DIMACS Technical Report 2004-43
September 2004

On Critical Trees Labeled with a Condition at Distance Two

by

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ABSTRACT

An $L(2,1)$-labeling of a graph is an assignment of nonnegative integers to its vertices so that adjacent vertices get labels at least two apart and vertices at distance two get distinct labels. A graph is said to be λ-critical if λ is the minimum span taken over all of its $L(2,1)$-labelings, and every proper subgraph has an $L(2,1)$-labeling with span strictly smaller than λ. Georges and Mauro have studied 5-critical trees with maximum degree $\Delta = 3$ by examining their path-like substructures. They also presented an infinite family of 5-critical trees of maximum degree $\Delta = 3$. We generalize these results for λ-critical trees with $\Delta \geq 4$.

Keywords: vertex labeling; distance two labeling; $L(2,1)$-labeling; λ-critical; $(\Delta + 2)$-critical tree.

1 Introduction

An *L(2,1)-labeling* of a graph G is an assignment of nonnegative integers to its vertices so that adjacent vertices get integers at least two apart and vertices at distance two get integers at least one apart. An $L(2,1)$-labeling that uses integers in the set $\{0, 1, \ldots, k\}$ is called a *k-labeling*. The *λ-number* of G, denoted by $\lambda(G)$, is the minimum k so that G has a k-labeling.

Since Griggs and Yeh [13] first introduced the $L(2,1)$-labelings in 1992, a number of authors have investigated various aspects of this graph coloring generalization, producing works which can be classified into three major categories. The first category focuses primarily on the number $\lambda(G)$ where exact values or upper bounds are found for several classes of graphs: paths, cycles, n-cubes, trees in [13, 20]; chordal graphs and unit interval graphs in [18]; unions and joins of graphs, trees, odd-sun-free- and sun-free-chordal graphs in [1]; products of complete graphs in [11, 12]; products of cycles and paths in [14, 15]; products of cycles in [19]; generalized Petersen graphs in [10]. The articles in this category were by and large motivated by the long-standing conjecture proposed by Griggs and Yeh [13] which set the upper bound for $\lambda(G)$ at Δ^2. The second category studies variations of $L(2,1)$-labelings. These include for example: assignments of nonnegative integers to the vertices so that, for positive integers $p \geq q$, adjacent vertices get integers at least p apart and vertices at distance two get integers at least q apart in [2, 8, 11]; k-labelings using all labels in the set $\{0, 1, \ldots, k\}$ in [3, 5]; k-labelings where the set of labels is considered circular, that is, the differences between labels are taken modulo k in [16]. The third category explores extremal problems involving $L(2,1)$-labelings. Only a few articles belong to this last category: the minimum and maximum size of a graph with fixed order and λ-number are given in [9]; graphs where every proper subgraph has strictly smaller λ-number have been considered in [4, 6, 7]. The present manuscript belongs to this last category.

A connected graph G is said to be *λ-critical* if $\lambda = \lambda(G)$ and $\lambda(G - e) < \lambda$ for every edge e of G. Georges and Mauro [7] introduced this criticality concept and noticed that λ-critical graphs with $\lambda = 2, 3, 4$ can be completely determined.

Theorem 1 *(Georges and Mauro [7]):*

(a) *The complete graph on two vertices, K_2, is the only 2-critical graph.*

(b) *The path on three vertices, P_3, is the only 3-critical graph.*

(c) *The complete bipartite graph $K_{1,3}$ together with K_3, C_4 and P_5 are the only 4-critical graphs.* •

Surprisingly, the first non-trivial case, when $\lambda = 5$ and maximum degree $\Delta = 3$, proved to be very complex. This case was studied independently in Georges and Mauro [7], and Fishburn and Roberts [4], which showed that the characterization of 5-critical graphs with $\Delta = 3$ appears to be an intractable problem even when restricted to the class of trees. Nevertheless, interesting results on the structure of λ-critical trees with $\Delta = 3$ were presented and were useful in the construction of infinite families of 5-critical trees with $\Delta = 3$. More specifically, path-like structures, called j-paths, were essential for reaching a better understanding of λ-criticality on trees. For $j \geq 1$, two vertices in a graph are said to be *j-path adjacent* if there is a path of length j connecting them so that all internal vertices have degree two in the graph; such a path is called a *j-path*.

Georges and Mauro [7] results on the length of j-paths in 5-critical trees with maximum degree 3 were instrumental in a recent work by Fishburn and Roberts [6] where they provide a linear-time algorithm to identify these trees based on a recursive characterization of such family of graphs. Consequently, it is natural to believe that the study of j-paths in λ-critical trees with arbitrary maximum degrees may prove useful in the search of a characterization for this family of trees and perhaps shed some light on the more general problem of characterizing λ-critical graphs. This latter characterization could ultimately provide additional insight into Griggs and Yeh's conjecture on the upper bound $\lambda(G) \leq \Delta^2$. (Notice that this conjecture is true if and only if it is true for λ-critical graphs.)

In the following sections we will consider λ-critical trees with $\Delta \geq 4$, generalizing some existing results for the case $\Delta = 3$. Notice that in view of the next two results we may focus on $(\Delta + 2)$-critical trees with $\Delta \geq 4$.

Theorem 2 *(Griggs and Yeh [13]) If T is a tree with maximum degree Δ then $\lambda(T)$ is $\Delta+1$ or $\Delta + 2$.* •

Theorem 3 *(Georges and Mauro [7]) For $\Delta \geq 1$, the star $K_{1,\Delta}$ is the only $(\Delta + 1)$-critical tree.* •

In Section 2 we analyze j-paths between vertices of various degrees in $(\Delta + 2)$-critical trees for $\Delta \geq 4$. An infinite family of $(\Delta + 2)$-critical trees is presented in Section 3 for any $\Delta \geq 4$.

2 The j-Paths in $(\Delta + 2)$-Critical Trees

In this section we attempt to describe some aspects of j-paths in $(\Delta + 2)$-critical trees for $\Delta \geq 4$. (For the case $\Delta = 3$, the known results will be stated without proof.) We will focus on maximal j-paths, or equivalently, j-paths with end vertices each having degree not equal to two. Georges and Mauro [7] show that every leaf of a $(\Delta+2)$-critical tree must be adjacent to a vertex of degree Δ, therefore maximal j-paths ending on a leaf must have $j = 1$. We shall restrict the remainder of our discussion to j-path adjacent vertices of degree at least three. The main result in this section bounds the length of such j-paths in $(\Delta + 2)$-critical trees.

Theorem 4 *Let T be a $(\Delta+2)$-critical tree. If T contains a pair of j-path adjacent vertices of degree at least three, then:*

(a) $j \leq 7$ *if* $\Delta = 3$ *(Georges and Mauro [7]);*

(b) $j \leq 5$ *if* $\Delta = 4$*;*

(c) $j \leq 4$ *if* $\Delta \geq 5$ *and odd;*

(d) $j \leq 3$ *if* $\Delta \geq 6$ *and even.* •

We shall use the following terminology for the remainder of this paper. The set of nonnegative integers $\{0, 1, 2, \ldots, k\}$ will be denoted by $[0..k]$. The *closed neighborhood* of an integer x is the set $\{x, x+1, x-1\}$, denoted by $N(x)$. We will use the letter L to represent a k-labeling of a graph and $L(v)$ to represent the label assigned by L to vertex v. The *dual* of L, denoted by L^*, is the k-labeling defined as $L^*(v) = k - L(v)$.

We need to establish some auxiliary results before proving Theorem 4. In general terms, the proof of Theorem 4 will rely on the following: for a general k and a sufficient large j, let v_0, v_1, v_2, ... v_{j-1}, v_j be a j-path in G, and suppose L_1 and L_2 are arbitrary k-labelings of the subgraph induced by $\{v_0, v_1\}$ and the subgraph induced by $\{v_{j-1}, v_j\}$, respectively. Then there exists a k-labeling L of the subgraph induced by the entire j-path such that L agrees with L_1 on $\{v_0, v_1\}$ and with either L_2 or L_2^* on $\{v_{j-1}, v_j\}$.

An ordered 4-tuple (A, B, C, D) with A, B, C, D in $[0..k]$ is said to be *(k,j)-compatible* if there is a k-labeling of a path of length j so that the first, second, next-to-last, and last vertices on the path are labeled A, B, C, D, respectively, or A, B, C^*, D^*, respectively. Lemmas 1, 2, 3, and 4 below establish conditions on A, B, C, D so that (A, B, C, D) is $(k, 5)$-compatible or $(k, 4)$-compatible, for each $k \geq 5$.

Lemma 1 *Let A, B, C, D be in $[0..5]$ so that B and C are at most 2, $|A - B| \geq 2$, and $|C - D| \geq 2$. If $(A, B, C, D) \neq (5, 1, 1, 5)$ then (A, B, C, D) is $(5, 5)$-compatible.*

Proof: Suppose without loss of generality that $B \leq C \leq 2$. Let $X = [0..5] - N(B) - \{A, C\}$, $Y = [0..5] - N(C) - \{B, D\}$, $X' = [0..5] - N(B) - \{A, C^*\}$, and $Y' = [0..5] - N(C^*) - \{B, D^*\}$. So (A, B, C, D) is $(5, 5)$-compatible if and only if there are either (x in X and y in Y) or (x in X' and y in Y') with x not in $N(y)$. We proceed by presenting such x and y for each possible values of B and C.

<u>Case 1</u>: $B = 0$ or $C = 0$ or 1. Then $|X| \geq 3$ and $|Y| \geq 2$. Let x_1, x_2, x_3 be distinct elements in X and y_1, y_2 be distinct elements in Y. If $X - N(y_1)$ is nonempty then let x be any element in $X - N(y_1)$ and $y = y_1$. If $X - N(y_1)$ is empty then $X = N(y_1)$ and consequently $X - N(y_2)$ is nonempty; let x be any element in $X - N(y_2)$ and $y = y_2$.

<u>Case 2</u>: $B = 0$ and $C = 2$. Then X' is a subset of $\{2, 4, 5\} - \{A\}$ and Y' is a subset of $\{1, 5\} - \{D^*\}$. If 1 is in Y' then $X' - N(1)$ is nonempty; let x be any element in $X' - N(1)$ and $y = 1$. If 1 is not in Y' and 2 is in X' then let $x = 2$, and $y = 5$ in Y'. If 1 is not in Y' and 2 is not in X', then $X = \{3, 4, 5\}$ and $Y = \{5\}$; let $x = 3$ and $y = 5$.

Case 3: $B = 1$ and $C = 1$. Then X' is a subset of $\{3, 5\} - \{A\}$ and Y' is a subset of $\{0, 2\} - \{D^*\}$. If 0 is in Y' then let x be any element in X' and $y = 0$. If 0 is not in Y' and 5 is in X' then let $x = 5$ and $y = 2$ in Y'. The case 0 not in Y' and 5 not in X' is not possible since otherwise $(A, B, C, D) = (5, 1, 1, 5)$, contradicting our assumption.

Case 4: $B = 1$ and $C = 2$. Then X is a subset of $\{3, 4, 5\} - \{A\}$ and Y is a subset of $\{0, 4, 5\} - \{D\}$. If 0 is in Y then select any x in X and $y = 0$. If 0 is not in Y then $X' = \{4, 5\} - \{A\}$ and $Y' = \{0\}$. Select any x in X' and $y = 0$.

Case 5: $B = C = 2$. Then X is a subset of $\{0, 4, 5\} - \{A\}$ and Y is a subset of $\{0, 4, 5\} - \{D\}$. If 0 is in Y then $X - N(0)$ is nonempty; let x be any element in $X - N(0)$ and $y = 0$. Similarly, if 0 is in X then $Y - N(0)$ is nonempty; let y be any element in $Y - N(0)$ and $x = 0$. If 0 is neither in X nor in Y, then $X' = \{4, 5\}$ and $Y' = \{0, 1\}$; let x be any element in X' and y be any element in Y'. •

Lemma 2 *Let $k \geq 6$ and A, B, C, D be in $[0..k]$ so that B and C are at most $k/2$, $|A-B| \geq 2$, and $|C - D| \geq 2$. Then (A, B, C, D) is $(k, 5)$-compatible.*

Proof: Suppose first that $B \leq C \leq k/2$. Let $X = [0..k] - N(B) - \{A, C\}$, $Y = [0..k] - N(C) - \{B, D\}$, $X' = [0..k] - N(B) - \{A, C^*\}$, and $Y' = [0..k] - N(C^*) - \{B, D^*\}$. So (A, B, C, D) is $(k, 5)$-compatible if and only if there are either (x in X and y in Y) or (x in X' and y in Y') with x not in $N(y)$. When $|X| \geq 3$ and $|Y| \geq 2$, we can argue as in Case 1 in the proof of Lemma 1 to show that there are x in X and y in Y so that x is not in $N(y)$, and consequently (A, B, C, D) is $(k, 5)$-compatible. A straightforward inspection shows that $|X| \geq 3$ and $|Y| \geq 2$ except in the case where $k = 6$, $B = 1$ and $C = 3$. For this case, $X = X' = \{4, 5, 6\} - A$ and either Y or Y' will contain 0, therefore $(k, 5)$-compatibility follows by selecting an arbitrary x in X and $y = 0$. •

Lemma 3 *Let A, B, C, D in $[0..7]$ so that B and C are at most 3, $|A - B| \geq 2$, and $|C - D| \geq 2$. If (A, B, C, D) is not in $\{(3, 1, 1, 3), (4, 1, 1, 4), (0, 2, 2, 0), (7, 2, 2, 7)\}$ then (A, B, C, D) is $(7, 4)$-compatible.*

Proof: Without loss of generality suppose that $B \leq C \leq 3$. Let $X = [0..7] - N(B) - N(C) - \{A, D\}$, $X' = [0..7] - N(B) - N(C^*) - \{A, D^*\}$. Since $B \leq C \leq 3, |N(B) \cup N(C)| \leq 5$ and consequently X is nonempty. Therefore, if $B \neq C$ then (A, B, C, D) is $(7, 4)$-compatible. If $B = C$ is 0 or 3 then $|N(B) \cup N(C^*)| \leq 4$ and consequently X' is nonempty and therefore (A, B, C, D) is $(7, 4)$-compatible. If ($B = C = 1$ and $\{A, D^*\} \neq \{3, 4\}$) or ($B = C = 2$ and $\{A, D^*\} \neq \{0, 7\}$) then $|N(B) \cup N(C^*) \cup \{A, D^*\}| \leq 7$ and consequently X' is nonempty and again (A, B, C, D) is $(7, 4)$-compatible. In the remaining cases (A, B, C, D) is in $\{(3, 1, 1, 3), (4, 1, 1, 4), (0, 2, 2, 0), (7, 2, 2, 7)\}$, which is not possible. •

Lemma 4 *Let $k \geq 9$, k odd and A, B, C, D be in $[0..k]$ so that B and C are at most $k/2$, $|A - B| \geq 2$, and $|C - D| \geq 2$. Then (A, B, C, D) is $(k, 4)$-compatible.*

Proof: Without loss of generality suppose that $B \le C \le k/2$. Let $X = [0..k] - N(B) - N(C) - \{A, D\}$, $X' = [0..k] - N(B) - N(C^*) - \{A, D^*\}$. Since $k \ge 9$, X and X' are both nonempty. Therefore the $(k, 4)$-compatibility of (A, B, C, D) follows because, since k is odd, either $B \ne C$ or $B \ne C^*$. •

We need to prove one last result before presenting the proof of Theorem 4. In general terms, this result allows us to modify a $(\Delta + 1)$-labeling of a tree under certain conditions so that the leaf incident to a given pendant edge will be relabeled with its dual label while the other end vertex will maintain its original label. In the process, additional vertices may be relabeled with their dual labels to guarantee that the final modified labeling is still a $(\Delta + 1)$-labeling of the tree.

Theorem 5 *Let T be a tree and x be a leaf in T. If T has a $(\Delta+1)$-labeling assigning labels A to x and B to the vertex y adjacent to x, so that A and its dual A^* are distinct and at least 2 apart from B, then T has a $(\Delta + 1)$-labeling assigning labels A^* to x and B to y.*

Proof. If $\Delta \le 2$ then the result is vacuously true. If $\Delta = 3$ then (A, A^*, B) is $(0, 4, 2)$ or $(4, 0, 2)$ and the result holds with the dual of the given labeling. Let us assume $\Delta \ge 4$, T is rooted at x, and L is the given $(\Delta + 1)$-labeling. Define the subsets of vertices S_0, S_1, S_2 $\ldots$, of T as follows: $S_0 = \{x\}$ and for $i \ge 1$, S_i is the set of descendants at distance two of vertices in S_{i-1} which are labeled A^* under L if i odd, or A under L if i even. For each z in S_i, i ≥ 1, let $T(z + 1), T(z), T(z - 1)$ be the subtrees (possibly empty) of T rooted at a child of z labeled $L^*(z) + 1$, $L^*(z)$, $L^*(z) - 1$ under L, respectively; let T_z be the subtree of T induced by the vertex z and the vertices in $T(z + 1), T(z), T(z - 1)$.

Let us consider L' which coincides with $L^* = (\Delta + 1) - L$ on T_z for each z in S_i, $i \ge 0$, and with L otherwise. So, $L'(x) = A^*$, $L'(y) = B$, L' labels the vertices in S_i with A if i odd, or A^* if i even, and L' is a $(\Delta+1)$-labeling when restricted to T_z for each z in S_i, $i \ge 1$. To prove that L' is a $(\Delta + 1)$-labeling of the entire T, it remains to be shown that for each z in S_i, $i \ge 0$ and w not in any T_y:

I. If w and z are adjacent, then $|L'(z) - L'(w)| \ge 2$;

II. If w and z are at distance two, then $L'(z) \ne L'(w)$;

III. If w and z are adjacent and v is the root of either $T(z + 1)$, $T(z)$, or $T(z - 1)$ then $L'(w) \ne L'(v)$.

Since z is in S_i, $i \ge 0$, by the definition of L' we have $L'(z) = L^*(z)$.

To show I., assume w and z are adjacent. So since w is not in T_z, $L'(w) = L(w)$. If z is a child of w then there is a vertex u in S_{i-1} adjacent to w. Therefore since $L'(z) = L^*(z) = L(u)$, we have $|L'(z) - L'(w)| = |L(u) - L(w)| \ge 2$. If on the other hand w is a child of z, then $L(w)$ is not in $\{L^*(z) + 1,\ L^*(z),\ L^*(z) - 1\}$ since w is not in T_z. Therefore $|L'(z) - L'(w)| = |L^*(z) - L(w)| \ge 2$.

To prove II., assume w and z are at distance two. If neither w nor z is a descendant of the other then there is a vertex u in S_{i-1} with a child v which is adjacent to both w and z. Since we are assuming w is not in any T_y, we have $L'(w) = L(w) \neq L(u) = L^*(z) = L'(z)$. If z is a descendant of w then w is in S_{i-1} and $L'(z) = L^*(z) \neq L^*(w) = L'(w)$. Assume w is a descendant of z. If $L(w) = L^*(z)$ then w is in S_{i+1} and therefore $L'(z) = L^*(z) \neq L(z) = L^*(w) = L'(w)$. If $L(w) \neq L^*(z)$ then $L'(z) = L^*(z) \neq L(w) = L'(w)$.

Finally assume w and z are adjacent and suppose by contradiction that III. does not hold. If v is the root of $T(z+1)$ and $L'(w) = L'(v)$, then $L(w) = L'(w) = L'(v) = (L^*(z)+1)^* = (L^*(z))^* - 1 = L(z) - 1$, a contradiction. The cases where v is the root of $T(z)$ or of $T(z-1)$ are similar. So, III. holds. •

We can finally establish Theorem 4.

Proof of Theorem 4. Let T be a $(\Delta+2)$-critical tree. In each of the items (b), (c), and (d), assume by contradiction that for a fixed $j > 5$, $j > 4$, and $j > 3$, respectively, there is a j-path $v_0\ v_1\ \ldots\ v_j$ where the degrees of v_0 and v_j are at least 3. Set $n = 4$ if $\Delta \geq 6$ and even, or $n = 5$ otherwise. Delete the vertex v_2 if $n = 4$, or the vertices v_2 and v_3 if $n = 5$, and let T_1 and T_2 be the resulting two components containing v_0 and v_n, respectively. By assumption T is $(\Delta+2)$-critical, so there are $(\Delta+1)$-labelings L_1 and L_2 of T_1 and T_2, respectively. Since the dual of a $(\Delta+1)$-labeling is also a $(\Delta+1)$-labeling, we may assume without loss of generality that both $L_1(v_1)$ and $L_2(v_{n-1})$ are at most $(\Delta+1)/2$. Let $A = L_1(v_0)$, $B = L_1(v_1)$, $C = L_2(v_{n-1})$, $D = L_2(v_n)$. Thus A, B, C, D are in $[0..\Delta+1]$ so that B and C are at most $(\Delta+1)/2$, $|A-B| \geq 2$, and $|C-D| \geq 2$.

By Lemmas 1, 2, 3, and 4, we can conclude that (A, B, C, D) is $(\Delta+1, n)$-compatible except for two cases that will be discussed separately:

I. $\Delta = 4$ and $(A, B, C, D) = (5, 1, 1, 5)$; or

II. $\Delta = 6$ and (A, B, C, D) is in the set $\{(3,1,1,3), (4,1,1,4), (0,2,2,0), (7,2,2,7)\}$.

If (A, B, C, D) is $(\Delta+1, n)$-compatible, then L_1 can be extended down the j-path, with the result that either L_1 will agree with L_2 or L_2^* at v_{n-1}. If the former holds, then L_1 along with its extension down the j-path and L_2 form a $(\Delta+1)$-labeling of T,a contradiction. If the latter, then L_1 along with its extension down the j-path and L_2^* form a $(\Delta+1)$-labeling of T, another contradiction.

In case I., since $j > 5$ the degree of v_5 is strictly smaller than $\Delta = 4$ and therefore there is a label C' in $\{0, 2, 3\}$ not assigned by L_2 to any neighbor of v_5. (Note: This part of the proof could be simplified by observing that the degree of v_5 is actually 2; the more general approach used here was chosen so that the same proof can also be used to establish Corollary 1 below.) If $C' = 0$ or 2, let L_2' coincide with L_2 on all vertices except for v_4 and $L_2'(v_4) = C'$. If $C' = 3$, let L_2' coincide with L_2^* on all vertices except for v_4 and $L_2'(v_4) = (C')^*$. Clearly L_2' is a 5-labeling of T_2. Set $D' = L_2'(v_5)$. Therefore A, B, C', D' is in $[0..5]$ so that B and C' are at most 2, $|A-B| \geq 2$, $|C'-D'| \geq 2$ and $(A, B, C', D') \neq (5, 1, 1, 5)$. By Lemma 1,

(A, B, C', D') is $(5,5)$-compatible, so either the pair L_1 and L_2' or the pair L_1 and $(L_2')^*$ can be extended to a 5-labeling of the entire T, contradicting our assumption.

In case II., notice first that C and C^* are different and at least two apart from D. By Theorem 5, there is a 7-labeling of T_2 which assigns labels C^* and D to v_3 and v_4, respectively. Let L_2' be the dual of such 7-labeling and set $D' = L_2'(v_4)$. Therefore A, B, C, D' are in $[0..7]$ so that B and C are at most 3, $|A - B| \geq 2$, $|C - D'| \geq 2$ and (A, B, C, D') is not in $\{(3,1,1,3), (4,1,1,4), (0,2,2,0), (7,2,2,7)\}$. By Lemma 3, (A, B, C, D') is $(7,4)$-compatible, so either the pair L_1 and L_2' or the pair L_1 and $(L_2')^*$ can be extended to a 7-labeling of the entire T, contradicting our assumption. •

Notice that essentially the same argument used to prove that item (b) in Theorem 4 holds can be used to lower the upper bound $j \leq 5$ when one of the j-path adjacent vertices of degree at least three has degree exactly three.

Corollary 1 *Let T be a $(\Delta + 2)$-critical tree with $\Delta = 4$. If T contains a pair of j-path adjacent vertices, one of degree at least three and the other of degree exactly 3, then $j \leq 4$.* •

For item (a) of Theorem 4 Georges and Mauro [7] provided examples of $(\Delta + 2)$-critical trees to show that the upper bound for j is tight. For $\Delta \geq 4$, we present in Figure 1 some examples of $(\Delta + 2)$-critical trees containing j-path adjacent vertices of degree at least 3 where j is equal to the upper bound in the corresponding item of Theorem 4. For the remainder of this paper, dark circles will represent the vertices with degree Δ and leaves adjacent to degree Δ vertices will not be shown. The proofs that the examples presented in Figure 1 are indeed $(\Delta+2)$-critical trees use techniques similar to the ones that will be used in the next section and in the interest of brevity will be omitted. It should be noted that a general example of $(\Delta + 2)$-critical trees containing a 4-path for $\Delta \geq 9$ and odd could not be found (the example for $\Delta = 7$ and $j = 4$ is not included here due to its large size).

The upper bounds in Theorem 4 can be lowered even more if one of the j-path adjacent vertices is external. A vertex is said to be *external* if it has degree Δ and it is adjacent to $\Delta - 1$ vertices of degree 1. Notice that a vertex of degree Δ can only receive labels 0 or $\Delta + 1$ in any $(\Delta + 1)$-labeling.

Theorem 6 *Let T be a $(\Delta+2)$-critical tree. If T contains a pair of j-path adjacent vertices of degree at least 3 such that one of them is external, then:*

(a) *$j \leq 2$ or $j = 4$, if $\Delta = 3$ (Georges and Mauro [7]);*

(b) *$j \leq 2$ if $\Delta \geq 4$.*

Proof. We shall prove item (b) by contradiction. Suppose that $v_0\ v_1\ \ldots\ v_j$ with $j \geq 3$, is a j-path in T where v_0 is external and v_j has degree at least 3. Let T_1 be the component of $T - \{v_1\}$ containing v_j. Since T is $(\Delta + 2)$-critical, there is a $(\Delta + 1)$-labeling L_1 of T_1

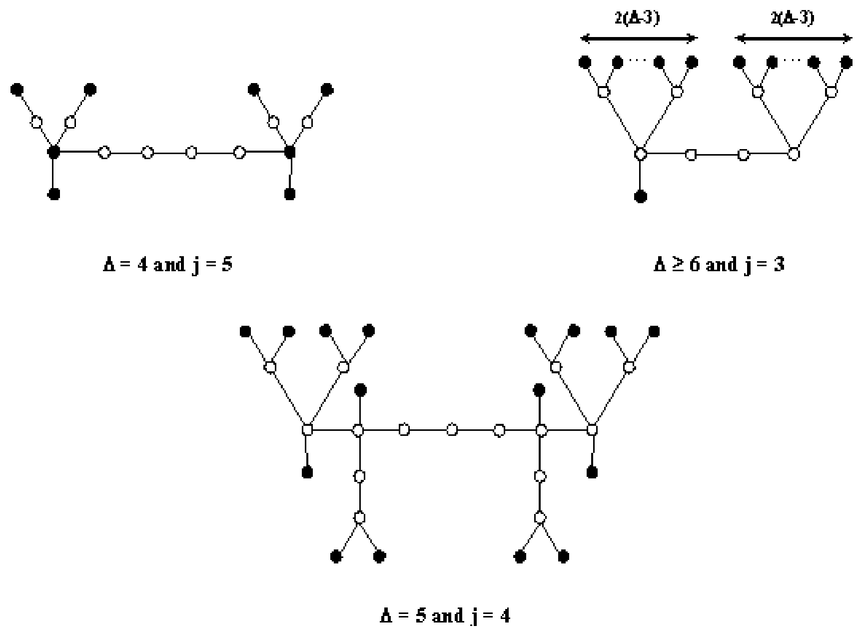

Figure 1: Examples of tight bounds for j in Theorem 4

with $L_1(v_2)$ at most $(\Delta+1)/2$. Let $C = L_1(v_2)$ and $D = L_1(v_3)$. We will show that there are A in $\{0, \Delta+1\}$ and B in $[0..\Delta+1]$ so that (A, B, C, D) is $(\Delta+1, 3)$-compatible. Let $X = [0..\Delta-1] - N(C) - \{D\}$. If $\Delta \geq 5$ then X is nonempty and we can set $A = \Delta+1$ and B in X. If $\Delta = 4$ and (C, D) is in $\{(1, 3), (2, 0)\}$ then set $A = 0$ and $B = 5$. If $\Delta = 4$ and (C, D) is not in $\{(1, 3), (2, 0)\}$ then X is nonempty and we can set $A = 5$ and B in X. Therefore (A, B, C, D) is $(\Delta+1, 3)$-compatible. We can extend L_1 by setting $L_1(v_0) = A$ and $L_1(v_1) = B$, and greedily labeling the leaves incident to v_0. The resulting assignment is a $(\Delta+1)$-labeling of T, a contradiction. •

The following result shows that a 2-path from an external vertex cannot end at a vertex with low degree.

Theorem 7 *Let T be a $(\Delta+2)$-critical tree with $\Delta \geq 3$. If T contains a pair of 2-path adjacent vertices, one external and the other of degree at least 3, then the latter must have degree Δ or $\Delta - 1$.*

Proof. Let $v_0\ v_1\ v_2$ be a 2-path in T where v_0 is external and suppose to the contrary that v_2 has degree at most $\Delta - 2$. Let T_1 be the component of $T - \{v_0\}$ containing v_1. Since T is $(\Delta+2)$-critical, there is a $(\Delta+1)$-labeling L_1 of T_1 with $L_1(v_1)$ at most $(\Delta+1)/2$. Let $A = L_1(v_1)$ and $B = L_1(v_2)$. If (A, B) is not in $\{(0, \Delta+1),\ (1, \Delta+1)\}$, then the set $X = \{0, \Delta+1\} - N(A) - \{B\}$ is nonempty. We can extend L_1 by assigning any label in X to v_0 and greedily label the leaves adjacent to v_0, obtaining a $(\Delta+1)$-labeling of T,contradicting our assumption. On the other hand if (A, B) is in $\{(0, \Delta+1),\ (1, \Delta+1)\}$, since v_2 has degree at most $\Delta - 2$, there is a label A' in $[2..\Delta - 1]$ that was not used by a neighbor of v_2 different from v_1. Modify L_1 by assigning label A' to v_1 and let L'_1 be the modified L_1. Therefore L'_1 is a $(\Delta + 1)$-labeling of T_1 and we can assign label 0 to v_0 and greedily label the leaves adjacent to v_0, obtaining a $(\Delta + 1)$-labeling of T, again contradicting our assumption. •

In Theorem 8, we examine j-path adjacent vertices of degree Δ and show that one of them will be external for certain combinations of values of Δ and low values of j. The case $\Delta = 3$ was studied in Georges and Mauro [7].

Theorem 8 *Let T be a $(\Delta + 2)$-critical tree and let u and w be two vertices of degree Δ.*

(a) *If u and w are adjacent, then one of them must be external for any $\Delta \geq 3$.*

(b) *If u and w are 2-path adjacent then one of them must be external for $\Delta = 3$ or 4.*

Proof. Let T be a $(\Delta + 2)$-critical tree and let u and w be two vertices of degree Δ.

Let us prove (a). Suppose to the contrary that u and w are adjacent and not external for some $\Delta \geq 3$. Thus T contains a path $v_0\ v_1\ v_2\ v_3\ v_4\ v_5$ where $v_2 = u$ and $v_3 = w$. Let T_1 be the component of $T - e_1$ containing v_1, where e_1 is the edge between v_0 and v_1. Since T is $(\Delta+2)$-critical there exists a $(\Delta + 1)$-labeling L_1 of T_1. Notice that $\{L_1(v_2), L_1(v_3)\} = \{0, \Delta + 1\}$ and $L_1(v_1)$ is not in $\{0, \Delta + 1\}$. Next let T_2 be the component of $T - e_2$ containing v_4, where e_2 is the edge between v_4 and v_5. Since T is $(\Delta + 2)$-critical there exists a $(\Delta + 1)$-labeling L_2 of T_2. Notice that $\{L_2(v_2), L_2(v_3)\} = \{0, \Delta + 1\}$ and $L_2(v_4)$ is not in $\{0, \Delta + 1\}$.We may assume that L_1 and L_2 agree on v_2 and v_3 (if not, consider the dual of L_2 instead). We produce a $(\Delta + 1)$-labeling L of T by letting L be equal to $L_1$when restricted to vertices in $T_1 - \{v_1, v_2\}$ and equal to L_2 when restricted to vertices in $T_2 - \{v_3, v_4\}$. This contradicts the assumption that T is $(\Delta + 2)$-critical.

To prove (b), suppose to the contrary that u and w are 2-path adjacent and not external with $\Delta = 3$ or 4. Thus T contains a path $v_0\ v_1\ v_2\ v_3\ v_4\ v_5\ v_6$ where $v_2 = u$, v_3 has degree 2, and $v_4 = w$. Let T_1 be the component of $T - e_1$ containing v_1, where e_1 is the edge between v_0 and v_1. Since T is $(\Delta + 2)$-critical there exists a $(\Delta + 1)$-labeling L_1 of T_1. Notice that $\{L_1(v_2), L_1(v_4)\} = \{0, \Delta+1\}$, $L_1(v_3)$ is in $[2..\Delta-1]$. Next let T_2 be the component of $T - e_2$ containing v_5, where e_2 is the edge between v_5 and v_6. Since T is $(\Delta+2)$-critical there exists a $(\Delta+1)$-labeling L_2 of T_2. Notice that $\{L_2(v_2), L_2(v_4)\} = \{0, \Delta+1\}$, $L_2(v_3)$ is in $[2..\Delta-1]$. We may assume that L_1 and L_2 agree on v_2 and v_4 (if not, consider the dual of L_2 instead). If L_1 and L_2 agree on v_3 then let L'_1 be the restriction of L_1 to $T_1 - \{v_1, v_2\}$. If L_1 and L_2 do not agree on v_3 then $\Delta = 4$ and one labeling must assign label 2 and the other must assign label 3 to v_3. In this case, since 2 and 3 are dual labels at distance at least 2 from $\{0, 5\}$, we can use Theorem 5 to modify the restriction of L_1 to $T_1 - \{v_1, v_2\}$ obtaining L'_1, so that it agrees with L_2 on v_3 and v_4.

We produce a $(\Delta + 1)$-labeling L of T by letting L be equal to L_2 when restricted to vertices in $T_2 - \{v_3, v_4, v_5\}$ and equal to $L'_1$when restricted to vertices in $T_1 - \{v_1, v_2\}$. This contradicts the assumption that T is $(\Delta + 2)$-critical. •

Item (b) in Theorem 8 could not be generalized for $\Delta \geq 5$. For instance when $\Delta = 5$, the tree in Figure 2 is 7-critical and contains 2-path adjacent vertices of degree 5 that are not external (e.g., pairs $(x, y), (z, y), (w, y)$).

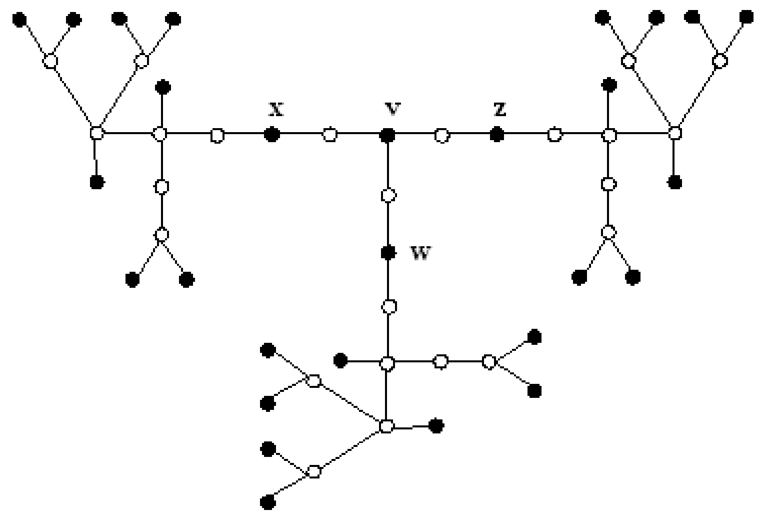

Figure 2: 7-critical tree with $\Delta = 5$ and non-external 2-path adjacent vertices

3 Infinite Family of $(\Delta + 2)$-Critical Trees

In this section, we will construct an infinite family of $(\Delta + 2)$-critical trees for each value of $\Delta \geq 4$. Let us first define the building blocks in our construction. For the remainder of this section we will assume that $\Delta \geq 4$. Let Q be the tree with vertex x of degree Δ such that x is adjacent to one leaf y, adjacent to one external vertex and 2-path adjacent to $\Delta - 2$ external vertices. Let R be the tree with vertex v of degree $\Delta - 2$ such that v is adjacent to two leaves u and w, and adjacent to $\Delta - 4$ vertices of degree 3, each adjacent to two external vertices. Finally, let S be the tree with vertex x of degree Δ such that x is adjacent to leaves z and y, adjacent to one external vertex, and 2-path adjacent to $\Delta - 3$ external vertices. The trees Q, R, and S are shown in Figure 3.

In Lemmas 5, 6 and 7, we show that the trees Q, R, and S have $(\Delta + 1)$-labelings and we limit the possible labels for specific vertices. Note that it is enough to provide $(\Delta + 1)$-labelings for the vertices shown in Figure 3 since they can be extended to the leaves adjacent to external vertices greedily.

Lemma 5 *The graph Q has a $(\Delta + 1)$-labeling L_Q which assigns labels 0, Δ to vertices x, y, respectively. In addition, every $(\Delta + 1)$-labeling of Q assigns either the sequence of labels 0, Δ or their duals $\Delta + 1$, 1 to x and y, respectively.*

Proof. The labeling L_Q given in Figure 4 is a $(\Delta + 1)$-labeling of Q.

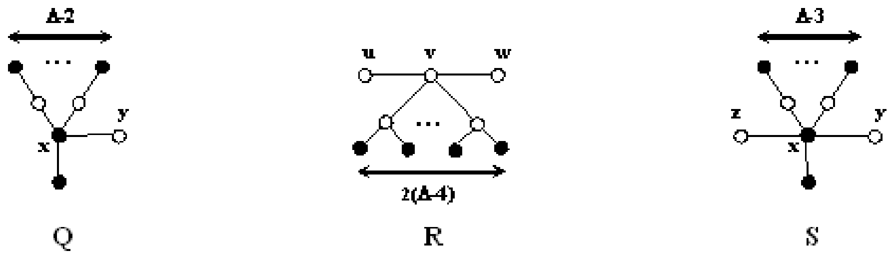

Figure 3: Trees Q, R, and S

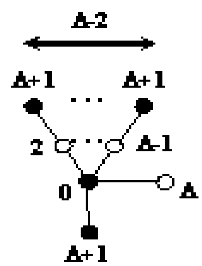

Figure 4: $(\Delta + 1)$-labeling L_Q of Q.

Let L be an arbitrary $(\Delta + 1)$-labeling of Q. Since each degree 2 vertex is adjacent to 2 vertices of degree Δ with labels 0 and $\Delta + 1$, it must get a label in $[2..\Delta - 1]$. There are exactly $\Delta - 2$ such vertices, all at distance 2 of each other. So all $\Delta - 2$ labels in $[2..\Delta - 1]$ are used by a neighbor of x of degree 2. Also x is a degree Δ vertex adjacent to another degree Δ vertex which forces any other neighbor of x to avoid labels 0 and $\Delta + 1$. So the only available labels for y are 1 and Δ. •

Lemma 6 *The graph R has a $(\Delta + 1)$-labeling L_R which assigns labels Δ, 1, $\Delta - 1$ to vertices u, v, w, respectively. In addition, every $(\Delta + 1)$-labeling of R assigns a label in $\{0, 1, 2, \Delta - 1, \Delta, \Delta + 1\}$ to vertex v.*

Proof. The labeling L_R given in Figure 5 is a $(\Delta + 1)$-labeling of R.

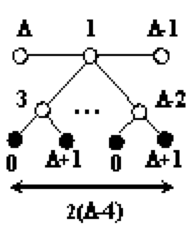

Figure 5: $(\Delta + 1)$-labeling L_R of R.

Let L be an arbitrary $(\Delta + 1)$-labeling of R. The $\Delta - 4$ vertices of degree 3 adjacent to pairs of degree Δ vertices are adjacent to v and must get different labels in $[2..\Delta - 1]$. If L

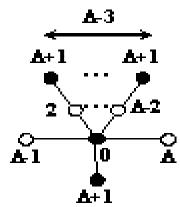

Figure 6: $(\Delta+1)$-labeling L_S of S.

assigns a label k in $[3..\Delta-2]$ to v then no vertex adjacent to v can get labels in $N(k)$, a subset of $[2..\Delta-1]$ and consequently only $\Delta-5$ labels in $[2..\Delta-1]$ are available to be assigned to neighbors of v, a contradiction. Therefore v must get a label in $\{0, 1, 2, \Delta-1, \Delta, \Delta+1\}$. •

Lemma 7 *The graph S has a $(\Delta+1)$-labeling L_S which assigns labels $\Delta-1$, 0, Δ to vertices z, x, y, respectively. In addition, every $(\Delta+1)$-labeling of S has either vertex z or vertex y labeled with a label in $\{1, \Delta\}$.*

Proof. The labeling L_S given in Figure 6 is a $(\Delta+1)$-labeling of S.

Let L be an arbitrary $(\Delta+1)$-labeling of S. Suppose by contradiction that neither z nor y receives a label in $\{1, \Delta\}$. Since z and y are adjacent to x of degree Δ and x is adjacent to another degree Δ vertex, then z and y must receive different labels in $[2..\Delta-1]$. But the $\Delta-3$ degree 2 vertices adjacent to pairs of degree Δ vertices must receive different labels in $[2..\Delta-1]$ different from the ones assigned to z and y totaling $\Delta-1$ vertices receiving different labels in $[2..\Delta-1]$. But there are only $\Delta-2$ labels in $[2..\Delta-1]$, a contradiction. Therefore either z or y must get a label in $\{1, \Delta\}$. •

We next combine copies of Q, R, and S as follows. For each integer $i \geq 0$ let Q_i, R_i, S_i, be isomorphic to Q, R, S, associating subscript i to each vertex u, v, w, x, y, z. Let W_0 be the graph obtained by identifying vertex y_0 in Q_0 with vertex u_0 in R_0. For $i \geq 1$ construct W_i by identifying vertex w_{i-1} in W_{i-1} with vertex z_i in S_i and identifying vertex y_i in S_i with vertex u_i in R_i. For each integer $n \geq 0$ define T_n by identifying vertex w_n in W_n with vertex y_{n+1} in Q_{n+1}. We will use Lemmas 8 and 9 to show that T_n is $(\Delta+2)$-critical for each $n \geq 0$ in Theorem 9.

Lemma 8 *For each integer $n \geq 0$, the graph W_n has a $(\Delta+1)$-labeling L_W which assigns labels 0, Δ to vertices x_n, y_n, respectively. In addition, every $(\Delta+1)$-labeling of W_n assigns either the sequence of labels 0, Δ or their duals $\Delta+1$, 1 to x_n, y_n, respectively.*

Proof. Let L_W be a labeling of W_n so that L_W restricted to:
Q_0 is equal to L_Q of Lemma 5;
R_i is equal to L_R of Lemma 6, for $i = 0, 1, ..., n$;
S_i is equal to L_S of Lemma 7, for $i = 1, 2, ..., n$.

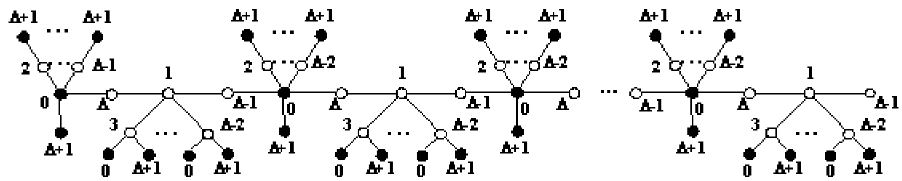

Figure 7: W_n with the $(\Delta+1)$-labeling L_W

It is not difficult to verify that L_W is a $(\Delta+1)$-labeling of W_n. See Figure 7 for a description of L_W.

Let us show by induction on n that every $(\Delta+1)$-labeling of W_n assigns either the sequence of labels $0, \Delta$ or their duals $\Delta+1, 1$ to x_n, y_n, respectively. Lemma 5 covers the case $n=0$. Suppose the result is true for $n \geq 0$ and let us show that it holds for $n+1$. Let L be a $(\Delta+1)$-labeling of W_{n+1}. From the induction hypothesis, vertices x_n, y_n, receive the sequence of labels 0, Δ or the sequence $\Delta+1$, 1, respectively. Let us assume that x_n, y_n, receive labels 0, Δ, respectively (the proof for the other case is similar). By Lemma 6 vertex v_n must receive a label in $\{0, 1, 2, \Delta-1, \Delta, \Delta+1\}$ but since v_n is adjacent to y_n labeled Δ and is at distance 2 of vertex x_n labeled 0, the only possible labels for v_n are 1 and 2. By Lemma 7, either z_{n+1} or y_{n+1} must receive a label in $\{1, \Delta\}$. The vertex $w_n = z_{n+1}$ cannot be labeled Δ since it is at distance 2 of vertex $y_n = u_n$ also labeled Δ, and it cannot be labeled 1 since it is adjacent to v_n labeled 1 or 2. So y_{n+1} must receive a label in $\{1, \Delta\}$. Therefore x_{n+1}, y_{n+1} receive the sequence of labels $0, \Delta$ or their duals $\Delta+1, 1$, respectively, since x_{n+1} has degree Δ. •

Lemma 9 *Let T be a tree with $\lambda(T) = \Delta+2$. Then T is $(\Delta+2)$-critical if and only if $T-e$ has a $(\Delta+1)$-labeling for every pendant edge e.*

Proof: The necessity is obvious. Suppose $T-e$ has a $(\Delta+1)$-labeling for every pendant edge e. Let f be an edge in T which is not pendant. The graph $T-f$ has 2 components T_1 and T_2 containing pendant edges e_1 and e_2 respectively. T_1 and T_2 are subgraphs of $T-e_2$ and $T-e_1$, respectively, which have $(\Delta+1)$-labelings. So $T-f$ has a $(\Delta+1)$-labeling and therefore T is $(\Delta+2)$-critical. •

Theorem 9 *For each integer $n \geq 0$, the graph T_n is $(\Delta+2)$-critical.*

Proof. Let us first show that $\lambda(T_n) = \Delta+2$. Suppose by contradiction that this is not the case. In view of Theorem 2, $\lambda(T_n) = \Delta+1$ and therefore T_n has a $(\Delta+1)$-labeling L. By Lemma 8, L either assigns the sequence of labels 0, Δ or their duals $\Delta+1$, 1 to both pairs x_n, y_n, and to x_{n+1}, y_{n+1}, respectively. Since y_n, y_{n+1} are at distance 2, $\{L(y_n), L(y_{n+1})\} = \{1, \Delta\}$. So vertex v_n which is adjacent to both y_n, y_{n+1} cannot get a label in $\{0, 1, 2, \Delta-1, \Delta, \Delta+1\}$, contradicting Lemma 6.

To establish the criticality of T_n, by Lemma 9 we need to show that $T_n - e$ has a $(\Delta+1)$-labeling for every pendant edge e. Notice first that every pendant edge e in T_n is incident to a degree Δ vertex. We divide the proof into five cases:

Case 1: e is a pendant edge in Q_{n+1} (or Q_0) connected to the degree Δ vertex that is adjacent to x_{n+1} (or x_0).

Let L_W be the $(\Delta+1)$-labeling of W_n defined in Lemma 8. Relabel vertex w_n (previously labeled $\Delta - 1$) with label $\Delta + 1$. This is still a $(\Delta+1)$-labeling of W_n. We next extend this labeling to a $(\Delta+1)$-labeling of $T_n - e$ by labeling $Q_{n+1} - e$ as shown in Figure 8.

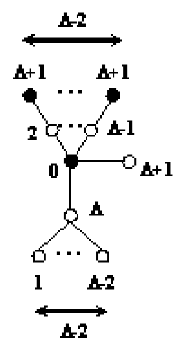

Figure 8: $(\Delta+1)$-labeling of $Q_{n+1} - e$.

Case 2: e is a pendant edge in Q_{n+1} (or Q_0) connected to a degree Δ vertex that is not adjacent to x_{n+1} (or x_0).

Let L_W^* be the dual of L_W, the $(\Delta+1)$-labeling of W_n defined in Lemma 8. We next extend this labeling to a $(\Delta+1)$-labeling of $T_n - e$ by labeling $Q_{n+1} - e$ as shown in Figure 9.

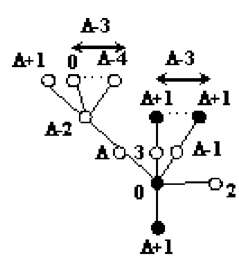

Figure 9: $(\Delta+1)$-labeling of $Q_{n+1} - e$.

Case 3: e is a pendant edge in R_i connected to a degree Δ vertex, for some $i \geq 0$.

We may assume that Δ is at least 5 (if $\Delta = 4$, such e does not exist). Let C_1 be the component of $T_n - e_1$ where e_1 is the edge connecting u_i and v_i, so that C_1 contains u_i. Let C_2 be the component of $T_n - e_2$ where e_2 is the edge connecting v_i and w_i so that C_2 contains w_i. The trees C_1 and C_2 are isomorphic to subgraphs of W_n. By Lemma 8 there are $(\Delta+1)$-labelings L_1 and L_2 of C_1 and C_2, respectively, so that L_1 assigns labels 0, Δ to x_i and $u_i = y_i$, respectively, and that L_2 assigns labels 1, $\Delta+1$ to $w_i = z_{i+1}$ and x_{i+1} , respectively (take the dual of the labeling in Lemma 8). We next extend labelings L_1 and L_2 to a $(\Delta+1)$-labeling of $T_n - e$ by labeling $R_i - e$ as shown in Figure 10.

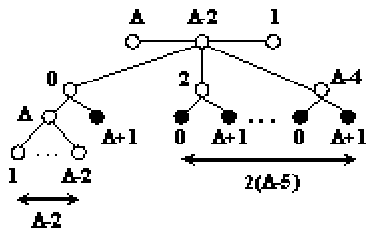

Figure 10: $(\Delta+1)$-labeling of $R_i - e$.

Case 4: e is a pendant edge in S_i connected to the degree Δ vertex that is adjacent to x_i, for some $i \geq 1$.

Let C_1 be the component of $T_n - e_1$ where e_1 is the edge connecting z_i and x_i so that C_1 contains z_i. Let C_2 be the component of $T_n - e_2$ where e_2 is the edge connecting x_i and y_i so that C_2 contains y_i. The trees C_1 and C_2 are isomorphic to subgraphs of W_n. By Lemma 8 there are $(\Delta+1)$-labelings L_1 and L_2 of C_1 and C_2, respectively, so that both assign labels 1, $\Delta - 1$ to the pairs of vertices v_{i-1}, w_{i-1}, and v_i, w_i, respectively. Modify L_2 by relabeling vertex w_i (previously labeled $\Delta - 1$) with $\Delta + 1$, as in Case 1. We next extend labelings L_1 and the modified L_2 to a $(\Delta+1)$-labeling of $T_n - e$ by labeling $S_i - e$ as shown in Figure 11.

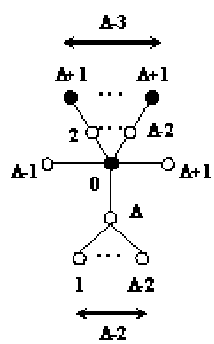

Figure 11: $(\Delta+1)$-labeling of $S_i - e$.

Case 5: e is a pendant edge in S_i connected to a degree Δ vertex that is not adjacent to x_i, for some $i \geq 1$.

Let C_1 be the component of $T_n - e_1$ where e_1 is the edge connecting z_i and x_i so that C_1 contains z_i. Let C_2 be the component of $T_n - e_2$ where e_2 is the edge connecting x_i and y_i so that C_2 contains y_i. The trees C_1 and C_2 are isomorphic to subgraphs of W_n. By Lemma 8 there are $(\Delta+1)$-labelings L_1 and L_2 of C_1 and C_2, respectively, so that both assign labels 1, $\Delta - 1$ to the pairs of vertices v_{i-1}, w_{i-1}, and v_i, w_i, respectively. Consider L_2^*, the dual of L_2, which assigns labels Δ, 2 to the pair of vertices v_i, w_i, respectively. We next extend

labelings L_1 and L_2^* to a $(\Delta + 1)$-labeling of $T_n - e$ by labeling $S_i - e$ as shown in Figure 12.
•

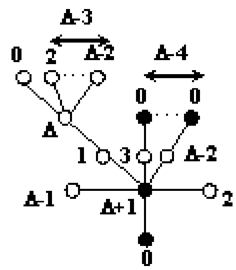

Figure 12: $(\Delta + 1)$-labeling of $S_i - e$.

4 Acknowledgements

The author would like to thank the Babson College Board of Research for its support. She would also like to thank Fred Roberts, director of DIMACS, and the National Science Foundation for the opportunity to present preliminary results on this work at the *DIMACS Connect Institute (DCI) 2002: Research Workshop on Labelings and Numberings of Graphs* (NSF grant ESI-99-11769).

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