the inductive hypothesis, there are at least $2^{n-1-2k}-1$ steps after this moment in $\mathcal{A}$. Counting the moves of $\langle n\rangle$, the total number of steps is at least $2^{n-2k}-1$. □

Proposition 3.4 *For any $n\leq 2k+2$, $S(n,k)=g(n,k)$.*

Proof. This is trivial when $n\leq k+2$. We show $S(n,k)\geq 4n-2k-5$ for all $n\geq k+2$. We color the disk $\langle x\rangle$ black if $\langle x\rangle$ is ever moved onto a smaller disk in $\mathcal{A}$. Otherwise we color $\langle x\rangle$ as white. We assume there are s black disks, $s\leq k$. Each black disk is moved at least twice: one violates the divine rule, and the last one does not. So the cost of all the black disks is at least $2s$. Let T be the set $[n-1]$. Consider the state S right before the first move of $\langle n\rangle$.
Case 1. $\langle n\rangle$ is white. All the disks in T are stacked on another peg P. Let $\langle x\rangle$ be any white disk in T. If $\langle x\rangle$ is at the bottom, the cost of $\langle x\rangle$ in the entire procedure is at least 2. Otherwise, $\langle x\rangle$ is on the top of a bigger disk $\langle y\rangle$. $\langle x\rangle$ is moved at least twice before, otherwise $\langle y\rangle$ can not go below $\langle x\rangle$ on P. $\langle x\rangle$ will be moved at least twice later, otherwise $\langle y\rangle$ can not go below $\langle x\rangle$ on the destination peg. So, the cost on the white pegs is at least $4(n-s-2)+2+1$.
Case 2. $\langle n\rangle$ is black. The other disks are stacked on the other two pegs. There are $n-s$ white disks. Similar to the previous case, at most two of them has cost 2, each of the other white disks costs at least 4. So, the cost on the white disks is at least $4(n-s-2)+4$.

In either case, the total cost is at least $4(n-s-2)+3+2s\geq 4n-2k-5$. □

We remark that once we fixed the k black disks, the lower bound holds even in the case where each black disk can violate the divine rule many times. If we want to achieve the lower bound, the bottom disk must be white. However, by choosing different black disks, as long as $\langle 1\rangle$ and $\langle n\rangle$ are white, and there are no adjacent white disks, we can achieve the lower bound in different ways.

While it is possible to provide a more careful induction like the proof of Proposition 3.3, we found the following way of proving the main theorem in this section more pleasing.

Theorem 3.5 *For any n and k, $S(n,k) = g(n,k)$.*

Proof. The case where $n \leq 2k+2$ is done in Proposition 3.4. We prove the theorem for $n \geq 2k+2$ by an induction on n for any fixed k.

The basis is $n = 2k+2$, this is done in Proposition 3.4. Now assume $n > 2k+2$. Consider the smallest disk $\langle 1 \rangle$. For any procedure, if we ignore all the moves of $\langle 1 \rangle$, it is a procedure for $H_S([n] \setminus \{\langle 1 \rangle\}, P_1, P_2, k)$. By the inductive hypothesis, the procedure needs at least $g(n-1,k) = 2^{n-1-2k} + 6k - 1$ steps on $[n] \setminus \{\langle 1 \rangle\}$. So, if $\langle 1 \rangle$ is ever moved more than $2^{n-1-2k} - 1$ steps, the induction is finished. We prove that the other case, where $\langle 1 \rangle$ is moved at most $2^{n-1-2k} - 1$ steps, is impossible.

Assume the contrary, there is a procedure $\mathcal{P}$ for the game $H_S([n], P_1, P_2, k)$ with $c(\mathcal{P}, \langle 1 \rangle) \leq 2^{n-1-2k} - 1$. Suppose the cost of the procedure $\mathcal{P}$ is N. We pick a large m so that $2^{m+1} > N$. Now, suppose D is a set of m disks smaller than $\langle 1 \rangle$. We define a procedure $\mathcal{P}'$ which simulates $\mathcal{P}$. Whenever we have a move $\langle 1 \rangle(P \to Q)$, we move the whole tower $\{\langle 1 \rangle\} \cup D$ from P to Q in the canonical way with $2^{m+1} - 1$ moves. It is clear that the procedure $\mathcal{P}'$ successfully plays the game $H_S([n] \cup D, P_1, P_2, k)$, which is the same as $H_S([n+m], P_1, P_2, k)$. Since $c(\mathcal{P}, \langle 1 \rangle) \leq 2^{n-1-2k} - 1$, we have $c(\mathcal{P}, \{\langle 1 \rangle\} \cup D) \leq (2^{n-1-2k} - 1)(2^{m+1} - 1)$. So, the total cost of $\mathcal{P}'$ is at most

$$(2^{n-1-2k} - 1)(2^{m+1} - 1) + N < 2^{m+n-2k} - 2^{m+1} + N \leq 2^{m+n-2k} - 1.$$

Contradicts Proposition 3.3. $\square$

4 Santa's Towers

Define $C(n,d)$ to be the minimum number of moves one needs to move n disks obeying the Santa's rule; and define $G(n,d)$ to be the minimum number of moves one needs to move n disks obeying the generous Santa's rule. In this section, we derive the exact formula for $C(n,d)$ and $G(n,d)$. Hence, we resolve the open question asked by D. Wood in 1981 ([11]), which asks for a general formula for $G(n,d)$. [1]

It is clear that the Santa's rule with discrepancy d is more restricted than the generous Santa's rule with discrepancy d, hence we have

Fact 4.1 $G(n,d) \leq C(n,d)$.

It is also easy to see, the generous Santa's rule is not too generous. There is an implicit restriction on the positions where the smaller disks can occur. We mention the following fact, although we will not need it in the rest of the article.

[1] As we reword the problem here, our $G(n,d)$ is equivalent to $f(n,d-1)$ in [11]. We found some mistakes in the table on page 23 of [11]. As we will show, $f(4,1) = 9$ and $f(5,2) = 11$, which are miscalculated as 11 and 13, respectively, in [11].

Fact 4.2 *For any $d > t > 0$, if $\langle x\rangle$ is placed onto a pile which contains the disk $\langle x-t\rangle$ under the generous Santa's rule with discrepancy d, then $\langle x-t\rangle$ must occur among the top $2d-t-1$ positions in that pile before the move.*

Proof. Observe that any disk $\langle y\rangle$ above $\langle x-t\rangle$ in that pile must satisfy $y < n-t+d$ and $y > n-d$. □

Our main result in this section is to prove $C(n,d) = G(n,d)$ and give their formula. The solution resides in two ways of grouping the disks. To get an upper bound, we group every d consecutive disks, and simulate the canonical `hanoi`, viewing each group as a disk. To prove the lower bound, we view each arithmetic progression with difference d as a class, and analyze the lower bound on each class.

4.1 A Procedure for $H_C([n], P_a, P_b, d)$

We study the following procedure.

Procedure 4.3 `santa([n], a, b, d)`

Let $n-1 = dq+r$, where $1 \le r \le d$. We group the set $[n-1]$ into q blocks of size d and one block of size r. To be precise, let $B_1 = \{\langle i\rangle : 1 \le i \le r\}$, and $B_j = \{\langle i\rangle : r+(j-2)d+1 \le i \le r+(j-1)d\}$, $2 \le j \le q+1$. We are going to view each block as a disk and simulate the canonical `hanoi` *procedure on the set of blocks. An instruction of moving B from P_i to P_j means we move all the disks in block B one by one from P_i to P_j; we call this a block move. Notice that each block move corresponds to up to d actual moves, and the order inside the block is reversed. We denote the set $\{B_i : 1 \le i \le q+1\}$ by $[[q+1]]$.*

```
c := 6-a-b;
hanoi([[q+1]], a, c);
move disk n from peg a to peg b;
hanoi([[q+1]], c, b);
```

Proposition 4.4 *The procedure* `santa([n], a, b, d)` *successfully plays the game under Santa's rule, $H_C([n], P_a, P_b, d)$.*

Proof. Notice that, when $i < j$, all the disks in block B_i are smaller than the disks in block B_j. We simulate the original Hanoi game on the blocks, never put B_j onto B_i when $i < j$, so the Santa's rule with discrepancy d is obeyed. Finally, the blocks are in the correct order on the peg P_b. We just need to check if the disks are in the right order within each block. This is so because each block move reversed the order within the block, and since we simulated the canonical Hanoi twice, each block is involved in an even number of block moves. □

The following proposition follows easily from Fact 2.2.

Proposition 4.5 *Let $n-1 = dq+r$, where $1 \le r \le d$. The number of moves that occur in Procedure 4.3 is $1 + d(2^{q+1}-2) + r2^{q+1}$.*

4.2 The Formula for $C(n, d)$ and $G(n, d)$

Theorem 4.6 *Let $n-1 = dq + r$, where $1 \le r \le d$;*

$$C(n,d) = G(n,d) = 1 + d(2^{q+1} - 2) + r2^{q+1}.$$

Proof. By Propositions 4.4, 4.5, and Fact 4.1, we have $G(n,d) \le C(n,d) \le 1 + d(2^{q+1} - 2) + r2^{q+1}$. It is enough to prove the lower bound for $G(n,d)$, i.e., Procedure 4.3 is the best one can achieve to play the game $H_G([n], P_a, P_b, d)$. We denote Procedure 4.3 by $\mathcal{A}$. We partition the disks into d classes. $C_k = \{\langle i\rangle : i \equiv k \pmod d\}$, $0 \le k < d$. Let C_{k_0} be the class to which $\langle n\rangle$ belongs, i.e., $n \equiv k_0 \pmod d$, $0 \le k_0 < d$. We give new names to the disks. For any class C_k with s disks, we label the disks in that class, from the biggest to the smallest, $\langle k\rangle_1$, …, $\langle k\rangle_s$.

Under the generous Santa's rule, the disks in each class can not reverse their order on any peg. So, if we disregard all the other disks, only observe the movement of disks in C_k, it is a canonical game $H(C_k, P_a, P_b)$. So, the cost involving that class is at least $2^{|C_k|} - 1$ in any successful play of the game $H_G([n], P_a, P_b, d)$. However, it is easy to see, $c(\mathcal{A}, \langle k\rangle_i) = 2^i$ if $k \ne k_0$. $c(\mathcal{A}, \langle k_0\rangle_i) = 2^{i-1}$. So, whenever $k \ne k_0$, $c(\mathcal{A}, C_k) = 2(2^{|C_k|} - 1)$. Here is a little remark about these values. If our rule does not require that all pegs are in order in the final state, one can easily see that there is a procedure requires just $2^{|C_k|} - 1$ steps for each C_k. The main difficulty in the proof is to show, in order to make the final order correct, we really need to double the number of moves for almost all the classes. The parity condition in Lemma 4.12(b) is crucial.

From now on, we fix $\mathcal{B}$ to be any procedure that moves $[n]$ from P_1 to P_2 under the generous Santa's rule with discrepancy d.

Definition 4.7 *Two consecutive moves in a procedure are called* redundant *if they are in the form $\langle x\rangle(P \to Q)$ and $\langle x\rangle(Q \to P)$. A procedure is called* reduced *if there are no redundant moves. From any procedure $\mathcal{P}$, we keep deleting redundant moves in any order, it is easy to see we get a unique reduced procedure, we call this the reduced procedure of $\mathcal{P}$, denoted $\rho\mathcal{P}$.*

Definition 4.8 *Let $\mathcal{P}$ be a procedure with a sequence of disk moves. Let X be a subset of the disks. The* induced procedure of $\mathcal{P}$ with respect to X, *denoted $\mathcal{P}\mid_X$, is the subsequence of $\mathcal{P}$ that involves the moves of disks from X. We call $\rho(\mathcal{P}\mid_X)$ the* reduced procedure of $\mathcal{P}$ with respect to X.

We will use the following easy fact frequently.

Fact 4.9 *Let $\mathcal{P}$ be any procedure that obeys the generous Santa's rule and moves a disk set D from P_1 to P_2. Let X and Y be subsets of D such that $X \subseteq Y$. Then (a) $\mathcal{P}\mid_X$ is a procedure that obeys Santa's rule and moves the disk set X from P_1 to P_2. The order of disks in the initial and final states are the same as those in the procedure $\mathcal{P}$. The same is true for the procedure $\rho(\mathcal{P}\mid_X)$. (b) $\rho(\mathcal{P}\mid_X)$ appears as a sub-sequence in $\rho(\mathcal{P}\mid_Y)$. Consequently, $c(\rho(\mathcal{P}\mid_X), \langle x\rangle) \le c(\rho(\mathcal{P}\mid_Y), \langle x\rangle)$ for any disk $\langle x\rangle$. (c) For any X and any disk $\langle x\rangle$, $c(\rho(\mathcal{P}\mid_X), \langle x\rangle)$ is of the same parity as $c(\mathcal{P}, \langle x\rangle)$.*

Definition 4.10 *Let $\mathcal{P}$ be a procedure with a sequence of m moves, $(\langle x_i \rangle (P_i \to Q_i) : i = 1 \cdots m)$. The* log *of $\mathcal{P}$ is the sequence $(\langle x_i \rangle : i = 1 \cdots m)$.*

Lemma 4.11 *Let $\mathcal{P}$ be any procedure that obeys the generous Santa's rule with discrepancy d. Let X be any subset of disks such that each disk in X is of size at least $x + d$. (a) If $y - x \geq d$ and $z - y \geq d$, then $\langle y \rangle$ and $\langle z \rangle$ can not both appear between any pair of consecutive $\langle x \rangle$'s in the log of $\mathcal{P}$. (b) Between any two consecutive $\langle x \rangle$-moves in $\rho(\mathcal{P} \mid_{X \cup \{\langle x \rangle\}})$, all the moves of disks in X are from the same source peg and to the same destination peg. These are the two pegs ,other than the peg containing $\langle x \rangle$, during this period. (c) For any $\langle y \rangle \in X$. There is at most one $\langle y \rangle$ between any pair of consecutive $\langle x \rangle$'s in the log of $\rho(\mathcal{P} \mid_{X \cup \{\langle x \rangle\}})$.*

Proof. Let S_1 be the state after an $\langle x \rangle$-move, and let S_2 be the state before the next time we move $\langle x \rangle$. From the state S_1 to S_2, $\langle x \rangle$ stays on the same peg, say, P_1.
(a). Suppose $\langle y \rangle$ is ever moved during the period from S_1 to S_2, then y is never on the peg P_1 during this period. A move for $\langle z \rangle$ is impossible: It would either move above $\langle y \rangle$, or already be above $\langle y \rangle$ before the move.
(b) and (c). During the period between S_1 and S_2, no disk in X can be moved to peg P_1. Because the procedure $\rho(\mathcal{P} \mid_{X \cup \{\langle x \rangle\}})$ is reduced, all the moves during this period have the same source and destination, say, from P_2 to P_3. So, any disk in X will be moved at most once. □

Lemma 4.12 *Let $\langle x \rangle$ and $\langle x' \rangle$ be two disks such that $x < x'$. Let X be a set of disks such any disk in X is of size at least $x' + d$. Let $T \subseteq X$ such that the difference of sizes between each pair of disks in T is at least d. Suppose $c(\rho(\mathcal{B} \mid_X), T) \geq s$, then (a) $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle\}}), \langle x \rangle) \geq s + 1$, $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x' \rangle\}}), \langle x' \rangle) \geq s + 1$. (b) If $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x' \rangle\}}), \langle x' \rangle) = s + 1$ and $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}}), \langle x \rangle) = s + 1$, then either $s + 1$ is an even number, or $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}}), \langle x' \rangle) \geq s + 3$.*

Proof. (a). Look at the procedure $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle\}})$. Clearly $\langle x \rangle$ is moved before we ever move any disk from T, and is also moved after we finish all the disks from T. Between any two consecutive appearances of symbols from T in the log, we must have at least one appearance of x, otherwise it contradicts Lemma 4.11. Since X is a subset of $X \cup \{\langle x \rangle\}$, there are at least s symbols from T in the log, so $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle\}}), \langle x \rangle) \geq s + 1$. The same proof applies to $\langle x' \rangle$ instead of $\langle x \rangle$.
(b). Let $\mathcal{P} = \rho(\mathcal{B} \mid_{X \cup \{\langle x' \rangle\}})$. $c(\mathcal{P}, \langle x' \rangle) \geq s + 1$, and the equality holds if $c(\mathcal{P}, T) = s$, and the $s + 1$ $\langle x' \rangle$'s and the s symbols from T appear alternatively in the log of $\mathcal{P}$. Let the i-th move from T be $\langle y_i \rangle (R_i \to Q_i)$, and let the i-th move of $\langle x' \rangle$ be $\langle x' \rangle (R'_i \to Q'_i)$. It is easy to see that $\{R_i, Q_i\} = \{P_1, P_2, P_3\} \setminus \{R'_{i+1}\}$, and $\{R_{i+1}, Q_{i+1}\} = \{P_1, P_2, P_3\} \setminus \{Q'_{i+1}\}$. Since $R'_{i+1} \neq Q'_{i+1}$, we conclude $\{R_i, Q_i, R_{i+1}, Q_{i+1}\} = \{P_1, P_2, P_3\}$.

The procedure $\mathcal{P}$ is a sub-sequence of $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$. Focus on the s moves $\langle y_i \rangle (R_i \to Q_i)$ in the procedure $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$, we call them the T-moves. The $s + 1$ x's must appear once before all the T-moves, once after all the T-moves, and once between each consecutive T-moves. The last statement is true, otherwise there are two consecutive T-moves $\langle y_i \rangle(R_i \to Q_i)$ and $\langle y_{i+1} \rangle(R_{i+1} \to Q_{i+1})$ while $\langle x \rangle$ stays on a certain peg, which is impossible since $\{R_i, Q_i, R_{i+1}, Q_{i+1}\} = \{P_1, P_2, P_3\}$.

In particular, we proved that no two $\langle x \rangle$-moves in $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$ are adjacent. This implies that if we delete all the $\langle x \rangle$-moves from $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$, it is either already reduced, i.e., it is $\rho(\mathcal{B} \mid_{X \cup \{\langle x' \rangle\}})$, or at least two $\langle x' \rangle$-moves are redundant. To see this, observe that the only way to start a reduction is that there is a move, say $\langle x \rangle(P_1 \to P_2)$, where the previous move, which is from $X \cup \{\langle x' \rangle\}$, and the next move reverse each other. However, if these are not $\langle x' \rangle$-moves, the previous move is between P_2 and P_3, the next move is between P_1 and P_3, since each disk in X is of size at least $x + d$.

As a consequence, if $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}}), \langle x' \rangle) < s + 3$, there are exactly $s + 1$ $\langle x' \rangle$-moves and exactly s T-moves in the procedure $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$. The $\langle x \rangle$-moves appear once before all the moves from T, once after all of them, and once between each consecutive moves from T. The same is true for x'. We further claim that the source and destination of each move is fully determined by the moves of elements in T. We prove this claim inductively from the first move to the last one. Let $\langle x \rangle(P \to Q)$ be the i-th move of $\langle x \rangle$. If it is the first one, then $P = P_1$, otherwise it is the destination of the last move. If it is the last move, $Q = P_2$. Otherwise, the next move from T is $\langle y_i \rangle(R_i \to Q_i)$, Q must be the peg other than R_i and Q_i.

Thus, we have $s + 1$ positions in the log of $\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle, \langle x' \rangle\}})$, one before all the symbols from T, one after them, and one between each pair of consecutive symbols from T. In each position, there is one $\langle x \rangle$ and one $\langle x' \rangle$, with the moves from the same source to the same destination. Assume $x < x'$, in the odd numbered positions x moves first, and in the even numbered positions x' moves first. If $s + 1$ is odd, x would be stacked under x' in the final state, a contradiction. □

Lemma 4.13 *Let X be a set of disks such that each disk in X is of size at least $x + d$. Let $T \subseteq X$ be a subset of size at least 2 such that the difference of sizes between each pair of disks in T is at least d, except for one pair $\langle y \rangle$ and $\langle y' \rangle$. Suppose $c(\rho(\mathcal{B} \mid_X), T \setminus \{\langle y \rangle, \langle y' \rangle\}) \geq s$, $c(\rho(\mathcal{B} \mid_X), \langle y \rangle) \geq p$, and $c(\rho(\mathcal{B} \mid_X), \langle y' \rangle) \geq p - 1$ and has a different parity as p. Then $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle\}}), x) \geq s + p + 2$.*

Proof. By Lemma 4.12 with the set $T \setminus \{\langle y' \rangle\}$, x appears at least $s+p+1$ times in $\rho(\mathcal{B} \mid_{X \cup \{x\}})$. Assume the contrary, x appears exactly $s + p + 1$ times, then there are exactly $s + p$ symbols from $T \setminus \{\langle y' \rangle\}$ in the log of $\rho(\mathcal{B} \mid_{X \cup \{x\}})$. Consider the $s + p$ spots, one between each consecutive pair of x's in the log. By Lemma 4.11, the $s + p$ symbols from $T \setminus \{\langle y' \rangle\}$ appears once in each spot, and $\langle y' \rangle$ can not be put in a spot twice or in the same spot with any disk other than $\langle y \rangle$. Since there are at most p such spots, and $c(\rho(\mathcal{B} \mid_{X \cup \{\langle x \rangle\}}), x)$ is not the same parity as p, we must have exactly $p - 1$ $\langle y' \rangle$-moves. We call a spot even if it contains both $\langle y \rangle$ and $\langle y' \rangle$, otherwise call it odd. Since neither $\langle y \rangle$ nor $\langle y' \rangle$ can be moved onto the peg containing $\langle x \rangle$, we must have the same source and destination for $\langle y \rangle$ and $\langle y' \rangle$ in each even spot. Consider the only odd spot. If it is the last move of $\langle y \rangle$, $\langle y \rangle$ and $\langle y' \rangle$ would be on different pegs in the final state. Otherwise $\langle y \rangle$ and $\langle y' \rangle$ can not move from the same source in the next even spot. We have a contradiction in either case. $\square$

Definition 4.14 *For any disk $\langle x \rangle$, define $U(\langle x \rangle)$ to be the set of disks that are not smaller than x, i.e., $U(\langle x \rangle) = \{\langle x \rangle, \langle x+1 \rangle, \ldots, \langle n \rangle\}$. Define the* downwards cost *of $\langle x \rangle$ to be $d(\langle x \rangle) = c(\rho(\mathcal{B} \mid_{U(\langle x \rangle)}), \langle x \rangle)$, i.e., the number of moves of $\langle x \rangle$ in the reduced procedure of $\mathcal{B}$ if we disregard all the smaller disks.*

Let C_k be any class where $k \neq k_0$. It is easy to see that $c(\mathcal{A}, C_k) = 2(2^{|C_k|} - 1)$. If $c(\mathcal{B}, C_k) < 2(2^{|C_k|} - 1)$, there must be a disk $\langle k \rangle_i$ such that $c(\mathcal{B}, \langle k \rangle_i) < 2^i$; so, the disk $\langle k \rangle_i$ satisfies $d(\langle k \rangle_i) < 2^i$. Thus, we have the following definition.

Definition 4.15 *We call a class C_k* bad *if $k \neq k_0$ and $c(\mathcal{B}, C_k) < 2(2^{|C_k|} - 1)$. Let C_k be a bad class, there must be a disk $\langle k \rangle_i$ such that $d(\langle k \rangle_i) < 2^i$. We call a disk $\langle k \rangle_i$* precious *if $d(\langle k \rangle_i) < 2^i$. We call the biggest precious disk in C_k the* prince *of C_k.*

Lemma 4.16 *(a) Let C_k be a bad class with prince $\langle k \rangle_x$, then $\langle k \rangle_x$ is the only precious disk in C_k, and $c(\mathcal{B}, \langle k \rangle_x) = 2^x - 1$. (b) Suppose C_{k_1} and C_{k_2} are two bad classes with princes $\langle k_1 \rangle_y$ and $\langle k_2 \rangle_z$, respectively. Then $y \neq z$.*

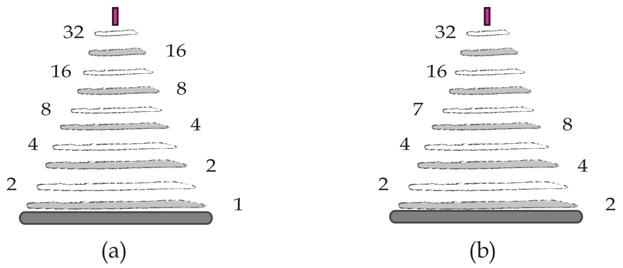

Figure 2: The gray disks are the class C_{k_0}, the white disks are the class C_k. (a) The cost of each disk in Procedure 4.3. (b) The least possible costs if C_k is a bad class with its third disk as its prince.

Proof. (a). By the definition of a prince, $d(\langle k \rangle_i) \geq 2^i$ for each $i < x$. Set $T = \{\langle k \rangle_i : i < x\}$ and $X = U(\langle k \rangle_{x-1})$. We have $c(\rho(\mathcal{B} \mid_X), \langle k \rangle_i) \geq d(\langle k \rangle_i) \geq 2^i$. By Lemma 4.12(a) we get $d(\langle k \rangle_x) \geq c(\rho(\mathcal{B} \mid_{X \cup \{\langle k \rangle_x\}}), \langle k \rangle_x) \geq 2^x - 1$. In order to make $\langle k \rangle_x$ a prince, we must have $d(\langle k \rangle_x) = 2^x - 1$ and $d(\langle k \rangle_i) = 2^i$ for each $i < x$.

We claim that one of two things must happen:

$$d(\langle k_0 \rangle_i) \geq 2^i, \forall i \leq x; \quad \text{or} \tag{1}$$

$$d(\langle k_0 \rangle_i) \geq 2^i, \forall i < x, \quad \text{and} \quad c(\rho(\mathcal{B} \mid_{U(\langle k \rangle_x)}), \langle k_0 \rangle_x) \geq 2^x + 1. \tag{2}$$

To prove the claim, assume there is $i \leq x$ such that $d(\langle k_0 \rangle_i) < 2^i$ and $d(\langle k_0 \rangle_j) \geq 2^j$ for all $j < i$. Set $T = \{\langle k_0 \rangle_j : j < i\}$ and $X = U(\langle k_0 \rangle_{i-1})$ in Lemma 4.12 we have $d(\langle k_0 \rangle_i) = 2^i - 1$ and $d(\langle k_0 \rangle_j) = 2^j$ for all $j < i$. (The case $i = 1$ is trivial.) We discuss two cases. In the first case, $i = x$. Consider the set $T = \{\langle k_0 \rangle_j : j < x\}$ and $X = U(\langle k_0 \rangle_{x-1})$. $c(\rho(\mathcal{B} \mid_X), T) \geq 2^x - 2$. By Lemma 4.12(a), $c(\rho(\mathcal{B} \mid_{X \cup \{\langle k_0 \rangle_x\}}), \langle k_0 \rangle_x) \geq 2^x - 1$. On the other hand, we have

$$c(\rho(\mathcal{B} \mid_{X \cup \{\langle k_0 \rangle_x\}}), \langle k_0 \rangle_x) \leq c(\rho(\mathcal{B} \mid_{U(\langle k_0 \rangle_x}), \langle k_0 \rangle_x) = d(\langle k_0 \rangle_x) = 2^x - 1.$$

So, $c(\rho(\mathcal{B} \mid_{X \cup \{\langle k_0 \rangle_x\}}), \langle k_0 \rangle_x) = 2^x - 1$. Similarly, we have $c(\rho(\mathcal{B} \mid_{X \cup \{\langle k_0 \rangle_x, \langle k \rangle_x\}}), \langle k \rangle_x) = 2^x - 1$. By Lemma 4.12 (b) we establish (2). In the second case where $i < x$, using Lemma 4.13 with $T = \{\langle k_0 \rangle_j : j \leq i\} \cup \{\langle k \rangle_j : i \leq j < x\}$, $X = U(\langle k \rangle_{x-1})$, $\langle y \rangle = \langle k \rangle_i$, and $\langle y' \rangle = \langle k_0 \rangle_i$, we get the cost $d(\langle k \rangle_x) \geq 2^x$. It is not a prince.

Having proved the claim, we now show that $d(\langle k \rangle_i) \geq 2^i$ for any $i > x$. We do this inductively on i. If there is an $t \leq x$ such that $c(\rho(\mathcal{B} \mid_{U(\langle k \rangle_x)}), \langle k_0 \rangle_t) \geq 2^t + 1$, we may set $T_i = \{\langle k_0 \rangle_j : j \leq x\} \cup \{\langle k \rangle_j : x < j < i\}$ for each i and use Lemma 4.12(a). Otherwise, by (1) and (2), $c(\rho(\mathcal{B} \mid_{U(\langle k \rangle_x)}), \langle k_0 \rangle_t) = 2^t$ for each $t \leq x$. Set $T_i = \{\langle k_0 \rangle_j : j \leq x\} \cup \{\langle k \rangle_j : x \leq j < i\}$, $X_i = U(\langle k \rangle_{i-1})$ and use Lemma 4.13. In both cases, the cost for $\langle k \rangle_j$ is at least 2^j for all $j > x$ by the inductive hypothesis, we get $c(\mathcal{B}, \langle k \rangle_i) \geq 2^i$.

Finally, we must have $c(\mathcal{B}, \langle k \rangle_x) = 2^x - 1$, otherwise C_k is not a bad class.

(b). We may assume the biggest disk in C_{k_1} is bigger than the biggest disk in C_{k_2}, so $\langle k_1 \rangle_i$ is bigger than $\langle k_2 \rangle_i$ for any i. Assume the contrary $y = z$. Consider the set $T = \{\langle k_1 \rangle_i : i < y\}$ and $X = U(\langle k_1 \rangle_{y-1})$, with $\langle k_1 \rangle_y$ and $\langle k_2 \rangle_y$. By Lemma 4.12, $c(\mathcal{B}, \langle k_1 \rangle_y) \geq 2^y + 1$, a contradiction. □

Lemma 4.17 *If there is any bad class, then* $c(\mathcal{B}, C_{k_0}) \geq c(\mathcal{A}, C_{k_0}) + 2^{|C_{k_0}|-1}$.

Proof. By Lemma 4.16(a), if there is any bad class, $c(\mathcal{B}, \langle k_0 \rangle_1) \geq 2$. In the procedure $\mathcal{B}$, before the first move of $\langle k_0 \rangle_1$, after the last of its moves, and between each consecutive pair of its moves, the set $C_{k_0} \setminus \{\langle k_0 \rangle_1\}$ is moved completely from a peg to another under the divine rule for at least three times. Notice that the same set is moved twice under the divine rule in $\mathcal{A}$. So, the extra cost on this set is at least $2^{|C_{k_0}|-1} - 1$. Adding the extra cost on $\langle k_0 \rangle_1$, the lemma follows. □

Now we finish our proof of the main theorem. We compare the cost of the procedures $\mathcal{A}$ and $\mathcal{B}$ on each individual disk. By Lemma 4.16 (a), $\mathcal{B}$ can only save one step against $\mathcal{A}$ for each prince. However, if there is any bad class, $\mathcal{A}$ saves at least $2^{|C_{k_0}|-1} \geq |C_{k_0}|$ steps against $\mathcal{B}$ for the disks in the class C_{k_0} by Lemma 4.17. By Lemma 4.16 (b), the number of princes is no more than $|C_{k_0}|$. In fact, $\mathcal{B}$ can not save as many as $|C_{k_0}|$ steps on the princes, since a prince in the range $[n-d, n]$ still needs to moved at least twice. □

5 Other Towers of Hanoi

There could be many reasonable variations of the Hanoi tower problem. We mention one here.

As the remark after the proof of Proposition 3.4 shows, when $n \leq 2k+2$, one needs $g(n,k)$ steps even when we fix k disks and each of them can violate the divine rule many times. We propose the following variation of our Sinner's mode.

> *Hanoi Tower with k Evildoers*: Among the n disks, one may pick k of them as evildoers then perform a procedure of disk moves. An ordinary disk can not be placed on the top of a smaller disk, but an evildoer can do this an unlimited number of times.

Let $E(n,k)$ be the minimum number of steps one need to move the entire stack from one peg to another in the evildoer's mode. Clearly $E(n,k) \leq S(n,k)$. Our computer program shows that $E(n,k) = S(n,k)$ for all $n < 8$ and any k. However, $E(8,1) = 57$ while $S(n,k) = 69$. It could be an interesting problem to find the formula for $E(n,k)$.

Acknowledgments We like to thank Doron Zeilberger, in whose delightful class we restarted to play with the towers of Hanoi, and our advisor Mario Szegedy, who happens to be an expert on this subject, for their encouragement and advice.

We leant that, at the same time as our research, Yefim Dinitz and Shay Solomon from Ben Gurion University discovered and solved the problem of Hanoi Towers with the generous Santa's rule independent from us. We owe our gratitude to Shay for pointing out errors in our early drafts.

References

[1] M. D. Atkinson, *The cyclic towers of Hanoi,* Inform. Process. Lett. 13 (1981), no. 3, pp. 118–119.

[2] X. Chen and J. Shen, *On the Frame-Stewart conjecture about the Towers of Hanoi,* SIAM J. Comput. 33 (2004), no. 3, pp. 584–589.

[3] N. Claus (= E. Lucas), *La Tour d Hanoi,* Jeu de calcul, Science et Nature 1 (8) (1884), pp. 127 – 128.

[4] J. S. Frame, *Solution to advanced problem 3918,* Amer. Math. Monthly, 48 (1941), pp. 216 – 217.

[5] S. Klavžar, U. Milutinović, and C. Petr, *On the Frame-Stewart algorithm for the multi-peg Tower of Hanoi problem,* Discrete Appl. Math. 120 (2002), no. 1-3, pp. 141–157.

[6] C. S. Klein and S. Minsker, *The super Towers of Hanoi problem: large rings on small rings,* Discrete Math. 114 (1993), no. 1-3, pp. 283–295.

[7] B. M. Stewart, *Advanced problem 3918,* Amer. Math. Monthly, 46 (1939), p. 363.

[8] B. M. Stewart, *Solution to advanced problem 3918,* Amer. Math. Monthly, 48 (1941), pp. 217 – 219.

[9] P. K. Stockmeyer *The Tower of Hanoi: A Historical Survey and Bibliography,* manuscript available at `http://www.cs.wm.edu/ pkstoc/biblio.ps`.

[10] M. Szegedy, *In how many steps the k peg version of the Towers of Hanoi game can be solved?,* STACS 99 (Trier), pp. 356 – 361, Lecture Notes in Comput. Sci., 1563, Springer, Berlin, (1999).

[11] D. Wood, *Towers of Brahma and Hanoi Revisited*, J. Recr. Math. 14 (1981), pp. 17–24.