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Generating All Minimal Integral Solutions to AND-OR Systems of Monotone Inequalities: Conjunctions are Easier than Disjunctions [1]

by

Endre Boros ,[2] Khaled Elbassioni [3] Vladimir Gurvich [4]
Leonid Khachiyan [5]

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[2]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: boros@rutcor.rutgers.edu
[3]Max-Planck-Institut für Informatik, Saarbrücken, Germany; email: elbassio@mpi-sb.mpg.de
[4]RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003; email: gurvich@rutcor.rutgers.edu
[5]Department of Computer Science, Rutgers University, 110 Frelinghuysen Road, Piscataway NJ 08854-8003; email: leonid@cs.rutgers.edu

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ABSTRACT

We consider monotone $\vee, \wedge$-formulae ϕ of m atoms, each of which is a monotone inequality of the form $f_i(x) \geq t_i$ over the integers, where for $i = 1, \ldots, m$, $f_i : \mathbb{Z}^n \mapsto \mathbb{R}$ is a given monotone function and t_i is a given threshold. We show that if the $\vee$-degree of ϕ is bounded by a constant, then for linear, transversal and polymatroid monotone inequalities all minimal integer vectors satisfying ϕ can be generated in incremental quasi-polynomial time. In contrast, the enumeration problem for the disjunction of m inequalities is NP-hard when m is part of the input. We also discuss some applications of the above results in disjunctive programming, data mining, matroid and reliability theory.

1 Introduction

Consider a system of linear inequalities

$$\sum_{j=1}^{n} a_{ij}x_j \geq t_i \text{ for } i = 1, ..., m, \tag{1}$$

where a_{ij} are given non-negative reals, and where we assume that the variables can take only binary values. Due to the non-negativity of the coefficients, if a vector $x \geq 0$ satisfies some of these inequalities and $y \geq x$, then y satisfies the same inequalities (and possibly some others as well), i.e., the system (1) is *monotone*. We say that $x \in \{0,1\}^n$ is a *minimal feasible solution* for a subset $I \subseteq \{1, ..., m\}$ of the inequalities (1), if x satisfies all inequalities belonging to I, and any binary vector $y \neq x$ such that $y \leq x$ violates at least one of these inequalities. Lawler, Lenstra and Rinnooy Kan [22] considered the problems of generating all minimal feasible solutions satisfying

(P1) all m inequalities of (1), and

(P2) at least one of the m inequalities of (1).

These or equivalent problems arise in a number of areas, including integer programming, scheduling, and polyhedral combinatorics (see e.g., [5, 6, 22, 25, 27]). Note that the number of minimal feasible solutions may not be limited by a polynomial of n and m. A generation algorithm is said to be *incrementally polynomial* (or quasi-polynomial[1], or exponential) if for an arbitrary subset $\mathcal{X}$ of minimal feasible solutions it can find an additional minimal feasible solution $x \notin \mathcal{X}$, or recognize that $\mathcal{X}$ contains all such solutions, in time polynomially (or quasi-polynomially, or exponentially) limited in n, m and $|\mathcal{X}|$. Equivalently, an algorithm is incrementally polynomial (quasi-polynomial, or exponential) if for arbitrary integer k, it can generate k minimal feasible solutions, or all of them if k is too large, in time polynomial (quasi-polynomial, exponential) in n, m and k.

When $m = 1$, problems (P1) and (P2) coincide, and incrementally efficient generation of all minimal feasible solutions is possible (see e.g., [22]). For the general case, when $m > 1$, only incrementally exponential algorithms were proposed for (P1) and (P2) in [22], and it was conjectured that no polynomial time algorithm can solve these problems, unless P=NP (i.e., that no algorithm can recognize the completeness of a subset $\mathcal{X}$ of the minimal feasible solutions in polynomial time, unless P=NP, and hence no algorithm can generate all minimal feasible solutions in time limited by a polynomial of n, m, and the number of minimal feasible vectors). These problems were reconsidered recently in [9], and contrary to the conjecture of [22], (P1) was shown to be tractable in incremental quasi-polynomial time (which makes it very unlikely to be NP-hard), while (P2) was shown to be tractable in incremental polynomial time for fixed m, but NP-hard for the general case.

[1] A function $f(x)$ is *quasi-polynomial* if $f(x) = O(2^{polylog(x)})$.

This motivated us to study more complex monotone systems, and generalize the above results in three directions.

First, we generalize the above enumeration problems to *integer* variables. More precisely, we consider the inequalities (1) with the variables running over an arbitrary integer box $\mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}$, where $c \in \mathbb{Z}_+^n$ is a given integer vector with possibly infinite coordinates.

Second, we consider not only conjunctions (like in (P1)), or disjunctions (like in (P2)), but *arbitrary monotone expressions* of linear inequalities. Specifically, let $Y_i : \mathcal{C} \to \{0,1\}$ be the characteristic variables corresponding to the inequalities of (1), i.e., $Y_i(x) = 1$ iff $\sum_{j=1}^n a_{ij}x_j \geq t_i$, $i = 1, ..., m$. Then we associate to any given monotone $\vee, \wedge$- formula ϕ in m propositional variables a system Σ_ϕ of inequalities for which a vector $x \in \mathcal{C}$ is feasible iff $\phi(Y_1(x), Y_2(x), ..., Y_m(x)) = 1$. Since Σ_ϕ is a monotone system, the notion of *minimal feasible solutions* of Σ_ϕ is well defined, and we can consider the corresponding enumeration problem:

(P3) Generate all minimal feasible solutions of Σ_ϕ.

Note that when $\phi = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m$, then (P3) is the integer variant of (P1), while for $\phi = Y_1 \vee Y_2 \vee \cdots \vee Y_m$, problem (P3) is the integer variant of (P2).

Let us further associate to each monotone $\vee, \wedge$-formula $\phi = \phi(Y_1, \ldots, Y_m)$ a polynomial $P_\phi \in \mathbb{Z}[y_1, \ldots, y_m]$ defined by replacing logical conjunctions by arithmetic additions, and logical disjunctions by arithmetic multiplications. For instance, if $\phi = Y_1 \vee (Y_2 \wedge (Y_3 \vee Y_4))$, then we have $P_\phi(y_1, y_2, y_3, y_4) = y_1(y_2 + (y_3y_4)) = y_1y_2 + y_1y_3y_4$. We call P_ϕ the *evaluation polynomial of* ϕ.

Theorem 1 *If the degree of the evaluation polynomial P_ϕ is bounded, then we can generate all minimal feasible solutions to system Σ_ϕ in incremental quasi-polynomial time in terms of n and m.*

Let us note that if the degree of F_Φ is not bounded, then generating minimal feasible solutions for Σ_ϕ is NP-hard already for the Boolean case $\mathcal{C} = \{0,1\}^n$ of (P2), see [9].

Finally, in a third direction, we extend Theorem 1 to *transversal* and *polymatroid* inequalities.

Given a subset $\mathcal{H} \subseteq \mathcal{C}$ and non-negative real weights $w : \mathcal{H} \to \mathbb{R}_+$, we call the function

$$f_{\mathcal{H},w}(x) = \sum\{w(a) \mid a \in \mathcal{H},\ a \not\geq x\}$$

a *(weighted) transversal function* over $\mathcal{C}$. Note that for any given $x \in \mathcal{C}$ we can compute the value $f_{\mathcal{H},w}(x)$ in $O(n|\mathcal{H}|)$ time, and that for the Boolean case $\mathcal{C} = \{0,1\}^n$ we can equivalently define $f_{\mathcal{H},w}(x)$ as the total weight of all hyperedges of the hypergraph $\mathcal{H}$, whose complements intersect the support of x.

An integer-valued monotone function $f : \mathcal{C} \to \mathbb{Z}_+$ is called *polymatroid* if $f(0, ..., 0) = 0$ and f is submodular, i.e., $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$ holds for all vectors $x, y \in \mathcal{C}$, where $x \vee y = (\max\{x_j, y_j\} \mid j = 1, .., n)$ and $x \wedge y = (\min\{x_j, y_j\} \mid j = 1, ..., n)$.

Let us note that transversal functions are both monotone and submodular, thus they are also polymatroid, whenever they take only integer values. In what follows we will be dealing with polymatroid functions whose value at any point $x \in \mathcal{C}$ can be evaluated in polynomial time. Some applications of monotone systems defined via transversal and polymatroid functions are discussed in Section 4.

Analogously to the case of linear inequalities, given a system of inequalities

$$f_i(x) \geq t_i, \quad i = 1, ..., m, \tag{2}$$

where $f_i(x)$, are monotone functions over $\mathcal{C}$ and $t_i \in \mathbb{R}$ for $i = 1, ..., m$, and given a monotone $\vee, \wedge$-formula ϕ in m variables, we can associate to (2) and ϕ a monotone system Σ_ϕ: A vector $x \in \mathcal{C}$ is called *feasible* for Σ_ϕ if $\phi(Y_1(x), ..., Y_m(x)) = 1$, where $Y_i(x) = 1$ iff $f_i(x) \geq t_i$, $i = 1, ..., m$. Let $\mathcal{F}_\phi$ denote the set of all minimal feasible solutions to system Σ_ϕ.

Theorem 2 *If (2) involves transversal inequalities, i.e., if we have $f_i = f_{\mathcal{H}^i, w^i}$ for $i = 1, ..., m$, and if the degree of the evaluation polynomial P_ϕ is bounded, then we can generate $\mathcal{F}_\phi$ in incremental quasi-polynomial time in terms of n, m and $\max_{1 \leq i \leq m} |\mathcal{H}^i|$.*

Note that for the Boolean case $\mathcal{C} = \{0, 1\}^n$, linear inequalities are transversal inequalities corresponding to the hypergraph $\mathcal{H}$ consisting of n singletons $\{1\}, \{2\}, \ldots, \{n\}$. In particular, if the degree of P_ϕ is not bounded, then no efficient generation of minimal feasible solutions for Σ_ϕ is possible, unless P=NP.

Theorem 3 *If (2) involves polymatroid inequalities, and if the degree of the evaluation polynomial P_ϕ is bounded, then we can generate $\mathcal{F}_\phi$ in incremental quasi-polynomial time in terms of n, m and $\max_{1 \leq i \leq m} t_i$.*

Let us remark that the above theorem provides efficient bounds only if $\max_{1 \leq i \leq m} t_i$ is bounded by a polynomial or quasi-polynomial expression of n and m. In fact, generating all minimal feasible solutions to a single polymatroid inequality over $\{0, 1\}^n$ is already NP-hard, if the right hand side is not bounded [8]. Due to this fact, integrality of polymatroid functions is essential in our analysis.

2 Our approach and further results

Our approach is to utilize a general enumeration method for minimal elements of a monotone system, the so called *joint generation*, proposed first in [7, 21], and analyzed at greater detail in [12]. For a subset $\mathcal{X} \subseteq \mathcal{C}$ let us denote by $\mathcal{I}(\mathcal{X})$ the set of all maximal vectors not above any vectors of $\mathcal{X}$, i.e., $\mathcal{I}(\mathcal{X}) = \{\text{maximal } y \in \mathcal{C} \mid \nexists x \in \mathcal{X} : x \leq y\}$. For instance, if $\mathcal{F}_\phi$ denotes the family of all minimal feasible solutions for the system Σ_ϕ, as before, then $\mathcal{I}(\mathcal{F}_\phi)$ is the set of all maximal infeasible solutions.

The method jointly generates $\mathcal{F} \cup \mathcal{I}(\mathcal{F})$ by iteratively extending two partial sets $\mathcal{X} \subseteq \mathcal{F}$ and $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F})$. This is done by solving the so-called *dualization problem*: either produce a point in $\mathcal{I}(\mathcal{X}) \setminus \mathcal{Y}$ or halt because $\mathcal{Y} = \mathcal{I}(\mathcal{X})$ (see e.g., [18]). In particular, incrementally extending a given subset $\mathcal{X} \subseteq \mathcal{F}$ requires solving at most $|\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(\mathcal{X})|$ dualization problems each of size at most $|\mathcal{X}| + |\mathcal{I}(\mathcal{F}) \cap \mathcal{I}(\mathcal{X})|$ with n variables. Let us denote by $\delta(n, M)$ the complexity of solving a dualization problem in n variables and of size $M = |\mathcal{X}| + |\mathcal{Y}|$. Even though no polynomial upper bound on $\delta(n, M)$ is currently available, it is known that $\delta(n, M) = poly(n) M^{o(\log M)}$, see [9, 19].

Note that the joint generation method utilizes the given inequalities in (2) only to check the feasibility of a given vector $x \in \mathcal{C}$, and that any method for generating $\mathcal{F} = \mathcal{F}_\phi$ that uses only such feasibility tests has to perform at least $|\mathcal{F}| + |\mathcal{I}(\mathcal{F})|$ tests (and in fact has to generate both $\mathcal{F}$ and $\mathcal{I}(\mathcal{F})$), see e.g.[20].

The above discussion shows that the quantity $q_\mathcal{F}(k)$ defined by

$$q_\mathcal{F}(k) = \max_{\mathcal{X} \subseteq \mathcal{F},\ |\mathcal{X}| = k} |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \tag{3}$$

is an important parameter intimately related to the generation of the monotone system $\mathcal{F}$. We shall call it the *duality index* of the monotone system $\mathcal{F}$. Using this notion, let us then summarize the above in the following statement.

Proposition 1 *Given a system of monotone inequalities (2), and a monotone Boolean formula ϕ, let Σ_ϕ denote the monotone system associated to these, and let $\mathcal{F}_\phi$ denote the set of all minimal feasible solutions to Σ_ϕ (as above). Then, for an arbitrary subset $\mathcal{X} \subseteq \mathcal{F}_\phi$, we can find $x \in \mathcal{F}_\phi \setminus \mathcal{X}$, or recognize that $\mathcal{X} = \mathcal{F}_\phi$ in time $\ell\delta(n, k+\ell)$ time, where $k = |\mathcal{X}|$ and $\ell = q_{\mathcal{F}_\phi}(|\mathcal{X}|)$.* □

In particular, $\mathcal{F}$ can be generated in incremental quasi-polynomial time whenever the duality index $q_\mathcal{F}(k)$ is bounded by a quasi-polynomial. Hence in order to derive Theorems 1, 2, and 3 from Proposition 1, it now suffices to bound the duality index of $\mathcal{F}_\phi$. In the rest of the paper we focus on the duality index of various monotone systems, and in particular on obtaining the duality index of complex monotone systems from the duality indices of their component systems.

Given a monotone subset $\mathcal{A} \subseteq \mathcal{C}$, i.e., for which $x \in \mathcal{A}$, $y \geq x$ imply $y \in \mathcal{A}$, let us denote by $\min \mathcal{A}$ the set of all minimal vectors of $\mathcal{A}$. Next, if $\mathcal{A}$ is an arbitrary subset of $\mathcal{C}$ then $\mathcal{A}^+ = \{x \in \mathcal{C} \mid x \geq y \text{ for some } y \in \mathcal{A}\}$ denotes the minimum monotone subset (ideal) of $\mathcal{C}$ containing $\mathcal{A}$. Thus, if $\mathcal{F}$ is the set of all minimal feasible solutions of a monotone inequality $f(x) \geq t$ over $\mathcal{C}$, then $\mathcal{F}^+$ is the set of all feasible solutions for the same inequality.

If $\mathcal{A}^+$ and $\mathcal{B}^+$ are monotone subsets of $\mathcal{C}$, then their *conjunction* and *disjunction* are defined as $\mathcal{A}^+ \wedge \mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}^+$ and $\mathcal{A}^+ \vee \mathcal{B}^+ = \mathcal{A}^+ \cap \mathcal{B}^+$. Let us note that the same operations can naturally be extended to their sets of minimal elements, $\mathcal{A} = \min \mathcal{A}^+$ and $\mathcal{B} = \min \mathcal{B}^+$, by defining $\mathcal{A} \wedge \mathcal{B} = \min\{a \vee b \mid a \in \mathcal{A},\ b \in \mathcal{B}\}$ and $\mathcal{A} \vee \mathcal{B} = \min \mathcal{A} \cup \mathcal{B}$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are the minimal feasible solutions of monotone inequalities over $\mathcal{C}$, then $\mathcal{A} \wedge \mathcal{B}$ consists of all minimal vectors satisfying both inequalities, while $\mathcal{A} \vee \mathcal{B}$ consists of all minimal vectors satisfying at least one of the inequalities. Using these definitions, we can thus talk about monotone formulae of arbitrary monotone systems.

Our first result shows that the duality index of conjunctions of monotone systems can effectively be limited in terms of the duality indices of the component systems.

Theorem 4 *Let $\mathcal{F}_i \subseteq \mathcal{C}$ be monotone systems for $i = 1, ..., m$, and let $\phi = Y_1 \wedge Y_2 \wedge \cdots \wedge Y_m$, i.e., $\mathcal{F}_\phi = \mathcal{F}_1 \wedge \mathcal{F}_2 \wedge \cdots \wedge \mathcal{F}_m$. Then, we have*

$$q_{\mathcal{F}_\phi}(k) \leq P_\phi(q_{\mathcal{F}_1}(k), \ldots, q_{\mathcal{F}_m}(k)) = \sum_{i=1}^{m} q_{\mathcal{F}_i}(k).$$

Proof. Clearly, it is enough to prove the statement for $m = 2$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two arbitrary families of vectors in $\mathcal{C}$. Then, we claim that, for all subfamilies $\mathcal{X} \subseteq \mathcal{F}_1 \wedge \mathcal{F}_2$ we have

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_1 \wedge \mathcal{F}_2)| \leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(|\mathcal{X}|).$$

To see the claim, let us consider the sets of vectors

$$\begin{aligned} \mathcal{F}_1' &= \{a \in \mathcal{F}_1 \mid a \vee b \in \mathcal{X} \text{ for some } b \in \mathcal{F}_2\} \quad \text{and} \\ \mathcal{F}_2' &= \{b \in \mathcal{F}_2 \mid a \vee b \in \mathcal{X} \text{ for some } a \in \mathcal{F}_1\}, \end{aligned}$$

let $z \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_1 \wedge \mathcal{F}_2)$ an arbitrary element, and define

$$X = \{a \in \mathcal{F}_1 \mid a \leq z\} \quad \text{and} \quad Y = \{b \in \mathcal{F}_2 \mid b \leq z\}.$$

Then, we must have either $X = \emptyset$ or $Y = \emptyset$, since otherwise we would have elements $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$ for which $a \vee b \leq z$, contradicting that $z \in \mathcal{I}(\mathcal{F}_1 \wedge \mathcal{F}_2)$.

Now, if $X = \emptyset$, then $z \in \mathcal{I}(\mathcal{F}_1) \cap \mathcal{I}(\mathcal{F}_1')$, since it is clear that z is then an independent element with respect to $\mathcal{F}_1$, and furthermore $z \in \mathcal{I}(\mathcal{X})$ implies that for each $i \in [n]$, such that $z_i \neq c_i$, we have an element $a \in \mathcal{F}_1' \subseteq \mathcal{F}_1$ such that $a \leq z + \mathbf{e}^i$, where $\mathbf{e}^i$ is the ith unit vector.

Similarly, if $Y = \emptyset$, then $z \in \mathcal{I}(\mathcal{F}_2) \cap \mathcal{I}(\mathcal{F}_2')$ follows.

Thus,

$$\begin{aligned} |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_1 \wedge \mathcal{F}_2)| &\leq |\mathcal{I}(\mathcal{F}_1') \cap \mathcal{I}(\mathcal{F}_1)| + |\mathcal{I}(\mathcal{F}_2') \cap \mathcal{I}(\mathcal{F}_2)| \\ &\leq q_{\mathcal{F}_1}(|\mathcal{F}_1'|) + q_{\mathcal{F}_2}(|\mathcal{F}_2'|) \\ &\leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(\max\{|\mathcal{F}_1'|, |\mathcal{F}_2'|\}) \\ &\leq (q_{\mathcal{F}_1} + q_{\mathcal{F}_2})(|\mathcal{X}|). \end{aligned}$$

□

The above theorem implies that if (2) involves monotone inequalities the duality indices of which are (quasi)-polynomially bounded, then, so is the duality index of the entire system, implying a quasi-polynomially efficient generation of minimal feasible solutions, by Proposition 1.

In fact, an efficient bound on the duality index is known for several types of monotone inequalities. Let us denote temporarily by $\mathcal{A}$ the set of all minimal feasible solutions to a single monotone inequality $f(x) \geq t$. For a monotone linear function $f(x) = \sum_{j=1}^{n} a_j x_j \geq t$, where $a_j \geq 0$ for $j = 1, ..., n$, we have by [9] that

$$q_{\mathcal{A}}(k) \leq nk. \tag{4}$$

If $f(x) = f_{\mathcal{H},w}(x) \geq t$ is a weighted transversal function, then it was shown in [13] that

$$q_{\mathcal{A}}(k) \leq |\mathcal{H}|k, \tag{5}$$

regardless of the weights. Note that both of these bounds are sharp, up to a constant factor. Finally, if f is a polymatroid function, then it was shown in [8] that

$$q_{\mathcal{A}}(k) \leq max\{n, k^{(\log t)/c(2n,k)}\}, \tag{6}$$

where $c(\alpha, \beta)$ is the unique positive root of the equation $2^c(\alpha^{c/\log \beta} - 1) = 1$ (note that the above bound implies that $q_{\mathcal{A}} \leq (2nk)^{\log t}$ and that $c(2n, k) \approx \log \log k$ for large k). The exponent of (6) is asymptotically tight [11].

Corollary 1 *If the conjunction of inequalities (2) consists of linear, weighted transversal and/or polymatroid functions (the latter ones with quasi-polynomially limited right-hand sides), then all minimal feasible solutions of (2) can be generated in quasi-polynomial incremental time.* □

Unfortunately, as the following claim shows, no result analogous to Theorem 4 can hold unconditionally for disjunctions of monotone sets.

Theorem 5 *For each $\ell \geq 3$ there exist monotone systems $\mathcal{A}, \mathcal{B} \subseteq \{0, 1\}^{4\ell}$, for which $q_{\mathcal{A}}(k) \leq (2\ell + 1)k$ and $q_{\mathcal{B}}(k) \leq (2\ell + 1)k$, while $q_{\mathcal{A}\vee\mathcal{B}}(|\mathcal{A} \vee \mathcal{B}|) = 2^{|\mathcal{A}\vee\mathcal{B}|}$.*

To prove Theorem 5, we need a few definitions and a lemma. Given a finite set V and a hypergraph $\mathcal{H} \subseteq 2^V$, let us define the independence-degree of a hyperedge $H \in \mathcal{H}$ by

$$d(H) = |\{\, I \in \mathcal{I}(\mathcal{H}) \mid |H \setminus I| = 1\}|,$$

where $\mathcal{I}(\mathcal{H})$ is the family of maximal independent sets of $\mathcal{H}$. For a threshold r let

$$\mathcal{H}_r = \{H \in \mathcal{H} \mid d(H) \leq r\}.$$

Lemma 1 *Let $\mathcal{H} \subseteq 2^V$ be a given hypergraph and r be a given integer threshold. If for all $I \in \mathcal{I}(\mathcal{H})$ there exists a vertex $x \in V \setminus I$ for which $I \cup \{x\}$ contains hyperedges only from $\mathcal{H}_r$, then we have for all positive integers k that*

$$q_{\mathcal{H}}(k) \leq rk.$$

Proof. Let $\mathcal{X}$ be an arbitrary subset of $\mathcal{H}$ of cardinality k. If $I \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{H})$, then there exists a vertex $x_I \notin I$ for which $I \cup \{x_I\}$ contains hyperedges only from $\mathcal{H}_r$, according to the assumption. For each $I \in \mathcal{I}(\mathcal{X})$, the set $I \cup \{x_I\}$ must contain a hyperedge from $\mathcal{X}$, by definition. Let us associate to each $I \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{H})$ such a hyperedge $X_I \subseteq I \cup \{x_I\}$, $X_I \in \mathcal{X}$. Then, we have $X_I \in \mathcal{X} \cap \mathcal{H}_r$ for all $I \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{H})$. Thus, we have $d(X_I) \leq r$ for each, i.e., there are at most r independent sets $I' \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{H})$ for which $X_I = X_{I'}$. Therefore,

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{H})| \leq r|\mathcal{X} \cap \mathcal{H}_r| \leq r|\mathcal{X}|$$

follows. □

Proof of Theorem 5. Let $X = \{1, ..., 2\ell\}$, $Y = \{2\ell+1, ..., 4\ell\}$, and $V = X \cup Y$. Furthermore, let

$$\mathcal{M}_a = \{\{i, i+\ell\} \mid i = 1, .., \ell\} \quad \text{and} \quad \mathcal{M}_b = \{\{i+2\ell, i+3\ell\} \mid i = 1, .., \ell\}$$

be maximum matchings on the sets X and Y, respectively, and let $\mathcal{T}$ denote the family of all maximal transversals to $\mathcal{M}_a \cup \mathcal{M}_b$. Let us finally define

$$\begin{aligned} \mathcal{A} &= \mathcal{M}_a \cup \{T \cup \{y\} \mid T \in \mathcal{T}, y \in Y \setminus T\}, \text{ and} \\ \mathcal{B} &= \mathcal{M}_b \cup \{T \cup \{x\} \mid T \in \mathcal{T}, x \in X \setminus T\}. \end{aligned}$$

It is easy to see that every hyperedge of $\mathcal{A} \cup \mathcal{B}$ contains an edge from $\mathcal{M}_a \cup \mathcal{M}_b$, and since this latter family is a subfamily of $\mathcal{A} \cup \mathcal{B}$,

$$\mathcal{A} \vee \mathcal{B} = \mathcal{M}_a \cup \mathcal{M}_b$$

follows, immediately implying the last equality in the statement.

To complete the proof, we need to verify that both $\mathcal{A}$ and $\mathcal{B}$ are uniformly dual-bounded, as claimed. Since the roles of X and Y are symmetric in $\mathcal{A}$ and $\mathcal{B}$, it is enough to show this for one of them, say for $\mathcal{A}$.

We shall show the claim by Lemma 1 with $r = 2\ell + 1$. To this end, let us first observe that $\mathcal{A}$ has three types of maximal independent sets, and we shall refer to them as *red*, *blue* and *yellow*, to simplify notations:

$$\begin{aligned} \mathcal{I}(\mathcal{A}) \;=\; \mathcal{T} \;\cup\; & \{Y \cup (T \cap X) \setminus \{x\} \mid T \in \mathcal{T}, x \in T \cap X\} \\ \cup\; & \{(T \cap X) \cup (Y \setminus e) \mid T \in \mathcal{T}, e \in \mathcal{M}_b\}. \end{aligned}$$

To see this, let us first observe that if $I \in \mathcal{I}(\mathcal{A})$ contains at least one endpoint of each of the sets of $\mathcal{M}_a \cup \mathcal{M}_b$, then it contains exactly one such endpoint from each, and $I \in \mathcal{T}$ (type red). If I does not contain either endpoints of an edge $e \in \mathcal{M}_a$, then it must contain exactly one endpoint from each other $e' \in \mathcal{M}_a$, and must contain Y (type blue). Finally, if I does not contain either endpoints of a set $e \in \mathcal{M}_b$, then it must contain exactly one endpoint of each $e' \in \mathcal{M}_a$, and contains $Y \setminus e$ (type yellow).

Let us next show that

$$\mathcal{A}_{2\ell+1} = \mathcal{A} \setminus \mathcal{M}_a.$$

Indeed, $d(\{i, i+\ell\}) \geq |\mathcal{T}| > 2\ell + 1$ for all $\ell \geq 3$, and hence $\mathcal{A}_{2\ell+1} \subseteq \mathcal{A} \setminus \mathcal{M}_a$. On the other hand, for $H = T \cup \{y\}$, $T \in \mathcal{T}$ and $y \in Y \setminus T$, we have $d(H) = 2\ell + 1$. To see this let us denote by $e \in \mathcal{M}_b$ the edge containing y. Then, for every every $x \in H \cap X$ we have a unique blue set containing $H \setminus \{x\}$; for both endpoints $i \in e$ the set $H \setminus \{i\}$ is a blue set; and finally for every $y' \in H \cap Y \setminus e$ there is a unique yellow set containing $H \setminus \{y'\}$.

Finally, we claim that for each $I \in \mathcal{I}(\mathcal{A})$ there exists a point $x_i \notin I$ such that $I \cup \{x_I\}$ contains hyperedges of $\mathcal{A}$ only from $\mathcal{A}_{2\ell+1}$, i.e., does not contain an edge of $\mathcal{M}_a$. For a red $I = T \in \mathcal{I}(\mathcal{A})$ any $x_I \in Y \setminus I$ is such a point (and there are $2\ell > 1$). For a blue $I = (T \cap X \setminus \{x\}) \cup Y \in \mathcal{I}(\mathcal{A})$ we can set $x_I = x$. Finally, for a yellow set $I = (T \cap X) \cup (Y \setminus e) \in \mathcal{I}(\mathcal{A})$ we can set x_I to be either endpoints of $e \in \mathcal{M}_b$.

Thus, Lemma 1 can be applied, and hence

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{A})| \leq (2\ell + 1)|\mathcal{X}|$$

follows for all subfamilies $\mathcal{X} \subseteq \mathcal{A}$. $\square$

However, for the special cases of linear, transversal or polymatroid systems the following result can be regarded as an analogue of Theorem 4.

Theorem 6 *Assume that either all functions $f_i(x)$, $i = 1, ..., m$ in (2) are transversal and/or linear, or all of them are polymatroid, and let $\phi = Y_1 \vee Y_2 \vee \cdots \vee Y_m$. Then*

$$q_{\mathcal{F}_\phi}(k) \leq P_\phi(q_1(k), \ldots, q_m(k)) = \prod_{i=1}^{m} q_i(k),$$

where P_ϕ is the evaluation polynomial of ϕ, and where $q_i(k)$ are the upper bounds on $q_{\mathcal{F}_i}(k)$ stated in (4), (5), and (6).

We claim next that even though we cannot mix different types of inequalities in Theorem 6, still a result analogous to Theorem 1 can be derived from Theorems 4 and 6.

Theorem 7 *If the system of inequalities (2) consists of either m linear and/or transversal inequalities or m polymatroid inequalities, and ϕ is an arbitrary monotone $\vee, \wedge$-formula in m propositional variables, then the inequality*

$$q_{\mathcal{F}_\phi}(k) \leq P_\phi(q_1(k), q_2(k), ..., q_m(k))$$

holds, where P_ϕ is the evaluation polynomial of ϕ, and where, as in Theorem 6, the functions $q_i(k)$ are the upper bounds on the duality indices of the individual inequalities stated in (4), (5), and (6).

Theorems 1, 2 and 3 readily follow from Theorem 7 in view of Proposition 1 and the bounds of (4), (5) and (6).

In the next section we state the main lemmas from which Theorems 6 and 7 can be derived, and which may be of interest on their own. In the last section we list several applications of monotone systems defined by linear, transversal and polymatroid inequalities.

3 Main lemmas

Aggregating polymatroid inequalities

Let $f_i : \mathcal{C} \to \mathbb{Z}_+$, be a polymatroid function, $t_i \in \mathbb{Z}_+$ be a given positive integer threshold, and denote by $\mathcal{F}_i \subseteq \mathcal{C}$ the set of all minimal feasible solutions to the polymatroid inequality $f_i(X) \geq t_i$, $i = 1, ..., m$. Let us further define

$$(f_1 \wedge \cdots \wedge f_m)(x) = \sum_{i=1}^{m} \min\{f_i(x), t_i\}, \qquad \text{and} \tag{7}$$

$$(f_1 \vee \cdots \vee f_m)(x) = \prod_{i=1}^{m} t_i - \prod_{i=1}^{m} (t_i - \min\{f_i(x), t_i\}) \tag{8}$$

for all $x \in \mathcal{C}$.

Lemma 2 *Both functions, $g = f_1 \wedge \cdots \wedge f_m$ and $h = f_1 \vee \cdots \vee f_m$ are polymatroid. Furthermore, the sets $\mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_m$ and $\mathcal{F}_1 \vee \cdots \vee \mathcal{F}_m$ respectively consist of all minimal feasible solutions of the polymatroid inequalities*

$$g(x) \geq \sum_{i=1}^{m} t_i \quad \text{and} \quad h(x) \geq \prod_{i=1}^{m} t_i.$$

Proof. We have to show that the monotonically increasing functions $g(x)$ and $h(x)$ are polymatroid. It is well known and easy to see that if $f_i(x)$ is polymatroid, then so is $f_i^*(x) = \min\{f_i(x), t_i\}$. Since a sum of polymatroid functions is polymatroid, we conclude that $g(x)$ is polymatroid. To prove that $h(x)$ is polymatroid as well, write $\mathcal{C}$ as the product of n chains $\mathcal{C}_1 \times \cdots \times \mathcal{C}_n$. Then it suffices to show that for any $i \in [n]$ and for any $z \in \mathcal{C}_i$, the difference

$$\partial(x, i, z) \ \stackrel{\text{def}}{=}\ h(x + \mathbf{e}^i) - h(x) = \prod_{i=1}^{m} [t_i - f_i^*(x)] \ - \ \prod_{i=1}^{m} [t_i - f_i^*(x + \mathbf{e}^i)]$$

is monotonically decreasing in $x \in \mathcal{C}_1 \times \ldots \times \mathcal{C}_{i-1} \times \{z\} \times \mathcal{C}_{i+1} \times \ldots \times \mathcal{C}_n$, where $\mathbf{e}^i$ is the ith unit vector. This readily follows from the representation

$$\partial(x,i,z) = \sum_{j=1}^{m} \Big(\prod_{k=1}^{m-j} [t_k - f_k^*(x+\mathbf{e}^i)] \cdot \prod_{k=m-j+1}^{m} [t_k - f_k^*(x)] \Big) \; [f_j^*(x+\mathbf{e}^i) - f_j^*(x)].$$

□

Lemma 2 implies that any monotone $\vee, \wedge$-formula of polymatroid inequalities can be replaced by an equivalent polymatroid inequality:

Corollary 2 *Let ϕ be a monotone $\vee, \wedge$-formula in m variables, $f_i(x)$ be a polymatroid function, $t_i \in \mathbb{Z}_+$ be non-negative integral threshold, and let $\mathcal{F}_i$ denote the set of all minimal feasible solutions of the polymatroid inequality $f_i(x) \geq t_i$, for $i = 1, ..., m$. Then, $\mathcal{F}_\phi = \phi(\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_m)$ is the set of all minimal feasible solutions of the system Σ_ϕ, and also of the single polymatroid inequality*

$$(\phi(f_1, f_2, ..., f_m))\,(x) \geq P_\phi(t_1, ..., t_m),$$

where P_ϕ is the evaluation polynomial of ϕ.

Monotone composition of transversal inequalities

Let us associate to a system of nonnegative weights $w(i,z)$, $i = 1, ..., n$, $z = 0, ..., c_i$ a separable function $f_w : \mathcal{C} \longrightarrow \mathbb{R}_+$ defined by

$$f_w(x) = \sum_{i=1}^{n} \sum_{z=0}^{x_i} w(i,z). \tag{9}$$

Clearly, the mapping f_w is a special, separable transversal function.

Denote by $\mathbb{N}$ the set of positive integers, and start with an elementary lemma, which we shall need later.

Lemma 3 *If $a, b, m \in \mathbb{N}$, $p, q \in \mathbb{N}^m$, and $w \in \mathbb{R}^m_+$, such that $w^T p \geq a$ and $w^T q \leq b$ then*

$$\max_{1 \leq i \leq m} \frac{p_i}{q_i} \geq \frac{w^T p}{w^T q} \geq \frac{a}{b}.$$

□

Lemma 4 *Let f_w be a nonnegative separable mapping as in (9), and let $t \in \mathbb{R}_+$. Assume further that $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$ are subsets of vectors for which $\mathcal{X} \neq \emptyset$, $(0, ..., 0) \notin \mathcal{X}$, and which satisfy the following separation constraints:*

(i) $f_w(x) \geq t$ holds for all $x \in \mathcal{X}$;

(ii) $f_w(y) < t$ for all $y \in \mathcal{Y}$.

Then, there exists a subfamily $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $x \not\leq y \vee y'$ for all $y, y' \in \mathcal{Y}'$ and $x \in \mathcal{X}$, and

$$|\mathcal{Y}'| \geq \frac{|\mathcal{Y}|}{\left(\sum_{x \in \mathcal{X}} \nu(x)\right)} \geq \frac{|\mathcal{Y}|}{n|\mathcal{X}|},$$

where $\nu(x)$ denotes the number of nonzero entries of $x \in \mathcal{C}$.

Proof. Let $L = \{(i, z) \mid i \in V, z \in \mathcal{C}_i \setminus \{\ell_i\}\}$, and let us prove the claim by induction on the size of L. Clearly, if $|L| \leq 1$, then $|\mathcal{X}| = 1$ and $|\mathcal{Y}| \leq 1$, and thus the claim holds trivially.

For $(i, z) \in L$ let us define

$$\mathcal{X}(i, z) = \{x \in \mathcal{X} \mid x_i < z\} \quad \text{and} \quad \mathcal{Y}(i, z) = \{y \in \mathcal{Y} \mid y_i < z\},$$

and let $U = \{(i, z) \mid \mathcal{X}(i, z) = \emptyset\}$.

Let us observe first that if $x \leq y \vee y'$ for some $x \in \mathcal{X}$, and $y, y' \in \mathcal{Y}(i, z)$ for some $(i, z) \in L$, then we must have $x \in \mathcal{X}(i, z)$.

This observation implies first of all that (b) holds for all families $\mathcal{Y}(i, z) \subseteq \mathcal{Y}$ with $(i, z) \in U$.

Let us also observe that the sets $\mathcal{X}(i, z) \subseteq \mathcal{C}$ and $\mathcal{Y}(i, z) \subseteq \mathcal{C}$ satisfy the conditions (i) and (ii) for every $(i, z) \in L \setminus U$ with a smaller i-th chain $\mathcal{C}'_i = \mathcal{C}_i \setminus \{z' \mid z' \geq z\}$, i.e., for which the associated set $L' = L \setminus \{(i, z') \mid z' \in \mathcal{C}_i, z' \geq z\}$ satisfies $|L'| < |L|$. Thus, by our inductive hypothesis, there exists a subset $\mathcal{Y}(i, z)' \subseteq \mathcal{Y}(i, z)$ which satisfies (b) by our first observation, and for which we have

$$|\mathcal{Y}(i, z)'| \geq \frac{|\mathcal{Y}(i, z)|}{\sum_{x \in \mathcal{X}(i,z)} \nu(x)}.$$

Thus, we can prove the claim by induction, if we can show that

$$\max\left\{\max_{(i,z) \in U} |\mathcal{Y}(i, z)|, \quad \max_{(i,z) \in L \setminus U} \frac{|\mathcal{Y}(i, z)|}{\sum_{x \in \mathcal{X}(i,z)} \nu(x)}\right\} \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)}.$$

If we have $|\mathcal{Y}(i, z)| \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)}$ for some $(i, z) \in U$, then we are ready, with $\mathcal{Y}' = \mathcal{Y}(i, z)$. Let us assume next that $U \neq \emptyset$ and that

$$|\mathcal{Y}(i, z)| < \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)} \quad \text{for all} \quad (i, z) \in U.$$

Clearly, if $z, z' \in \mathcal{C}_i$, and $z \geq z'$, then $\mathcal{Y}(i,z) \supseteq \mathcal{Y}(i,z')$. Thus, by defining $z_i = \max\{z \mid (i,z) \in U\}$ for $i \in V$, setting $\mathbf{z} = (z_1, z_2, ..., z_n)$ and introducing

$$\tau = \nu(\mathbf{z}) = \sum_{\substack{i \in V \\ z_i > \ell_i}} 1 \qquad \text{and} \qquad \widetilde{\mathcal{Y}} = \mathcal{Y} \setminus \bigcup_{(i,z) \in U} \mathcal{Y}(i,z),$$

we have

$$|\widetilde{\mathcal{Y}}| \geq \left(1 - \frac{\tau}{\sum_{x \in \mathcal{X}} \nu(x)}\right) |\mathcal{Y}|. \tag{10}$$

Defining further $\widetilde{\mathcal{C}}_i = \{z' \in \mathcal{C}_i \mid z' \geq z_i\}$ for $i \in V$, and setting $\widetilde{\mathcal{C}} = \bigotimes_{i \in V} \widetilde{\mathcal{C}}_i$, we can see that the the sets $\mathcal{X} \subseteq \widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{Y}} \subseteq \widetilde{\mathcal{C}}$ satisfy (i) and (ii) with an associated set $\widetilde{L} = L \setminus U$, i.e., for which $|\widetilde{L}| < |L|$ in this case (we assumed $U \neq \emptyset$). Thus, by our inductive hypothesis there exists a subset $\mathcal{Y}' \subseteq \widetilde{\mathcal{Y}}$ for which (b) holds and for which we have

$$|\mathcal{Y}'| \geq \frac{|\widetilde{\mathcal{Y}}|}{\sum_{x \in \mathcal{X}} \widetilde{\nu}(x)} \tag{11}$$

where $\widetilde{\nu}(x) = \sum_{\substack{i \in V \\ x_i > z_i}} 1$ (note that $z_i = \min \widetilde{\mathcal{C}}_i$).

Let us also note that

$$\sum_{x \in \mathcal{X}} \widetilde{\nu}(x) \leq \sum_{x \in \mathcal{X}} \nu(x) \ - \tau. \tag{12}$$

This is because $\mathcal{X}(i,z) \neq \emptyset$ for all $z > z_i$, by the definition of z_i, and thus for all $i \in V$ there exists a vector $x^i \in \mathcal{X}$ for which $x^i_i = z_i$, and therefore $\widetilde{\nu}(x^i) \leq \nu(x^i) - 1$ for all indices for which $z_i > \ell_i$.

From inequalities (10), (11) and (12) we can conclude that

$$|\mathcal{Y}'| \geq \frac{\left(1 - \frac{\tau}{\sum_{x \in \mathcal{X}} \nu(x)}\right) |\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \widetilde{\nu}(x)} = \left(\frac{\sum_{x \in \mathcal{X}} \nu(x) \ - \tau}{\sum_{x \in \mathcal{X}} \widetilde{\nu}(x)}\right) \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)} \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)},$$

implying that the set $\mathcal{Y}'$ satisfies both (a) and (b).

Let us finally assume that $U = \emptyset$, and prove

$$\max_{(i,z) \in L} \frac{|\mathcal{Y}(i,z)|}{\sum_{x \in \mathcal{X}(i,z)} \nu(x)} \geq \frac{|\mathcal{Y}|}{\sum_{x \in \mathcal{X}} \nu(x)} \tag{13}$$

by Lemma 3, using the weights $\frac{w(i,z)}{W-t}$ for $(i,z) \in L$, where $W = \sum_{(i,z) \in L} w(i,z)$. Observe that due to the strict inequality in (ii), we may assume that $W > t$.

Let us note first that

$$\sum_{(i,z) \in L} \frac{w(i,z)}{W-t} |\mathcal{Y}(i,z)| = \sum_{y \in \mathcal{Y}} \sum_{\substack{(i,z) \in L \\ y_i < z}} \frac{w(i,z)}{W-t} = \sum_{y \in \mathcal{Y}} \frac{W - f_w(y)}{W-t} > |\mathcal{Y}|, \tag{14}$$

is implied by the inequalities $W - f_w(y) > W - t$ for all $y \in \mathcal{Y}$, which follow by (ii).

Let us note next that

$$\sum_{(i,z)\in L} \frac{w(i,z)}{W-t} \sum_{x\in\mathcal{X}(i,z)} \nu(x) = \sum_{x\in\mathcal{X}} \sum_{\substack{(i,z)\in L\\ x_i<z}} \frac{w(i,z)}{W-t}\, \nu(x)$$
$$= \sum_{x\in\mathcal{X}} \frac{W-f_w(x)}{W-t}\, \nu(x) \le \sum_{x\in\mathcal{X}} \nu(x), \tag{15}$$

is implied by the inequalities $W - f_w(x) \le W - t$ for all $x \in \mathcal{X}$, which follow by (i).

Thus, (14) and (15) imply (13) by Lemma 3, and thus conclude our proof. □

Corollary 3 *Assume that $f^i_{w^i}$ is a nonnegative separable mapping as in (9), and that $t_i \in \mathbb{R}_+$, for $i = 1, ..., m$. Assume further that $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{C}$ are nonempty collections of vectors satisfying the following separation condition:*

(i) For all $x \in \mathcal{X}$ we have $f^i_{w^i}(x) \ge t_i$ for at least one of the indices $i \in \{1, ..., m\}$.

(ii) For all $y \in \mathcal{Y}$ we have $f^i_{w^i}(y) < t_i$ for all indices $i \in \{1, ..., m\}$.

Then, there exists a subset $\mathcal{Y}' \subseteq \mathcal{Y}$ such that $x \not\le y \vee y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}'$, and

$$|\mathcal{Y}'| \ge \frac{|\mathcal{Y}|}{(n|\mathcal{X}|)^m}.$$

Proof. We shall show the claim by induction on m. If $m = 1$, then the statement follows from Lemma 4, and the fact that $\nu(x) \le n$ for all $x \in \mathcal{C}$ (note that $0 \notin \mathcal{X}$ since otherwise $\mathcal{Y}$ would be empty). In the general case, let us consider a partition of $\mathcal{X}$,

$$\mathcal{X} = \bigcup_{i\in[m]} \mathcal{X}^i$$

such that $f^i_{w^i}(x) \ge t_i$ for all $x \in \mathcal{X}^i$ and for all $i \in [m]$. Assume without loss of generality that $cX^i \ne \emptyset$ for $i = 1, \ldots, m'$, for some $m' \le m$. Then for the sets $\mathcal{Y}$ and $\mathcal{X}^{m'}$ together with the separable mapping $f^{m'}$ and threshold $t_{m'}$, the conditions of Lemma 4 hold. Thus there exists a subfamily $\widetilde{\mathcal{Y}} \subseteq \mathcal{Y}$ for which $x \not\le y \vee y'$ for all $y, y' \in \widetilde{\mathcal{Y}}$ and $x \in \mathcal{X}^m$ holds and for which

$$|\widetilde{\mathcal{Y}}| \ge \frac{|\mathcal{Y}|}{\sum_{x\in\mathcal{X}^{m'}} \nu(x)}.$$

Let us next consider the families $\widetilde{\mathcal{Y}}$, $\mathcal{X}^1 \cup \cdots \cup \mathcal{X}^{m'-1}$, together with $f^i_{w^i}$ and t_i for $i \in [m'-1]$. Then conditions (i) and (ii) hold for this input, and thus we can apply our inductive hypothesis, and conclude that there exists a subfamily $\mathcal{Y}' \subseteq \widetilde{\mathcal{Y}}$ such that $x \not\leq y \vee y'$ for all $y, y' \in \mathcal{Y}'$ and $x \in \mathcal{X}^1 \cup \cdots \cup \mathcal{X}^{m'-1}$ hold, and for which

$$|\mathcal{Y}'| \geq \frac{|\widetilde{\mathcal{Y}}|}{\prod_{i=1}^{m'-1}\left(\sum_{x\in\mathcal{X}^i}\nu(x)\right)}.$$

Putting these together completes the proof of the claim. □

Let us remark that the bound in Corollary 3 cannot be improved by more than a factor of $O(m^{2m})$, i.e., it is tight whenever m is constant. Let us add that this bound, and consequently all bounds in our previous theorems can be improved by a factor of m^m: In our claims the evaluation polynomial P_ϕ associated to a Boolean expression ϕ could be replaced by Q_ϕ obtained from ϕ by replacing conjunctions by arithmetic addition, and disjunctions by the arithmetic $\diamond$ operation defined by $a_1 \diamond a_2 \diamond \cdots \diamond a_r = a_1 a_2 \cdots a_r / r^r$.

Proof of Theorem 6. Let $\mathcal{X} \subseteq \mathcal{F}_\phi$ be an arbitrary subfamily of minimal feasible solutions of the system of size k, and let $\mathcal{Y} = \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_\phi)$. Clearly, if the functions $f_1, \ldots, f_m$ are polymatroid, then the statement follows immediately from Corollary 2. Assume next that the functions $f_1, \ldots, f_m$ are linear and/or transversal. In this case we apply Lemma 4 as follows. Let $\mathcal{X}^1 \cup \ldots \cup \mathcal{X}^m = \mathcal{X}$ be a partition of $\mathcal{X}$ such that $f_i(x) \geq t_i$ for $x \in \mathcal{X}^i$ and $i \in [m]$. Let $I \subseteq [m]$ be the set of indices i for which the function f_i is linear. For $i \in I$, if the function $f_i(x) = \sum_{j=1}^n a_{ij} x_j$ for $x \in \mathcal{C}$, where $a_{ij} \geq 0$ for all $j = 1, \ldots, n$, we define the non-negative weight $w^i(j, z) = a_j$ for $j \in [n]$ and $z \in \mathcal{C}_j$. Then clearly, for all $x \in \mathcal{C}$, $f_i(x) \geq t_i$ if and only if $f^i_{w^i}(x) \geq t_i$. Now if $\cup_{i\in I}\mathcal{X}^i \neq \emptyset$, we apply Corollary 3 to conclude that there is a subset $\widetilde{\mathcal{Y}} \subseteq \mathcal{Y}$ of size

$$|\widetilde{\mathcal{Y}}| \geq \frac{|\mathcal{Y}|}{\prod_{i\in I} q_i(k)},$$

such that $x \not\leq y \vee y'$ for all $x \in \mathcal{X}$ and $y, y' \in \widetilde{\mathcal{Y}}$. If $\cup_{i\in I}\mathcal{X}^i = \emptyset$, we let $\widetilde{\mathcal{Y}} \stackrel{\text{def}}{=} \mathcal{Y}$. For $i \in [m] \setminus I$, if the function f_i is transversal with respect to the family of vectors $\mathcal{H}_i \subseteq \mathcal{C}$, and corresponding weight $w^i : \mathcal{H}_i \mapsto \mathbb{R}_+$, then we define a non-negative weight function $w^i(a, z)$ over $a \in \mathcal{H}_i$ and $z \in \{0, 1\}$ as follows: $w^i(a, z) = 0$ if $z = 0$, and $w^i(a, z) = w^i(a)$ if $z = 1$. For $x \in \mathcal{C}$, define the vector $\eta^i(x) \in \{0, 1\}^{\mathcal{H}_i}$ whose a-component, for $a \in \mathcal{H}_i$, is given by $\eta^i_a(x) = 1$ if and only if $a \not\geq x$. Note that for $x \in \mathcal{C}$, we have $f_i(x) \geq t_i$ if and only if $f^i_{w^i}(\eta^i(x)) \geq t_i$, for $x \in \mathcal{C}$. Thus, if $\mathcal{X}^i \neq \emptyset$, we can apply Lemma 4 to the families $\eta^i(\mathcal{X}) \stackrel{\text{def}}{=} \{\eta^i(x) \mid x \in \mathcal{X}^i\}$ and $\eta^i(\widetilde{\mathcal{Y}}) \stackrel{\text{def}}{=} \{\eta^i(y) \mid y \in \widetilde{\mathcal{Y}}\}$ to conclude that there exists a subset $\mathcal{Y}^i \subseteq \widetilde{\mathcal{Y}}$ of size $|\eta^i(\mathcal{Y}^i)| \geq |\eta^i(\widetilde{\mathcal{Y}})|/q_i(k)$, such that $\eta^i(x) \not\leq \eta^i(y) \vee \eta^i(y')$ for all $y, y' \in \mathcal{Y}^i$ and $x \in \mathcal{X}^i$. Note that, since $\eta^i(y) \vee \eta^i(y') = \eta^i(y \vee y')$ for all $y, y' \in \mathcal{C}$ and $\eta^i(x)$ is monotone mapping in $x \in \mathcal{C}$, we get that $x \not\leq y \vee y'$ for all $x \in \mathcal{X}^i$ and $y, y' \in \mathcal{Y}^i$. Continuing this way, for all $i \in [m] \setminus I$, we get finally a subfamily $\mathcal{Y}' \subseteq \mathcal{Y}$ of size at least $|\mathcal{Y}| / \prod_{i=1}^m q_i(k)$, such that $x \not\leq y \vee y'$ for all $x \in \mathcal{X}$ and $y, y' \in \mathcal{Y}'$. Note, however, that $\mathcal{X} \subseteq \mathcal{F}$ and $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F})$ imply that, for any distinct $y, y' \in \mathcal{Y}$, there exists an $x \in \mathcal{X}$, such that $y \vee y' \geq x$. Consequently, there cannot be more than one element in $\mathcal{Y}'$, i.e. $|\mathcal{Y}'| \leq 1$, and the theorem follows. □

Proof of Theorem 7. First, we observe that if P_ϕ is the evaluation polynomial for a monotone formula ϕ, and P'_ϕ is the evaluation polynomial for the *conjunctive normal form (CNF)* representation of ϕ then

$$P'_\phi(x_1, \ldots, x_m) \leq P_\phi(x_1, \ldots, x_m),$$

for all positive integers $x_1, \ldots, x_m$. Thus we may assume without loss of generality that the formula $\phi(Y_1, \ldots, Y_m) = \bigwedge_{j=1}^{r} \bigvee_{i \in I_j} Y_j$ is given in CNF form, i.e. $\mathcal{F}_\phi = \mathcal{F}_1 \wedge \cdots \wedge \mathcal{F}_r$, where $\mathcal{F}_1, \ldots, \mathcal{F}_r$ are minimal feasible solutions satisfying monotone disjunctions of linear/transversal or polymatroid inequalities. From Theorem 6, we conclude that

$$q_{\mathcal{F}_j}(k) \leq \prod_{i \in I_j} q_i(k), \qquad \text{for } j = 1, \ldots, r.$$

Form Theorem 4, we conclude that

$$q_{\mathcal{F}}(k) \leq \sum_{j=1}^{r} q_{\mathcal{F}_j}(k) = \sum_{j=1}^{r} \prod_{i \in I_j} q_i(k) = P_\phi(q_1(k), \ldots, q_m(k)).$$

□

4 Applications

Disjunctive Programming. Let $A_1 \in \mathbb{R}_+^{r_1 \times n}, \ldots, A_m \in \mathbb{R}_+^{r_m \times n}$ be non-negative real matrices, and $b^1 \in \mathbb{R}_+^{r_1}, \ldots, b^m \in \mathbb{R}_+^{r_m}$ be positive real vectors. Consider the following disjunctive normal form (DNF) of linear monotone inequalities [5]:

$$\bigvee_{i=1}^{m} (A_i x \geq b_i), \qquad x \in \mathbb{R}_+^n. \tag{16}$$

It follows from Theorem 1 that, if m is bounded by a constant then all minimal integer solutions of (16) can be enumerated in incremental quasi-polynomial time. In contrast, when m is unbounded but $\max\{r_1, \ldots, r_m\} \leq const$, Theorem 1 implies a quasi-polynomial incremental algorithm for enumerating all maximal infeasible vectors for (16).

Data mining. Given a binary database $\mathcal{D} \subseteq \{0,1\}^n$, an integer threshold t, a subset $X \subseteq [n]$ is said to be t-frequent if $s(X) = |\{Y \in \mathcal{D} \ : \ Y \supseteq X\}| \geq t$ and t-infrequent if $s(X) < t$. It is easy to see that the function $f(X) = |\mathcal{D}| - s(X)$ is the transversal function for the hypergraph $\mathcal{H} = \{[n] \setminus Y \ : \ Y \in \mathcal{D}\}$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be m binary databases, $t_1, \ldots, t_m$ be real thresholds, and consider the family

$$\mathcal{F} = \min\{X \subseteq [n] \mid \exists i \in [m] : X \text{ is } t_i\text{-infrequent with respect to } \mathcal{D}_i\}.$$

For instance, each database $\mathcal{D}_i$ may represent the set of items purchased in each weekday $i = 1, \ldots, m = 7$, with $\mathcal{F}$ representing the family of minimal collections of items that lie below a specified purchase threshold in at least one of the 7 days of the week. Clearly, $\mathcal{F}$ is the family of minimal true vectors for the disjunction of transversal inequalities, and thus, for constant m, the elements of $\mathcal{F}$ can be enumerated in incremental quasi-polynomial time by Theorem 2. The generation of maximal frequent and minimal infrequent sets arises in the generation of association rules in data mining applications, see e.g. [2, 3, 20].

Sparse boxes. Another notion related to data mining applications is that of sparse boxes. Let $\mathcal{S}$ be a set of points in n dimensions, and $t \leq |\mathcal{S}|$ be a given integer. A *maximal t-box* is a closed n-dimensional interval which contains at most t points of $\mathcal{S}$ in its interior, and which is maximal with respect to this property (i.e., cannot be extended in any direction without strictly enclosing more points of $\mathcal{S}$). Typically, the set of points $\mathcal{S}$ represents the set of records in a quantitative database, and maximal sparse boxes correspond to empty or nearly empty regions in the data space [4, 15, 17, 23, 24]. It is not difficult to see that the family $\mathcal{F}_{\mathcal{S},t}$ of maximal sparse boxes, with respect to a given set of n-dimensional points $\mathcal{S}$ and a given threshold $t \in \mathbb{Z}_+$, can be represented as the set of minimal feasible vectors of a transversal inequality over a $2n$-dimensional box $\mathcal{C}$ [10]. Given m databases $\mathcal{S}_1, \ldots, \mathcal{S}_m$, and thresholds $t_1, \ldots, t_m$, one may be interested in finding all maximal regions in space which are sparse with respect to at least one of the databases, i.e. finding the disjunction $\mathcal{F} = \bigvee_{i=1}^m \mathcal{F}_{\mathcal{S}_i,t_i}$. Theorem 2 implies that the family $\mathcal{F}$ can be generated in quasi-polynomial time if the number of databases m is constant. In contrast, mining all maximal boxes that are sparse for all m databases can be done in incremental quasi-polynomial time regardless of m. Let us add that only exponential algorithms were previously known in the literature for mining sparse boxes.

Matroid intersections. Given m matroids $M_1, \ldots, M_m$, defined on the common ground set V by m independence oracles, Lawler, Lenstra and Rinnooy Kan [22] considered the problem of enumerating all maximal sets $X \subseteq V$ independent in all the matroids. They gave an exponential-time enumeration algorithm whose running time is $O(|V|^{m+2})$ per each generated maximal independent set. Since a set $X \subseteq V$ is independent in a matroid if and only if $rank^*(V \setminus X) \geq rank^*(V)$, where $rank^*(\cdot)$ is the rank function for the dual matroid, Theorem 3 implies that the above problem can be solved in incremental quasi-polynomial time regardless of m. (Specifically, k maximal sets independent in $M_1, \ldots, M_m$ can be generated in $K^{o(\log K)}$ time and $poly(K)$ independence tests, where $K = \max\{k, |V|, m\}$.) When m is fixed, Theorem 3 also implies an incremental quasi-polynomial-time for generating all maximal sets X independent in at least one of the matroids $M_1, \ldots, M_m$. Our next example deals with graphic matroids.

Reliability. Let R be a finite set of vertices, $R_1, \ldots, R_m \subseteq R$ be m possibly intersecting subsets of R, and $E_1, \ldots, E_n \subseteq \binom{R}{2}$ be a collection of n sets of edges on R. Given a set

$X \subseteq [n]$ and $i \in [m]$, define $c_i(X)$ to be the number of connected components of the graph $(R_i, \bigcup_{i \in X}(E_i \cap \binom{R_i}{2}))$. Then for any integral threshold t_i, the inequality $f_i(X) = |R_i| - c_i(X) \geq t_i$ is polymatroid. In particular, if $t_i = |R_i| - 1$ and $c_i([n]) = 1$ then the family $\mathcal{F}_i$ is the set of all minimal collections of the input sets of edges $E_1, \ldots, E_n$ which interconnect all vertices in R_i. In network reliability applications (see e.g., [1, 14, 16, 26]), the sets of edges $E_1, \ldots, E_n$ correspond to relays, each controlled by a single switch which may work or fail, and the sets of vertices $R_1, \ldots, R_m$ correspond to regions, or sets of nodes in the network, whose connectivity is to be observed. It may be the case that the connectivity of the whole network is measured by the connectivity of these regions, e.g. the network is considered working properly if at least one of the regions R_i is connected, or more generally if a certain monotone Boolean expression ϕ on the connectivity of these regions is satisfied. It follows from Theorem 3 that if the number of prime implicants of ϕ is bounded by a constant, then all minimal collections of relays maintaining the connectivity of the network, as defined by the Boolean expression ϕ, can be enumerated in incremental quasi-polynomial time.

Statistics. Let $(S, 2^S, \mu_1), \ldots, (S, 2^S, \mu_m)$ be m probability spaces defined on some finite sample space S. Given a set $\mathcal{H} \subseteq 2^S$ of events, we are interested in finding all minimal collections $\mathcal{X} \subseteq \mathcal{H}$ of events the probability of the union of which exceeds some threshold t, with respect to at least one of the measures $\mu_1, \ldots, \mu_m$, i.e.

$$\left(\Pr_{\mu_1}[\bigcup_{X \in \mathcal{X}} X] \geq t\right) \vee \cdots \vee \left(\Pr_{\mu_m}[\bigcup_{X \in \mathcal{X}} X] \geq t\right).$$

The above condition is an example of the disjunction of transversal functions, and for constant m, the family of minimal such collections can be enumerated in quasi-polynomial time. This is also true for arbitrary monotone $\vee, \wedge$ -conditions of bounded $\vee$-degree.

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