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On the misere version of game Euclid and miserable games[1]

by

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ABSTRACT

Recently, Gabriel Nivasch got the following remarkable formula for the Sprague-Grundy function of game Euclid: $g^+(x,y) = \lfloor |x/y - y/x| \rfloor$ for all integer $x, y \geq 1$. We consider the corresponding misere game and show that its Sprague-Grundy function $g^-(x,y)$ is equal to $g^+(x,y)$ for all positions (x,y), except for the case when (x,y) or (y,x) equals (kF_i, kF_{i+1}), where F_i is the i-th Fibonacci number and k is a positive integer. It is easy to see that these exceptional *Fibonacci* positions are exactly those in which all further moves in the game are forced (unique) and hence, the results of the normal and misere versions are opposite; in other words, for these positions g^+ and g^- take values 0 and 1 so that $g^- = 1 - g^+ = (-1)^i + g^+$.

Let us note that the good old game of Nim has similar property: if there is at most one bean in each pile then all further moves are forced. Hence, in these *forced* positions the results of the normal and misere versions are opposite, while for all other positions they are the same, as it was proved by Charles Bouton in 1901. Respectively, in the forced positions g^+ and g^- take values 0 and 1 so that $g^- = 1 - g^+ = (-1)^i + g^+$, where i is the number of non-empty piles, while in all other positions $g^+ \equiv g^-$

Key words combinatorial games, impartial games, misere, Nim, Wythoff's game, game Euclid, Sprague-Grundy function

1 Game Euclid and its misere version

In 1969 Cole and Davie [5] introduced the following game based on the Euclidean algorithm. Positions are pairs (x, y) of positive integers. Two players move alternating. Each move consists of subtracting any positive multiple of the smaller number from the larger one, provided the result is still positive. The game ends when no more moves are possible, that is in the position (z, z), where $z = gcd(x, y)$. The player who can not move is the loser. Respectively, in the misere version this player is the winner. In [5] it was shown that (x, y) is a P-position (that is the player who made the *previous* move wins) if $x < \phi y$ and $y < \phi x$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio; if $x > \phi y$ or $y > \phi x$ then (x, y) is an N-position (that is the player who makes the *next* move wins). See [5, 12, 15, 17, 21, 23] for further analysis of this game, its variants and generalizations.

In this paper we consider the misere version of game Euclid. Let F_j be the j-th Fibonacci number, $F_j = 1, 1, 2, 3, 5, 8, ...$ for $j = 1, 2, 3, 4, 5, 6, ...$ Let us call (x, y) a *Fibonacci position* of rank i if (x, y) or (y, x) equals (kF_i, kF_{i+1}), where i and k are positive integers. It is easy to see that in such a position there is only one move and it results in a Fibonacci position of rank $i - 1$, since $(kF_i, kF_{i+1} - kF_i) = (kF_i, kF_{i-1})$. Then the next move is again unique and it results in the Fibonacci position of rank $i - 2$, etc.

Lemma 1 *([15]). The following two claims are equivalent:*
(i) (x, y) is a Fibonacci position,
(ii) beginning from (x, y), each further move is forced.

Proof . Obviously, both claims are equivalent to
(iii) y/x expands into a continued fraction whose every incomplete quotient is 1. □

In [15] Tamas Lengyel showed that the winning strategy in position (x, y) can be defined in terms of the incomplete quotient of y/x for game Euclid, as well as for several similar and more general games.

Proposition 1 *Given a Fibonacci position (x, y) of rank i, if i is odd then (x, y) is a P-position in the normal version of the game and an N-position in the misere version; if i is even then (x, y) is an N-position in the normal version and a P-position in the misere one.*

Proof follows immediately from Lemma 1 □

For example, (k, k) and $(k, 2k)$ are the Fibonacci positions of rank 1 and 2 respectively. According to Proposition 1, for the Fibonacci positions the results of the normal and misere versions are opposite. Let us show that for all other positions these results are the same.

Lemma 2 *From a given position (x, y) either no move enters a Fibonacci position, or exactly two "longest" moves enter Fibonacci positions of ranks i and $i + 1$.*

For example, from (22,4) one can move to (18,4),(14,4)(10,4),(6,4), and (2,4). The last two, (2,4) and (4,6), are the Fibonacci positions of rank 2 and 3 respectively. From (16,3) one can move to (13,3),(10,3),(7,3),(4,3),and (1,3) none of which is a Fibonacci position.

Proof of Lemma 2. Assume that $y > x$ and consider the maximum $l > 0$ such that $z = y - lx > 0$. It is clear that $z \leq x$. If (z, x) is a Fibonacci position of rank i then $(y - (l-1)x, x) = (z + x, x)$ is a Fibonacci position of rank $i + 1$, while $(y - (l-m)x, x) = (z + mx, x)$ are not Fibonacci positions for $m = 2, \ldots, l-1$. If (z, x) is not a Fibonacci position then $(y - (l-m)x, x) = (z + mx, x),\ m = 1, \ldots, l-1$ are not Fibonacci positions, either. □

Proposition 2 *A non-Fibonacci position is a P-position (respectively, N-position) in the normal version of the game if and only if it is a P-position (respectively, N-position) in the misere version.*

Proof . According to Lemma 2, if from a non-Fibonacci position (x, y) a player by some move can enter a Fibonacci position of even rank then there is an alternative move which enters a Fibonacci position of odd rank and vice versa. According to Proposition 1, one of these two positions is a P-position if and only if the other one is an N-position. Hence, just choosing one of these two moves from (x, y) properly, the player can win the standard version of the game as well as the misere one. In other words, in both versions (x, y) is an N-position.

Also we know that (i) every play ends in a Fibonacci position (z, z) of rank 1 and (ii) after a play enters the set $\mathcal{F}$ of Fibonacci positions it never leaves this set, so (iii) the player who enters $\mathcal{F}$ can win in both versions of the game.

Hence, if the game begins from a non-Fibonacci position then a winning strategy for the normal version can be easily transformed to a winning strategy for the misere version and vice versa as follows: make the same moves untill you can enter a Fibonacci position, at this moment make an alternative move which leads to another Fibonacci position. Lemma 2 claims that such an alternative move always exists and it is unique. If you never have a chance to enter a Fibonacci position then your opponent can win both versions of the game. □

2 Miserable games

Similar concepts can be applied in general. An *impartial* or *combinatorial* game is a game of two players, with perfect information, and without random moves. It is also assumed that each player can move in each position and the set of legal moves is the same for both. An impartial game is modeled by a digraph $G = (V, A)$. Its vertices $v \in V$ and arcs $a \in A$ are called *positions* and *moves*, respectively. We assume that G is acyclic, i.e. it contains no directed cycle. Though G may be infinite, yet, we assume that it is locally finite, that is the set of all positions which can be reached from v is finite for every $v \in V$. A position v is called *final* if there is no move in it, that is $outdeg(v) = 0$. A *play* is a directed path which ends in a final position. By definition, a player who has no move is the loser; respectively, in the misere version of the game this player is the winner.

A *kernel* is a set of positions $P \subseteq V$ which is (i) independent, i.e. there is no move (v, v') such that both $v, v' \in P$ and (ii) absorbing, i.e. for each $v \notin P$ there is a move (v, v') such that $v' \in P$. Every acyclic digraph has a unique kernel P, [16]. By (ii), each final position belongs to P. It is easy to show that a player who came to a position $v \in P$, say player II, can win. Indeed, by (i), player I must leave P, then by (ii), player II can reenter P, then I must leave P again, and II can again reenter, etc. Note that by (ii), player I can not enter a final position, so player II, sooner or later, will enter it, since digraph G is acyclic and locally finite. Let us set $N = V \setminus P$. Positions from P and N are called P- and N-positions, respectively. Letters P and N stand for "previous" and "next": in a P-position (respectively, N-position) the player who made the previous (respectively, makes the next) move can win.

The misere version can also be analyzed in terms of P- and N-positions. Let us add to the digraph G just one new position v_0 and a move from each final position of G to v_0. Clearly, the kernel P' in the obtained graph G' defines the P-positions of the misere version. Though in general P' may be very different from P, yet, in some interesting cases P and P' almost coincide.

A position v of an impartial game is called *forced* if only one play begins in v, i.e. there is a unique move in v, say to v', and there is a unique move in v', say to v'', etc.

Proposition 3 *A forced position v is an N-position (respectively P-position) in the normal version of the game if and only if v is a P-position (respectively, N-position) in the misere version.*

Proof is straightforward, see the proof of Proposition 1. □

For example, the forced positions in game Euclid are exactly the Fibonacci positions, according to Lemma 1. The classical game of Nim provides a similar example. Nim is played as follows. There are n piles of beans of sizes $(x_1, \ldots, x_n)$. Two players make turns. By one move it is allowed to take any positive number of beans but only from one pile. The player who takes the last bean is the winner. Respectively, in the misere version this player is the loser. A position $(x_1, \ldots, x_n)$ is forced if and only if $x_i \leq 1$ for all $i \in \{1, \ldots, n\}$, that is every pile contains at most one bean. (Strictly speaking, up to $n!$ plays may begin in such a position but all these plays are equivalent due to the obvious symmetry.) Lemma 1 and Proposition 1 easily generalize to this case by simple substituting "Fibonacci" for "forced".

By definition, every move from a forced position v' leads to another forced position v''. Such two forced positions are called *neighbors*. We will call a game *miserable* if for every move from a non-forced position v to a forced position v' there is an alternative move from v to another forced position v'' such that v' and v'' are neighbors.

Proposition 4 *In a miserable game every non-forced position v is a P-position (respectively, N-position) in the normal version of the game if and only if v is a P-position (respectively, N-position) in the misere version.*

Proof is exactly the same as for Proposition 2. □

Game Euclid is miserable, by Lemma 2. Let us note that Nim is miserable, too. Indeed, from a non-forced position $(x_1, \ldots, x_n)$ there is a move to a forced position if and only if $x_i > 1$ for exactly one $i \in \{1, \ldots, n\}$. In this case x_i can be reduced either to 1 or to 0. In 1901, Charles Bouton [4] applied this argument to solve the misere version of Nim.

There are many other interesting games for which the solution (that is the partition $(V = P \cup N)$ for the normal and misere versions "differ just slightly"; see [6] Chapter 12, [2], Chapters 13,14, and [1, 7, 8, 14, 19, 20, 25]. As an example, let us consider well-known Wythoff's game, [24]. It is played as follows. There are two piles of beans of sizes $x, y \geq 0$. Two players move alternating. By each move it is allowed either to remove an arbitrary positive number of beans from one pile, or the same positive number of beans from both piles. In 1907 Wythoff characterized all P-positions by the following nice formula: $(\lfloor \phi i \rfloor, \lfloor \phi^2 i \rfloor)$ and $(\lfloor \phi^2 i \rfloor, \lfloor \phi i \rfloor)$; $\quad i = 0, 1, \ldots,$ where $\phi = (1+\sqrt{5})/2$ is the golden ratio. Note that $\phi^2 = \phi + 1$.

It is not difficult to show (see e.g. [2], Chapters 13) that the misere version of this game W^- has "almost the same" set of P-positions as the normal version W^+:

(1,2),(2,1), and (0,0) are P-positions in W^+ and N-positions in W^-,

(0,1),(1,0), and (2,2) are N-positions in W^+ and P-positions in W^-;

while all other positions have the same status (P or N) in both games W^+ and W^-. To see this, let us set $S = \{(0,0), (0,1), (1,0), (1,2), (2,1), (2,2)\}$ and note that from every position $v \notin S$ either there is no move to S, or there are exactly two such moves: one to a $v' \in \{(1,2), (2,1), (0,0)\}$ and another to a $v'' \in \{(0,1), (1,0), (2,2)\}$. However, not all positions in S are forced: (1,2),(2,1), and (2,2) are not. Hence, Wythoff's game is not miserable.

Aviezry Fraenkel gave an elegant analysis of generalized Wythoff's games, [9, 10, 11], and their misere versions, [11].

3 Sprague-Grundy functions

Given two games G_1 and G_2, we define their sum $G_1 + G_2$, as follows. On each turn a player chooses either G_1 or G_2 and moves in it leaving the position in the other game untouched. The game $G_1 + G_2$ is over as soon as final positions are reached in both G_1 and G_2. The player who can not move is the loser. Respectively, in the misere version this player is the winner. Obviously, the operation $G_1 + G_2$ is associative, hence the sum $G_1 + \ldots + G_n$ is well defined. How to play the sum ? Even if we know both kernels P_1 and P_2 we still do not know P- and N-positions for in the sum $G_1 + G_2$. Indeed, if $v_1 \in P_1$ and $v_2 \in P_2$ then $(v_1, v_2) \in P$, if $v_1 \in P_1$ and $v_2 \in N_2$ then $(v_1, v_2) \in N$, if $v_1 \in N_1$ and $v_2 \in P_2$ then $(v_1, v_2) \in N$, and yet, if $v_1 \in N_1$ and $v_2 \in N_2$ then the status of (v_1, v_2) is unknown, it can be the N- or P-position. The theory of playing the sums was developed by Sprague, 1936 [22], and Grundy, 1939 [13]. They generalized the concept of kernel as follows. Given $S \subseteq \mathbb{Z}_+ = \{0, 1, \ldots\}$, the *minimum excluded value* of S is defined as $mexS = min(\mathbb{Z}_+ \setminus S)$. Given a digraph $G = (V, A)$, its *Sprague-Grundy function* (SGf) g is defined recursively: $g(v) = 0$ if v is a final position of G and $g(v) = mex\{g(w)|(v, w) \in A\}$, that is the SGf's value of v is the smallest non-negative integer which does not appear among SGF's values of the immediate successors of v. The first result of the Sprague-Grundy theory claims that

P-positions of a game G **are exactly the zeros of its Sprague-Grundy function**.

Yet, not only zeros are important. Given $a, b \in \mathbb{Z}_+$, write them as binary numbers and add them bitwise mod 2. The obtained $c = a \oplus b$ is called the *Nim-sum* of a and b. For example, $2 \oplus 3 = 10_2 + 11_2 = 01_2 = 1$, $3 \oplus 1 = 11_2 + 01_2 = 10_2 = 2$, $1 \oplus 2 = 01_2 + 10_2 = 11_2 = 3$. Since nim-sum is associative, it is well defined for n numbers. The second result of the Sprague-Grundy theory claims that

the Sprague-Grundy function of the sum of n **games** $G_1 + \ldots + G_n$ **is the Nim-sum of the corresponding Sprague-Grundy functions, that is** $g(v_1, \ldots, v_n) = g_1(v_1) \oplus \ldots \oplus g_n(v_n)$.

For example, Nim with n piles of beans is the sum of n games with one pile each. Of course, one-pile Nim is trivial: the first player can win immediately. Yet, the one-pile Nim SGf is $g(x_i) = x_i$ and hence, the SGf of the n pile NIM is $g(x_1, \ldots, x_n) = g_1(x_1) \oplus \ldots \oplus g_n(x_n)$. The zeros of this function are exactly the P-positions of Nim. These arguments lead to the solution of Nim obtained by Bouton, [4].

Given an impartial game G, let g^+ and g^- be the SGfs of its normal and misere versions. (Recall that by definition $g^+(v) = 0$ and $g^-(v) = 1$ for each final position v of G.) Functions g^+ and g^- are related in a very simple way whenever game G is miserable.

Theorem 1 *For an arbitrary game in the forced positions g^+ and g^- take values 0 and 1 so that $g^+ + g^- \equiv 1$. Further, $g^+ \equiv g^-$ for all non-forced positions of a miserable game.*

Proof . The first claim just repeats Proposition 3. We will prove the second one by induction on the length cl of the longest play which begins in a given (non-forced) position v. If cl $= 0$ then v is final and hence it is forced. If cl $= 1$ then v is again forced, since the game is miserable. Given a non-forced position v (with cl ≥ 2), consider two cases. If there is no move from v to a forced position then $g^+(v) = g^-(v)$ by the induction hypothesis. If there is a move from v to a forced position v' then, since the game is miserabile, there is another move from v to v'' such that v' and v'' are forced neighbor positions. Both SGfs g^+ and g^- take at these two positions two values 0 and 1 so that $g^+ + g^- = 1$. Hence, by induction hypothesis, in this case too $g^+(v) = g^-(v) \geq 2$. □

Recently Nivasch [17] proved that the SGf of game Euclid is given by

$$g^+(x, y) = \lfloor |x/y - y/x| \rfloor \ \ \forall x, y \geq 1$$

.

Since game Euclid is miserable, the SGf can be easily determined for its misere version, too. Namely, $g^-(x,y) = 1 - g^+(x,y) = (-1)^i + g^+(x,y)$ whenever (x,y) or (y,x) equals (kF_i, kF_{i+1}) and $g^- \equiv g^+$ for all other positions.

Unlike Euclid, Wythoff has a difficult SGf which is described as "chaotic" in [2]. However, this SGf has some nice properties, too. As we already mentioned, a simple formula for its zeros (P-positions of the game) was found by Wythoff, [24]. Blass and Fraenkel [3] proved that each non-negative integer appears exactly once as a value of Wythoff's SGf in every row ($y = const$), in every column ($x = const$), and in every diagonal ($y - x = const$). A survey and many new results can be found in [18]).

Let g^+ and g^- be the SGfs of the standard and misere versions of Wythoff's game. These two functions are related in a very simple way, though no closed formula (and no polynomial evaluation algorithm) is known for either of them and Wythoff's game is not miserable.

Proposition 5 *Functions g^+ and g^- coincide in all positions, except for the following six:*
$S = \{(0,0),(0,1),(1,0),(1,2),(2,1),(2,2)\}$, *where*
$g^+(0,0) = g^+(1,2) = g^+(2,1) = 0,\ g^-(0,0) = g^-(1,2) = g^-(2,1) = 1,$
$g^+(0,1) = g^+(1,0) = g^+(2,2) = 1,\ g^-(0,1) = g^-(1,0) = g^-(2,2) = 0.$

Proof By the definition of SGf let us compute recursively $g^+(x,y)$ and $g^-(x,y)$ for all $x, y \leq 2$. We get all values given above and also
$g^+(0,2) = g^+(1,1) = g^+(2,0) = g^-(0,2) = g^-(1,1) = g^-(2,0) = 2,.$

Further, let us note that from each position $(x,y) \notin S$ either there is no move to S, or there are exactly two such moves: one to $S^+ = \{(0,0),(0,1),(1,0)\}$ and another to $S^- = \{(1,2),(2,1),(2,2)\}$. (There is only one exceptional position (1,1) from which one can move to (0,1),(1,0), and (0,0). Anyway, $g^+(1,1) = g^-(1,1) = 2$.) From this observation we derive by induction on $x + y$ that $g^+(x,y) = g^-(x,y)$ for all $(x,y) \notin S$. Given a position (x_0, y_0) such that the values of both SGfs $g^+(x,y)$ and $g^-(x,y)$ are already known for all positions (x,y) which can be reached from (x_0,y_0), by induction we assume that $g^+(x,y) = g^-(x,y)$ whenever $(x,y) \notin S$. If there is no move from (x_0,y_0) to S then obviously $g^+(x_0,y_0) = g^-(x_0,y_0)$. If there is such a move then, as we know, there is another move from (x_0,y_0) to S and in the corresponding two positions both SGfs g^+ and g^- take both values 0 and 1 so that $g^+ + g^- = 1$. Hence, in this case too $g^+(x_0,y_0) = g^-(x_0,y_0) \geq 2$. $\square$

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