DIMACS Technical Report 2004-35
July 2004

ON CLAW- AND NET-FREE GRAPHS

by

Alexander Kelmans
Rutgers University
New Brunswick, New Jersey
and
University of Puerto Rico
San Juan, Puerto Rico

ABSTRACT

In this paper we describe some structural properties of claw- and net-free graphs. Using these properties, we find (among other results) necessary and sufficient conditions for a claw- and net-free graph G with $s, t \in V(G)$, $s \neq t$, $e \in E(G)$, and disjoint paths S and T in G to have:

$(c1)$ a Hamiltonian st–path,
$(c2)$ a Hamiltonian s–path containing e,
$(c3)$ a Hamiltonian st–path containing e if G is 3–connected,
$(c4)$ a Hamiltonian cycle containing S if G is k–connected, $k \geq 2$, and $v(S) \leq k$,
$(c5)$ a Hamiltonian path containing e and S if G is $(k+1)$–connected, $k \geq 2$, and $v(S) \leq k$,
$(c6)$ a Hamiltonian st–path containing S and T if G is k–connected, $k \geq 3$, $v(S)+v(T) \leq k$, and s, t are end–vertices of paths S and T, respectively.

From the above mentioned results we obtain, in particular:

Let G be a claw- and net-free graph. If G is 4–connected, then for any two different vertices s, t and any edge e in G there is a Hamiltonian st–path in G containing e and, in particular, every two edges in G belong to a Hamiltonian cycle. If G is 3–connected then every two non-adjacent edges in G belong to a Hamiltonian cycle.

Keywords: claw, net, graph, Hamiltonian path, Hamiltonian cycle, polynomial time algorithm.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [1].

A *claw* is a graph having exactly three edges that are incident to a (the) common vertex. A claw looks like the letter Y. A *net* is a graph obtained from a triangle by attaching to each vertex a new dangling edge. We will consider *claw- and net-free graphs* (or simply, *cn–free graphs*), i.e. graphs having no induced subgraph isomorphic to either a claw or a net. There are many papers devoted to the study of Hamiltonicity of claw-free graphs, and, in particular, claw- and net-free graphs (e.g. [3, 5, 6, 8, 9] and survey papers [2, 4]).

In this paper we describe some structural properties of claw- and net-free graphs. Using these properties, we establish some new Hamiltonicity results on claw- and net-free graphs.

An *st–path* is a path with the end–vertices s and t. An *s–path* is a path with an end–vertex s.

Let G be a claw- and net-free graph, $s, t \in V(G)$, $s \neq t$, $e \in E(G)$, and S and T disjoint paths in G. We give necessary and sufficient conditions for G to have:

($c1$) a Hamiltonian *st*–path (see **3.6** and **5.15** below),

($c2$) a Hamiltonian s–path containing e (**3.8** and **5.8**),

($c3$) a Hamiltonian *st*–path containing e if G is 3–connected (**5.20**),

($c4$) a Hamiltonian cycle containing S if G is k–connected, $k \geq 2$, and $v(S) \leq k$ (**6.4**),

($c5$) a Hamiltonian path containing e and S if G is $(k+1)$–connected, $k \geq 2$, and $v(S) \leq k$ (**6.7**),

($c6$) a Hamiltonian *st*–path containing S and T if G is k–connected, $k \geq 3$, $v(S) + v(T) \leq k$, and s, t are end–vertices of paths S and T, respectively (**6.3**).

From the above mentioned results we obtain the following corollaries.

1.1 (Corollary of **5.8**) *Let G be a 3–connected claw- and net-free graph. Then for any vertex s and any edge e in G there is a Hamiltonian s–path in G containing e.*

1.2 (Corollary of **5.20**) *Let G be a 4–connected claw- and net-free graph. Then for any two different vertices s, t and any edge e in G there is a Hamiltonian st–path in G containing e.*

1.3 (Corollary of **5.20**) *Let G be a claw- and net-free graph.*
If G is 4–connected then every two edges in G belong to a Hamiltonian cycle.
If G is 3–connected then every two non-adjacent edges in G belong to a Hamiltonian cycle.

1.4 (Corollary of **6.1**) *Let G be a k–connected graph, $k \geq 5$, C be a cycle in G, $v(C) \leq k-2$, $s, t \in V(G)$, $s \neq t$, and $V(C) \cap \{s, t\} = \emptyset$. Then for every $e \in E(C)$, there is a Hamiltonian st–path in G containing $C - e$ (and so, avoiding e).*

1.5 (Corollary of **6.4**) *In a k–connected claw- and net-free graph, $k \geq 2$, every $(k-1)$–vertex path belongs to a Hamiltonian cycle.*

1.6 *In a k–connected claw- and net-free graph, $k \geq 2$, every k–vertex path belongs to a Hamiltonian path.*

1.7 (Corollary of **6.7**) *Let G be a k–connected claw- and net-free graph, $k \geq 2$, e an edge and S a path in G, and $v(S) \leq k-2$. Then G has a Hamiltonian path containing e and S.*

1.8 (Corollary of **6.3**) *Let G be a $(k+1)$–connected claw- and net-free graph, $k \geq 2$, S and T two disjoint paths in G, s and t end–vertices of paths S and T, respectively, and $v(S) + v(T) \leq k$. Then G has a Hamiltonian st–path containing S and T.*

Our proofs provide polynomial time algorithm for solving the corresponding Hamiltonicity problems.

Section 2 provides the main notions and notation. Some known results are described in Section 3. Structural properties of claw- and net-free graphs are given in Section 4. The main Hamiltonicity results for k–connected graph with $k \leq 3$ are presented in Section 5. In Section 6 we give some Hamiltonicity results for k–connected graph with $k \geq 3$.

The results of this paper were presented at the Discrete Mathematics Seminar at the University of Puerto Rico in November 1999 (see also [6]).

2 Main notions and notation

We use the following notation:

$V(G)$ and $E(G)$ are the sets of vertices and edges of a graph G, respectively,

$v(G) = |V(G)|$ and $e(G) = |E(G)|$,

$c(G)$ is the vertex connectivity of G, and

$N(x, G)$ is the set of vertices in G adjacent to a vertex x.

A *uv–path* (*v–path*) is a path with the end–vertices u and v (an end-vertex v, respectively). A uv–path P is sometimes denoted by uPv. If a and b are vertices of P then aPb denote the subpath of P with the end–vertices a and b.

Let $X, Y \subseteq V(G)$. We say that G is *Hamiltonian (X, Y)–connected* if G has a Hamiltonian xy–path for every distinct vertices $x \in X$ and $y \in Y$. If G is Hamiltonian $(V(G), V(G))$–connected then G is called *Hamiltonian connected*.

Let H be a subgraph of G. A vertex x of H is called an *inner vertex of H* if x is adjacent to no vertices in $G - H$, and a *boundary vertex of H*, otherwise. Let $Bnd(H)$ and $In(H)$ denote the set of boundary and inner vertices of H, respectively. A path P in H is called an *inner path of H* if one of the end-vertices of P is an inner vertex of H. In particular, an edge e of H is called an *inner edge of H* if e is incident to an inner vertex of H.

A *non-trivial* (*or 2–connected*) *block* of G is a maximal 2–connected subgraph of G. A *trivial block* of G is a subgraph of G induced by an edge–cut.

A block B of G is called an *end–block* of G if B has exactly one boundary vertex, and an *inner block*, otherwise.

A *2–graph* xHy is a graph H with two specified vertices x and y. A *hammock* of a 2–connected graph G is a 2–graph xHy such that H is a 2–connected induced subgraph of G and $G - \{x, y\}$ is not connected. A hammock xHy of G is *minimal* if H has no proper subgraph which is also a hammock of G. Obviously, a hammock xHy of G is minimal if and only if either H is a triangle or $H \cup xy$ is 3–connected.

We write $G = AXB$ if A and B are induced subgraphs of G, $V(A) \cap V(B) = X$, $E(A) \cap E(B) = \emptyset$, and so $E(A) \cup E(B) = E(G)$. If $X = \{x\}$, we write simply AxB instead of AXB. If B is an edge xy, we write simply Axy or yxA instead of AxB.

3 Some known results

3.1 [3] (Corollary of **3.5** below) *Every connected claw- and net-free graph has a Hamiltonian path.*

3.2 [3] (Corollary of **3.9**) *Every 2–connected claw- and net-free graph has a Hamiltonian cycle.*

3.3 [9] (Corollary of **5.15**) *Every 3–connected claw- and net-free graph is Hamiltonian connected.*

In [7] we proved the following lemma which allowed us to give short proofs of **3.1** and **3.2** and also to find some new Hamiltonicity results for this class of graphs.

3.4 [7] *Let G be a connected claw- and net-free graph and $z \in V(G)$. Suppose that $G - z$ has a Hamiltonian xy–path P and there exists $e_z = zp \in E(G)$ (and so G is connected). Let e_x and e_y be the end-edges of P. Then G has a Hamiltonian ab–path Q such that $\{a, b\} \subset \{x, y, z\}$, $e_z \in E(Q)$ and $\{e_x, e_y\} \cap E(Q) \neq \emptyset$.*

From **3.4** we have immediately the following strengthening of **3.1**.

3.5 [7] *Let G be a connected claw- and net-free graph. Then*

$(a1)$ *G has a Hamiltonian path and*

$(a2)$ *if $sz \in E(G)$ and $G - z$ is connected then sz belongs to a Hamiltonian path of G.*

Let G be a connected graph and $x, y \in V(G)$, $x \neq y$. Let $Z(G, xy)$ denote the set of edges ab in G such that one of the following holds:
$(c1)$ $\{a, b\} = \{x, y\}$ and $V(G) \neq \{x, y\}$,
$(c2)$ $G - \{a, b\}$ is not connected and one of the components of $G - \{a, b\}$ does not meet $\{x, y\}$.

By using **3.4**, we also proved the following strengthening of **3.5**.

3.6 [7] *Let G be a connected claw- and net-free graph having $k \geq 2$ blocks. Let A_j, $j = 1, 2$, be end–blocks of G, a_j the boundary vertex of A_j, $a'_j \in A_j - a_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \ldots, k-2\}$. Then G has a Hamiltonian $a'_1a'_2$–path containing U if and only if*

(c1) $\alpha_j \notin Z(A_j, a'_1a_1)$ for every $j \in \{1, 2\}$ and

(c2) if $v(B_i) \geq 3$ then β_i is an inner edge of B_i, $i \in \{1, \ldots, k-2\}$.

From **3.6** we have, in particular:

3.7 [7] *Let G be a connected claw- and net-free graph having at least two blocks. Let A_1 and A_2 be the end–blocks of G with the boundary vertices a_1 and a_2, respectively. Then*

(c1) G is Hamiltonian $(V(A - a), V(B - b))$–connected,

(c2) G has a Hamiltonian xy-path if and only if x and y are inner vertices of different end–blocks of G, and

(c3) G has a Hamiltonian xy-path containing an edge e if and only if x and y are inner vertices of different end–blocks of G and $e \in Z(G, xy)$.

From **3.7** we have:

3.8 [7] *Let G be a claw- and net-free graph, $c(G) = 1$, $s \in V(G)$, and $e \in E(G)$. Then G has a Hamiltonian s–path containing e if and only if s is an inner vertex of an end-block in G and $e \in H(G, sb)$ where b is the boundary vertex of the end–block B avoiding s.*

By using **3.4**, we proved the following strengthening of **3.2**.

3.9 [7] *Let G be a 2–connected claw- and net-free graph and $e = sz \in E(G)$. Then the following are equivalent:*

(c1) e belongs to a Hamiltonian cycle of G and

(c2) $G - \{s, z\}$ is connected.

Moreover, if e does not satisfies (c2), say $G - z = AaDdF$ where $s = a$ and A, D are blocks of $G - z$, then for every $a' \in V(A - a)$ and $d' \in V(D - \{a, d\}$ there exists a Hamiltonian $a'd'$–path of G containing e.

Obviously **3.9** is a strengthening of **3.2** because 2–connected graph G has an edge sz such that $G - \{s, z\}$ is connected.

From **3.9** we have, in particular:

3.10 [7] *Let G be a 2–connected claw- and net-free graph. Then every edge in G belongs to a Hamiltonian path of G.*

From **3.9** we also have:

3.11 [7] (Corollary of **3.3**) *Let G be a 3–connected claw- and net-free graph. Then every edge in G belongs to a Hamiltonian cycle of G.*

4 On the structure of claw- and net-free graphs

We start with the following easy observations.

4.1 *Let G be a connected claw-free graph and S a minimal vertex cut in G. Then $G - S$ has exactly two components.*

Proof Suppose, on the contrary, that $G - S$ has at least three components. Let A, B, C be the components of $G - S$. Since S is a minimal vertex cut of G, a vertex s in S is adjacent to vertices, say a, b, and c, in A, B, and C, respectively. Then $\{s; a, b, c\}$ induces a claw in G centered at s, a contradiction.

4.2 [7] *Let G be a graph. Then G is claw- and net-free if and only if G has no connected induced subgraph with at least three end–blocks.*

A connected graph G is called a *chain* if G has either exactly one edge or exactly two end–blocks. If G is a chain and $e(G) \geq 2$ then $G = B_1 x_1 B_2 \ldots x_{k-1} B_k$ where $k \geq 2$, each B_i is a block of G (and so B_i is either a 2–connected graph or a one edge graph), $\{x_1, \ldots, x_k\}$ are distinct vertices, and each $x_i = V(B_{i-1}) \cap V(B_i)$.

Obviously from **4.2** we have:

4.3 *Let G be a claw- and net-free graph and $c(G) = 1$. Then G is a chain.*

4.4 *Let H be a 2–connected graph non-isomorphic to a triangle and S a minimal vertex cut of H. Let a graph G be obtained from H by adding two new vertices x_1 and x_2, and two new edges $x_1 h_1$ and $x_2 h_2$, where each $h_i \in V(H)$ and $h_1 \neq h_2$. Suppose that G is a claw- and net-free graph. Then no component of $H - S$ avoids $\{h_1, h_2\}$, and so $H \cup h_1 h_2$ is 3–connected.*

Proof (uses **4.1** and **4.2**). Suppose, on the contrary, there is a component C of $H - S$ such that $V(C) \cap \{h_1, h_2\} = \emptyset$, and so C is also a component of $G - C$. Since G is claw-free, $N(h_i, H)$ induces in a complete graph and **4.1**, $G - S$ has exactly two components. Let C' be the component of $G - S$ distinct from C. Since H is 2–connected, $|N(h_i, H)| \geq 2$. Therefore $S \cap \{h_1, h_2\} = \emptyset$, and so. $\{h_1, h_2\} \subseteq V(C')$. Let $s \in S$. Since S is a minimal cut, s is adjacent to a vertex, say c, in C. Then $G - (C - c)$ is an induced subgraph of G with three end–blocks cs, $x_1 h_1$, and $h_2 x_2$. Since G is claw-free, by **4.2**, $G - (C - c)$ has at most two end–blocks, a contradiction.

A chain G is called a *string* if for every inner block B either B has one edge or B is a triangle or $b_1 B b_2$ is a minimal hammock of G (i.e. $B \cup b_1 b_2$ is 3–connected).

4.5 *Suppose that G is a claw- and net-free graph and $c(G) = 1$. Then G is a string.*

Proof (uses **4.3** and **4.4**). Let B be an inner block of G and b_1 and b_2 the two boundary vertices of B. Suppose that B is not a triangle and not of one edge. Then B is a 2–connected graph. By **4.3**, G is a chain. Since B is an inner block of G, there are two disjoint edges x_1b_1 and x_2b_2 in G such that $x_1b_1Bb_2x_2$ is an induced subgraph of G. Since G claw- and net-free and $x_1b_1Bb_2x_2$ is an induced subgraph of G, $x_1b_1Bb_2x_2$ also claw- and net-free. Therefore by **4.4**, $(B \cup b_1b_2) - x$ is 3–connected. Thus, G is a string.

A "circle" sequence $R = (x_0, B_0, x_1, B_1, \ldots x_{k-1}, B_{k-1}, x_k)$, where $x_0 = x_k$ and $k \geq 2$, is a *circle representation of a graph* G if

$(r1)$ each B_i is either a 2–connected graph or a one edge graph,

$(r2)$ $E(B_i) \cap E(B_j) = \emptyset$ if $i \neq j$, and $V(B_i) \cap V(B_j) = \emptyset$ if $|i - j| \geq 2$,

$(r3)$ if $k \geq 3$ then $V(B_{i-1}) \cap V(B_i) = x_i$, $i = 1, \ldots k$, and if $k = 2$ then $V(B_1) \cap V(B_2) = \{x_1, x_2\}$ and $v(B_1) \geq 3$, $v(B_2) \geq 3$,

$(r4)$ $\{x_1, \ldots, x_{k-1}\}$ are distinct vertices, and

$(r5)$ $G = x_0B_0x_1B_1 \ldots x_{k-1}B_{k-1}x_0$ (and so each 2–connected B_i is a hammock of G).

We call each B_i a *brick of* R and denote by $r(R)$ the number of bricks of R.

If G has the circle representation R above, we will say simply that G is a *circle* $x_0B_0x_1B_1 \ldots x_{k-1}B_{k-1}x_0$.

Obviously, a graph G has a circle representation if and only if $c(G) = 2$. A graph G with $c(G) = 2$ may have different circle representations.

If G is a 2–connected graph then let $S(G) = \{x \in V(G) : cn(G - x) = 1\}$. Obviously, G is 3–connected if and only if $S(G) = \emptyset$.

A circle representation R of G is called a *ring representation* if $r(R) = |S(G)|$, and so every two ring representations of G have the same length. Thus if R is a ring representations of G, then let $r(G) = r(R)$.

Obviously

4.6 *Let G be a 2–connected graph and $R = (x_0, B_0, x_1, B_1, \ldots x_{k-1}, B_{k-1}, x_0)$ a circle representation of G, and so $k \geq 3$.*

Then the following are equivalent:

$(a1)$ *R is a ring representation of a graph G,*

$(a2)$ *$S(G) = \{x_0, \ldots, x_{k-1}\}$, and*

$(a3)$ *each B_i either has one edge or is a minimal hammock (i.e. either B_i is a triangle or $B_i \cup x_ix_{i+1}$ is 3–connected).*

A 2–connected graph G is called a *ring* if G is either a triangle or has a ring representation and $r(G) \geq 3$. Obviously, a cycle is a minimal ring.

It is easy to see the following.

4.7 *Suppose that G is a ring and $v(G) \geq 4$. Then G has a unique ring representation.*

In Section 5 we will need the following notions of the *brick–length of a path* and the *brick–distance* between two vertices in a graph which is either a chain or has a circle representation.

Let G have a circle representation R and xPy a path in G. We define the *β_R–length $\beta_R(P)$ of of path P* as follows. If P is an inner path of a brick B in R then let $\beta_R(P) = 0$. If P is not an inner path of a brick in R then let $\beta_R(P) - 1$ be the number of bricks B in R such that both boundary vertices of B belong to P (i.e. the number of bricks in G, path P 'passes through'). A *β_R–shortest xy*–path in G is an xy–path with the smallest β_R–length among all xy–paths in G. Given $x, y \in V(G)$, the *β–distance $\beta_R(x, y)$ between x and y in G* is the β_R–length of a β_R–shortest xy–path in G. If G is a ring then by **4.7**, G has a unique circle representation. Therefore in this case we write β_G (or simply β) instead of β_R.

The corresponding notions for a chain are defined similarly.

A sequence $Q = (B_0, X_1, B_1, \ldots, X_k, B_k)$ where $k \geq 1$, is called a *2–chain representation* of a graph G if

$(q1)$ each B_i is a graph,

$(q2)$ $E(B_i) \cap E(B_j) = \emptyset$ if $i \neq j$, and $V(B_i) \cap V(B_j) = \emptyset$ if $|i - j| \geq 2$,

$(q3)$ $V(B_{i-1}) \cap V(B_i) = X_i = \{x_i, x'_i\}$, $x_i \neq x'_i$ for every $i \in \{1, \ldots, k-1\}$,

$(q4)$ $X_i \neq X_j$ for $i \neq j$, and

$(q5)$ $G = B_0X_1B_1 \ldots X_kB_k$.

Let $s(Q) = k$. A 2–chain representation Q is called a *2–string representation* if $S(G) = \cup\{X_i : i \in \{1, \ldots, k\}\}$.

Let $\bar{B}_i = B_i \cup \{x_ix'_i, x_{i+1}x'_{i+1}\}$, $\bar{B}_0 = B_0 \cup x_1x'_1$, and $\bar{B}_k = B_k \cup x_kx'_k$. Obviously, each X_i is a vertex 2-cut of G. Each $\bar{B}_i$, $1 \leq i \leq k-1$, is called an *inner brick*, and $\bar{B}_0$, $\bar{B}_k$ are called the *end–bricks* of the 2–chain representation Q of G.

A graph G is called a *2–string* if G has a 2–string representation and is not a ring.

One can easily prove the following.

4.8 *Let G be 2–connected graph and $Q = (B_0, X_1, B_1, \ldots, X_k, B_k)$ a 2–chain representation of G and $k \geq 2$. Then the following are equivalent:*

$(a1)$ *Q is a 2–string representation of a graph G,*

$(a2)$ *each $\bar{B}_i$ is either a triangle, or a square, or a 3–connected graph.*

Here are some examples illustrating the notions of a ring and a 2–string.

Let $G = x_0B_0x_1B_1x_2B_2x_3B_3x_4$ where $x_4 = x_0$ and each B_i is either an edge or a minimal hammock of G. Then G has a ring representation. If one or two B_i's are minimal hammocks of G then G has also a 2–chain representation. In both cases we call G a ring.

Let $G = x_0B_0x_1B_1x_2B_2x_3$ where $x_3 = x_0$ and each B_i is either an edge or a minimal hammock of G. If exactly two B_i's are minimal hammocks of G then G has both a ring and a 2–string representations. In this case we call G a ring. Now suppose that B_0 is the only minimal hammock of G. Then G has a circle representation R with $r(R) = 3$ but no ring representation with $r(R) = 3$. On the other hand, G has a 2–string representation Q with $s(Q) = 2$, and so G is a 2–string. If each B_i is an edge then G is a triangle, and so G is a ring.

It is also easy to see the following.

4.9 *If G has a 2–string representation then G has only one 2–string representation of minimum number of bricks up to the edges xy such that $G - \{x, y\}$ is not connected.*

We need the following simple lemma.

4.10 *Suppose that*

$(h1)$ *G is a claw- and net-free graph,*

$(h2)$ *G has a circle representation R, and*

$(h3)$ *one of the following holds:*

$(h3.1)$ $r(R) \geq 5$,

$(h3.2)$ $r(R) = 4$ *and R has at least two non-trivial bricks having a common vertex, and*

$(h3.3)$ $r(R) = 3$ *and all three bricks of R are non-trivial.*

Then G is a ring and R is the ring representation of G.

Proof (uses **4.7** and **4.4**). Let $b_1 B b_2$ be a hammock of a circle representation R of G. Assumption $(h3)$ implies that there are two disjoint edges $x_1 b_1$ and $x_2 b_2$ in G such that b_1 and b_2 are the two boundary vertices of B and $x_1 b_1 B b_2 x_2$ is an induced subgraph of G. From **4.4** it follows that either B is of one edge or B is a triangle or $B \cup b_1 b_2$ is a 3–connected graph. Therefore G is a ring and R is a ring representation of G. By **4.7**, R is the only ring representation of G.

There are infinitely many examples showing that if the assumption $(h3)$ in **4.10** is not satisfied then the claim of **4.10** is not true.

If a claw and net free graph G has a minimal 2–cut then G can be decomposed as follows.

4.11 *Let $G = x_1 A_1 x_2 A_2 x_1$ where A_1 and A_2 are connected subgraphs. Let T be a triangle x_1, x_2, z where $z \notin V(G)$ and $A'_i = (A_i - (A_j - A_i)) \cup T$ where $\{i, j\} = \{1, 2\}$. If G is claw- and net-free then each A'_i is also claw- and net-free.*

Proof Let $A \in \{A'_1, A'_2\}$. Suppose, on the contrary, A is not claw- and net-free. Then A has an induced subgraph H which is either a claw or a net.

Since G is claw- and net-free, clearly $E(H) \cap E(T) \neq \emptyset$. Obviously $T \not\subseteq H$, and so $|E(H) \cap E(T)| = 1$, say $E(H) \cap E(T) = h$.

Suppose that $h = xz$ for some $x \in \{x_1, x_2\}$. Since each A_i is connected, there is $v \in V(G - A)$ such that $xv \in E(G)$. If H is a claw (a net) then $H' = (H - z) \cup xv$ is an induced claw (a net, respectively) in G, a contradiction.

Now suppose that $h = x_1 x_2$. If $h \in E(G)$ then H is an induced subgraph of G, a contradiction. So we assume that $h \notin E(G)$.

Suppose that H is either a claw or a net such that h is not in the triangle of H. Let $x = x_1$ be the center of the claw H. Then $H' = (H - x_2) \cup xv$ is an induced claw or net in G, respectively, a contradiction.

Now suppose that H is a net and h is in the triangle of H. Then $H - h$ contains an induced claw of G, a contradiction.

It is easy to prove the following.

4.12 *Let G be a 2–chain and $\bar{B}_i$, $i = 0, \ldots, k$, be the bricks of G. Let $0 \le i < j \le k$. Suppose that G is claw-free. If $j - i = 2$ and $\bar{B}_i \cap \bar{B}_j \ne \emptyset$ then $\bar{B}_{i+1}$ is a triangle. If $j - i \ge 3$ then $\bar{B}_i \cap \bar{B}_j = \emptyset$.*

4.13 *Let G be a claw- and net-free graph and $c(G) = 2$. Then G is either a ring or a 2–string.*

Proof (uses **4.10** and **4.11**). We prove our claim by induction on $v(G)$. If $v(G) = 3$ then the claim is obviously true. Let $v(G) \ge 4$. Since $c(G) = 2$, clearly G has a vertex 2–cut $X = \{x_1, x_2\}$, and so $G = A_1 \cup A_2$ where $V(A_1) \cap V(A_2) = X$. Since G is claw–free, $G - X$ has exactly two components. Let T be a triangle x_1, x_2, z where $z \notin V(G)$. Let $A'_i = (A_i - (A_j - A_i)) \cup T$ where $\{i, j\} = \{1, 2\}$. Since G is 2–connected, each A'_i is also 2–connected. Since G is claw- and net-free, by **4.11**, each A'_i is also claw- and net-free-free.

(**p1**) Suppose that $v(A_i) \ge 4$ for some $i \in \{1, 2\}$, say $i = 2$. Clearly $c(A'_1) = 2$ and $v(A_1) < v(G)$. By the induction hypothesis, A'_1 is either a ring or a 2–string.

(**p1.1**) Suppose that A'_1 is a ring. Since T is a hammock of A'_1, A'_1 is not a cycle.

Suppose that $r(A'_1) \ge 5$. Then G has a circle representation R of length at least five, and by **4.10**, G is a ring.

Suppose that $r(A'_1) = 4$. Suppose that A'_1 has at least two hammocks with a vertex in common. Then G has a circle representation R either of length five or of length four and with at least two adjacent hammocks. By **4.10**, G is a ring. Now suppose that A'_1 has one or two non-adjacent hammocks. Then $v(A_1) \ge 4$, and so $v(A'_2) < v(G)$. By the induction hypothesis, A'_1 is either a ring or a 2–string. If A'_1 is a ring, then $r(A'_1) \ge 3$, and so G has a circle representation of length at least five. By **4.10**, G is a ring. If A'_1 is a 2–string, then G is also a 2–string.

Now suppose that $r(A'_1) = 3$, and so A'_1 has at least two hammocks. Then $v(A_1) \ge 4$, and so $v(A'_2) < v(G)$. By the induction hypothesis, A'_1 is either a ring or a 2–string. If A'_1 is a ring, then obviously G is a ring.

(**p1.2**) Now suppose that A'_1 is a 2–string. If A'_1 has three hammocks then G has a circle representation with three hammocks, and by **4.10**, G is a ring. If A'_1 has two hammocks and $A_2 \cup x_1x_2$ is 3–connected then G is a ring with two hammocks. If A'_1 has two hammocks and $A_2 \cup x_1x_2$ is not 3–connected then G is a 2–string.

(**p2**) Now suppose that $v(A_1) = v(A_2) = 3$. Then G is either a 4–cycle, and so G, or a diamond, and so G is a 2–string.

5 On Hamiltonicity of claw- and net-free graphs

We will use the following lemmas from [7].

5.1 [7] *Let G be a 2–connected graph and $x'xGyy'$ is a claw- and net-free graph, and so $x \ne y$. Let $sz \in E(G)$ and $\{s, z\} \ne \{x, y\}$. Then $x'xGyy'$ has a Hamiltonian $x'y'$–path containing sz.*

Proof (uses **3.1**, **3.4**, and **4.3**). Let $G' = G - z$. Since G is 2–connected, clearly G' is connected. Since $x'xGyy'$ is claw- and net-free, $x'xG'yy'$ is also claw- and net-free. By **4.3**, $x'xG'yy'$ is a string. Therefore G' is either a 2–connected graph or a string with x and y in different end–blocks of G'. By **3.1**, $x'xG'yy'$ has a Hamiltonian $x'y'$–path. By **3.4**, $x'xGyy'$ has a Hamiltonian ab–path containing sz where $\{a,b\} \subset \{x,y,z\}$. If $\{a,b\} \neq \{x,y\}$ then clearly $x'xGyy'$ has no Hamiltonian ab–path. Therefore $x'xGyy'$ has a Hamiltonian $x'y'$–path containing zs.

Let G be a connected graph and $x, y \in V(G)$, $x \neq y$. Let $Z(G, xy)$ denote the set of edges ab of G such that one of the following holds:

$(c1)$ $\{a,b\} = \{x,y\}$ and $V(G) \neq \{x,y\}$,

$(c2)$ $G - \{a,b\}$ is not connected and one of the components of $G - \{a,b\}$ does not meet $\{x,y\}$.

5.2 [7] *Let G be a 2–connected graph and Gyy' be a claw- and net-free graph. Let $x \in V(G - y)$ and $e \in E(G)$. Then Gyy' has a Hamiltonian xy'–path containing e if and only if $e \notin Z(G, xy)$.*

The proof of **3.6** in [7] uses **3.7**, **5.1**, and **5.2**.

It is easy to prove the following.

5.3 [9] *Let H be a connected graph and Hxy a claw- and net-free graph. Let X be the set of vertices in H that are on the same distance from x. Then X induces a complete subgraph in H.*

Although we do not use this lemma, we mention it here because this lemma may be used to provide alternative proofs for some claims below.

From **5.1** we have:

5.4 *Let G be a claw- and net-free graph–ring having k bricks, and so $k \geq 3$. Let B_i be a brick of G, $\beta_i \in E(B_i)$, and $U = \{\beta_i : i = 1, \ldots, k\}$. Suppose that β_i is incident to an inner vertex in B_i if $v(B_i) \geq 3$. Then G has a Hamiltonian cycle containing U.*

We need the following two simple lemmas.

5.5 *Let G be a 2–connected claw- and net-free graph and $x \in V(G)$. Suppose that G is not a triangle and $N(x, G)$ induces a clique Q in G. Then there is an edge st in Q such that $G - \{x, s, t\}$ is connected.*

Proof Since G is 2–connected, $|N(x, G)| \geq 2$.

(**p1**) Suppose that $|N(x, G)| \geq 3$, say $\{s, t, t'\} \subseteq N(x, G)$. Suppose that $G - \{s, t\}$ and $G - \{s, t'\}$ are not connected. Let C and C' be components of $G - \{s, t\}$ and $G - \{s, t'\}$, respectively, that do not contain x. Then $V(C) \cap V(C') = s$. Since G is 2–connected, there exist $c \in V(C - s)$ and $c' \in V(C' - s)$ such that $cs, c's \in E(G)$. Then $\{s; c, c', x\}$ induces a claw in G centered at s, a contradiction. Therefore we can assume that $G - \{s,t\}$ is connected. Since $N(x,G)$ is a complete graph, $G_x - \{s,t\}$ is connected.

(**p2**) Now suppose that $|N(x,G)| = 2$, say $N(x,G) = \{s,t\}$. Since G is not a triangle, $G - \{s,t\} \neq \emptyset$. Suppose that $G - \{x,s,t\}$ is not connected. Then $G - \{s,t\}$ has at least three components. Since G is claw-free, $G - \{s,t\}$ has at most two components, a contradiction.

5.6 *Let G be a 2–connected graph, Gyy' a claw- and net-free graph, $x \in V(G-y)$, $N(x,G)$ induce a complete graph Q in $G-y$, and $s \in V(G - \{x,y\})$. Then there is an edge q in Q such that $G-x$ has a Hamiltonian sy–path, containing q, and G has a Hamiltonian sy–path.*

Proof (uses **4.4**, **5.2**, and **5.5**). Let $z = y$. By **4.4**, $\bar{G} = G \cup xy$ is 3–connected. Therefore $G' = G - x$ is 2–connected and $|N(x, G-y)| \geq 2$. By **5.5**, there is an edge q in Q such that $G - \{x,u,v\}$ is connected. By **5.2**, $G - x$ has a Hamiltonian sy–path P containing q, say $P = sPuvPx$ where $q = uv$. Then $sPuxvPx$ is a Hamiltonian sy–path in G.

Let G be a graph. Let $F(G)$ denote the set of pairs (s,e) such that $s \in V(G)$, $e \in E(G)$, and G is a circle $x_1B_1 \ldots x_{k-1}B_kx_1$, and the β–distance between s and each end–vertex of e in this circle is at least two, (and so $k \geq 3$), namely, $e = x_1x_2$, and either $s = x_r$ or $s = B_r - \{x_{r-1}, x_r\}$ where $3 \leq r \leq k-1$.

Obviously

5.7 *Let G be a 2–connected graph. If $(s,e) \in F(G)$ then G has no Hamiltonian s–path containing e.*

It turns out that for 2–connected claw- and net-free graphs, converse is also true.

5.8 *Let G be a 2–connected claw- and net-free graph, $s \in V(G)$ and $e \in E(G)$. Then the following are equivalent:*

($a1$) G has a Hamiltonian s–path containing e and

($a2$) $(s,e) \notin F(G)$.

Proof (uses **3.5**, **3.9**, **4.2**, and **5.2**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction $v(G)$. If $v(G) = 3$ then our claim is obviously true. So let $v(G) \geq 4$. Let $e = uv$ and $G' = G - s$.

(**p1**) Suppose that $s \in \{u,v\}$, say $s = u$.

If G' is 2–connected, then by **3.9**, G' has a Hamiltonian v–path P. Then svP is a Hamiltonian s–path in G containing uv.

If G' is not 2–connected then by **4.2**, G' is a chain and s is adjacent to an inner vertex s' of an end–block in G'. By **3.5**, G' has a Hamiltonian s'–path P. Then $ss'P$ is a Hamiltonian s–path in G containing uv.

(**p2**) Now suppose that $s \notin \{u,v\}$.

(**p2.1**) Suppose that G' is 2–connected. If there is $y \in N(s,G)$ such that $(y,e) \notin F(G')$, then by the induction hypothesis there is a Hamiltonian s'–path P containing uv. Then $ss'P$ is a Hamiltonian s–path in G containing uv. Therefore we assume that for every $y \in N(s,G)$, $(y,e) \in F(G')$. Then $G' = x_1B_1 \dots x_{k-1}B_kx_1$ is a ring with $k \geq 3$ Since G is 2–connected $|N(s,G)| \geq 2$. Since $(y,e) \in F(G')$ for every $y \in N(s,G)$, clearly $N(s,G) \cap V(B_kx_1B_1x_2B_2 - \{x_3, x_{k-1}\}) = \emptyset$. Then $(s,e) \in F(G)$, a contradiction.

(**p2.2**) Now suppose that G' is not 2–connected. The G' is a chain and since G is 2–connected, s is adjacent to inner vertices of the end–blocks of the chain. Suppose that e belongs to an end–block B with the boundary vertex b. If B is 2–connected then there is a vertex b' in B such that $e \notin Z(G, bb')$. Therefore by **5.2**, B has a Hamiltonian $b'b$–path P containing e. Let $s' \in N(s,G)$ be an inner vertex of the end–block of G' distinct from B. By **3.5**, $G' - (B - b)$ has a hamiltonian $s'b$–path P'. Then $ss'P'bPb'$ is a Hamiltonian s–path in G containing e.

From **5.8** we have:

5.9 *Let G be a 2–connected claw- and net-free graph. Then every edge belongs to a Hamiltonian path in G.*

From **5.8** we also have:

5.10 *Let G be a 3–connected claw- and net-free graph. Then for every $s \in V(G)$ and $e \in E(G)$ there exists a Hamiltonian s–path in G containing e.*

5.11 *Let G be a claw- and net-free graph–ring and $c(G) = 2$. Let $u, v \in V(G)$ and $u \neq v$. Then the following are equivalent:*

($a1$) *G has a Hamiltonian uv–path and*

($a2$) *$(G; u, v)$ satisfies exactly one of the following conditions:*

($a2.1$) *if G is a ring then either $\beta(u,v) \leq 1$ or $\beta(u,v) = 2$ and there is a β–shortest uv–path containing a brick–edge of G,*

($a2.2$) *if G is a 2–string then $G - \{u, v\}$ is connected.*

Proof (uses **3.5**, **3.7**, **4.2**, **4.5**, **4.10**, **4.12**, **4.13**, **5.4**, **5.5**, and **5.6**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$. If G is a cycle or a diamond then our claim is obviously true. Therefore let $v(G) \geq 5$ and G is not a cycle. Since G is a claw- and net-free graph–ring and $c(G) = 2$, by **4.13**, G is either a ring or a 2–string.

(**p1**) Suppose that G is a ring. Obviously $(G; u, v)$ satisfies $(a2.1)$ if and only if exactly one of the following holds:

($h1$) aAx and xBb are two adjacent bricks of G and either $u, v \in V(A \cup B - \{a, b\})$ or $v(A) = 2$, $u = a$, and $v \in V(B - b)$,

($h2$) aAa' and $b'Bb$ are two bricks of G connected by an edge $a'b'$ and $u \in V(A - \{a, a'\})$, $v \in V(B - \{b, b'\})$.

We can assume that $v \in V(B)$.

(**p1.1**) Suppose that ($h1$) holds.

(**p1.1.1**) Suppose that $u \in A - \{a, x\}$ and $v \in B - \{b, x\}$. Then $v(A) \geq 3$ and $v(B) \geq 3$. Let $G_x = G - x$, $A_x = A - x$, and $B_x = B - x$. Also $u \in A_x - a$ and $v \in B_x - b$. Since G is claw-free, $N(x, A)$ and $N(x, B)$ are complete graphs. Since A is a brick of a ring and $v(A) \geq 3$, clearly $\bar{A}$ is either a triangle or 3–connected, and so A_x is either of one edge or 2–connected. Similarly B_x is either of one edge or 2–connected. Obviously G_x is claw- and net-free and $c(G_x) = 1$. Therefore by **4.5**, G_x is a string and, clearly, A_x and B_x are the end–blocks of G_x. By **5.5**, there are two vertices $s, t \in N(x, B)$, such that st is a vb–good edge in B_x. Now by **3.5**, G_x has a Hamiltonian uv–path P containing st, say $uPv = uPstPv$ (possibly $t = v$ and/or $s = b$). Then $uPsxtPv$ is a Hamiltonian uv–path in G.

(**p1.1.2**) Suppose that $u \in B$ and $v \in V(B - \{b, x\})$ (the case $u \in A - x$ and $v = x$ is similar). Then $v(B) \geq 3$, and so B is 2–connected. Let $G_v = G - v$ and $B_v = B - v$. If there is a vertex z adjacent to v such that G_v has a Hamiltonian uz–path P, then $uPzv$ is a Hamiltonian uv–path in G. So it is sufficient to show that a vertex z with the above properties exists.

Clearly G_v and B_v are claw- and net-free. Since B is a brick of the ring G and $v(B) \geq 3$, we have: $\bar{B} = B \cup bx$ is either a triangle or 3–connected. Therefore $G_v = G - v$ is 2–connected and since G is a ring, $c(G) = 2$. Thus by **4.13**, G_v is either a ring or a 2–string.

Suppose that $\bar{B}$ is a triangle. Then B_v has one edge bx and $u \in \{b, x\}$, say $u = x$. By **5.4**, every Hamiltonian cycle C of G_v contains bx. Then $(C - bx) \cup bv$ is a Hamitonian uv–path in G.

Now suppose that $\bar{B}$ is 3–connected. Then $c(B_v) \geq 1$. If $c(B_v) = 1$ then since B_v is claw- and net-free, by **4.5**, B_v is a string $D_1 d_1 \ldots d_{s-1} D_s$ where $s \geq 1$, each D_i either has one edge or is 2–connected. Clearly $x \in D_1 - d_1$ and $b \in D_s - d_{s-1}$. Obviously $c(G) = 2$, $\bar{B}$ is 3–connected, and $v \in V(B - \{b, x\})$ imply $c(G_v) = 2$. Therefore G_v has a circle representation R such that T is a brick of R if and only if T is either a brick of G distinct from B or a block of B_v. If, in particular, B_v is 2–connected then clearly B_v is a brick of R.

(**p1.1.2.1**) Suppose that G_v is a ring.

Suppose that B_v has a block D containing both u and a vertex z adjacent to v. By the induction hypothesis, G_v has a Hamiltonian uz–path uPz.

Now suppose that B_v has no block containing both u and a vertex z adjacent to v, and so B_v is a string.

If there is a *good* vertex v', i.e. a vertex v' such that $vv' \in E(G)$ and the pair $\{u, v'\}$ satisfies (h_1) or ($h2$) with respect to the blocks of B_v containing them, then by the induction hypothesis, G_v has a Hamiltonian uv'–path uPv'. Therefore assume, for the contraction, that there is no good vertices.

Since B_v is 2–connected, $N(x, G) \cap V(D_1 - d_1) \neq \emptyset$ and $N(x, G) \cap V(D_s - d_{s-1}) \neq \emptyset$. Since G has no induced claw centered at x, clearly $N(x, G) \subseteq V(D_1) \cup V(D_s)$.

Since there is no good vertices and $N(x, G) \cap V(D_1 - d_1) \neq \emptyset$, clearly $u \notin V(D_1 \cup D_2 \cup D_{s-1} \cup D_s - \{d_2, d_{s-2}\})$, and so $s \geq 4$. Then $G - \{a, d_{s-1}\}$ is connected and has three end–blocks. Since G is claw- and net-free, $G - \{a, d_{s-1}\}$ is also claw- and net-free. By **4.2**,

$G - \{a, d_{s-1}\}$ has at most two end–blocks, a contradiction.

(**p1.1.2.2**) Now suppose that G_v is not a ring, and so G_v is a 2–string. Then by **4.10**, one of the following holds:

$(c1)$ $G = xaCcbBx$ where B is a non-trivial brick of R,

$(c2)$ $G = aAxBba$ where both A and B are non-trivial bricks of R.

Since G_v is not a ring, $B_v = d_1Dd_2Fd_1$ where $\{b, x\} \subseteq V(F)$. Since G_v is claw- and net-free, $N(x, B_v)$ and $N(b, B_v)$ induce a complete subgraph in B_v. Therefore $\{b, x\} \subseteq V(F - \{d_1, d_2\}$. Since $\bar{B}$ is 3–connected and G has no induced claw centered at v, $D \cup d_1d_2$ and $F \cup \{bx, d_1d_2\}$ are 3–connected and there are $d \in V(D - \{d_1, d_2\}) \cap N(v, G)$ and $f \in V(f - \{d_1, d_2\}) \cap N(v, G)$. Therefore G_v is a 2–string with at least three bricks. If $u = d$, then $G_v - \{u, f\}$ is connected. If $u \neq d$, then $G_v - \{u, d\}$ is connected. Hence in both cases there is $z \in \{d, f\}$ such that $G_v - \{u, z\}$ is connected, and so $(G_v; u, z)$ satisfies $(a2.2)$. Thus by the induction hypothesis, G_v has a Hamiltonian uz–path P. Then $uPzv$ is a Hamiltonian uv–path in G.

(**p1.2**) Now suppose that $(h2)$ holds. By **5.6**, A and B have Hamiltonian paths $aA'u$ and $bB'v$, respectively. By **3.7**, $G - ((A \cup B) - \{a, b\})$ has a Hamiltonian ab–path P. Then $uA'aPbB'v$ is a Hamiltonian uv–path in G.

(**p2**) Now suppose that G is a 2–string. Let $G_u = G - u$. Clearly G_u is claw- and net-free and $v(G_u) < v(G)$.

Since G is a 2–string, we have: $G = B_1X_1B_2X_2 \ldots X_{k-1}B_k$ where $E(B_i) \cap E(B_j) = \emptyset$ for every $i \neq j$, $V(B_i) \cap V(B_{i+1}) = X_i = \{x_i, x'_i\} \subseteq V(G)$, all X_i's are different, and $\bar{B}_1 = B_1 \cup x_1x'_1$, $\bar{B}_k = B_k \cup x_{k-1}x'_{k-1}$, and $\bar{B}_i = B_i \cup \{x_ix'_i, x_{i+1}x'_{i+1}\}$ for $i \notin \{1, k\}$ are either triangles or squares or 3–connected graphs. If $X_i \cap X_j \neq \emptyset$ for $i < j$ then by **4.12**, G is claw-free implies that $j = i + 2$ and $\bar{B}_{i+1}$ is a triangle.

(**p2.1**) Suppose that u belongs to a 2–cut of G or u is an inner vertex of an end–brick which is a square. Then $c(G_u) = 1$, and so by **4.5**, G_u is a string. Since G is a 2–string, G_u satisfies one of the following conditions:

$(c1)$ $G_u = A_1xA_2$, where A_1 and A_2 are the blocks (and the end–blocks) of G_u (we put $x = a_1 = a_2$),

$(c2)$ $G_u = A_1a_1a_2A_2$ where A_1 and A_2 are the end–blocks, and $a_1 \in V(A_1)$, and $a_2 \in V(A_2)$.

Since $\{u, v\}$ is not a 2–cut of G, we can assume that $v \in V(A_2 - a_2)$. Since G is 2–connected, there is $a \in N(u, G) \cap V(A_1 - a_1)$. By **3.7**, G_u has a Hamiltonian av–path aPv. Then $uaPv$ is a Hamiltonian uv–path in G.

(**p2.2**) Now suppose that neither u nor v belongs to a 2–cut of G, i.e. both u and v are inner vertices of some bricks of G. Let u be an inner vertex of B_i in G, and so $N(u, G) = N(u, B_i)$ and $\bar{B}_i$ is either a triangle or a square or 3–connected, and if B_i is an inner brick then $\bar{B}_i$ is 3–connected. Then G_u is 2–connected. By **4.13**, G_u is either a ring or a 2–string or 3–connected.

(**p2.2.1**) Suppose that G_u is a ring. Then the 2–string G has at most three bricks.

(**p2.2.1.1**) Suppose that G has three bricks and $B_i = B_2$, i.e. u is an inner vertex of B_2. Then $V(B_2) = Bnd(B_2) \cup u$, and so $N(u, G) = Bnd(B_2)$. Since v is an inner vertex of a brick of G and $v \neq u$, clearly v is an inner vertex of an end–brick of G. The following are equivalent: B is a non-trivial brick of G_u, B is an end-brick of G, and $\bar{B}$ is 3–connected, and so G_u has at most two non-trivial bricks.

Suppose that G_u has no non-trivial bricks. Then G_u is a cycle, $5 \leq v(G_u) \leq 8$, and there is a vertex $z \in N(u, G)$ such that $vz \in E(G_u)$. Therefore G_u has a Hamiltonian vz–path P. Then $uzPv$ is a Hamiltonian uv–path in G.

Suppose that G_u has exactly one non-trivial brick. Then G_u has at most six bricks and v is either an inner vertex of the non-trivial brick of G_u or a vertex of degree two in G_u. In both cases there is a vertex $z \in N(u, G)$ such that $(G_u; v, z)$ satisfies (a2.1). By the induction hypothesis, G_u has a Hamiltonian vz–path P. Then $uzPv$ is a Hamiltonian uv–path in G.

Now suppose that G_u has exactly two non-trivial bricks. Then G_u has three or four bricks and v is an inner vertex of these bricks, and so there is a vertex $z \in N(u, G)$ such that $(G_u; v, z)$ satisfies (a2.1). By the induction hypothesis, G_u has a Hamiltonian vz–path P. Then $uzPv$ is a Hamiltonian uv–path in G.

(**p2.2.1.2**) Now suppose that G has three bricks and $B_i = B_1$,

Then $\bar{B}_1$ is a triangle, $V(B_1) = \{u, x_1, x_1'\}$, B_2 is a string $D_1 d_1 \dots d_{s-1} D_s$, $x_2 \in D_1 - d_1$, $x_2' \in D_s - d_{s-1}$. Thus G_u is the ring $x_2 D_1 d_1 \dots d_{s-1} D_s x_2' B_3 x_2$.

If $s \geq 4$ then $\bar{B}_2$ is neither a triangle nor a square nor 3–connected, and so G is not a 2–string, a contradiction.

(**p2.2.1.2.1**) Suppose that $s = 2$, and so the ring G_u has three bricks. If both D_1 and D_2 have one edge (i.e. $\bar{B}_2$ is a triangle) then G_u is not a ring, a contradiction. Therefore D_i is 2–connected for some $i \in \{1, 2\}$, say for $i = 1$. Then $\bar{B}_2$ is 3–connected, and so $\{x_1, x_1'\} \cap V(D_1 - \{d_1, x_2\} \neq \emptyset$, say $x_1 \in V(D_1 - \{d_1, x_2\}$. Since v is an inner vertex of a brick in G, clearly $v \neq x_1$. Then $G_u - \{v, z\}$ is connected. Since G_u has three bricks,$(G_u; v, x_1)$ satisfies (a2.1). By the induction hypothesis, G_u has a Hamiltonian vx_1–path P. Then ux_1Pv is a Hamiltonian uv–path in G.

(**p2.2.1.2.2**) Now suppose that $s = 3$, and so the ring G_u has four bricks. If some D_i is 2–connected, then the inner brick B_2 of the 2–string G has an inner vertex, and so $\bar{B}_2$ is 3–connected. If exactly one of D_i's is 2–connected or D_2 is 2–connected then $\bar{B}_2 = B_2 \cup \{x_1x_1', x_2x_2'\}$ is not 3–connected because $N(u, G) = \{x_1, x_1'\}$, a contradiction.

Suppose that D_1 and D_3 are 2–connected and D_2 has one edge, and so $x_2 D_1 d_1 D_2 d_2 D_3 x_2' = x_2 D_1 d_1 d_2 D_3 x_2'$. Since $\bar{B}_2$ is 3–connected $N(x, G)$ contains inner vertices of D_1 and D_3. Therefore there is $z \in N(u, G)$ such that $G_u - \{v, z\}$ is connected. Since G_u has four bricks, $(G_u; v, z)$ satisfies (a2.1). By the induction hypothesis, G_u has a Hamiltonian vz–path P. Then $uzPv$ is a Hamiltonian uv–path in G.

Now suppose that none of D_i's is 2–connected. Then $x_2 D_1 d_1 D_2 d_2 D_3 x_2'$ is a 3–edge path $x_2 d_1 d_2 x_2'$ and $N(u, G) = \{x_1, x_1'\} = \{d_1, d_2\}$. Since v is an inner vertex of a brick in G, clearly $G_u - \{v, z\}$ is connected. Since G_u has four bricks, $G_u; v, x_1)$ satisfies (a2.1). By the induction hypothesis, G_u has a Hamiltonian vx_1–path P. Then ux_1Pv is a Hamiltonian

uv–path in G.

(**p2.2.1.2**) Now suppose that G has two bricks. Then $B_1 - u$ is a string $D_1 d_1 \dots d_{s-1} D_s$, $x_2 \in D_1 - d_1$, $x'_2 \in D_s - d_{s-1}$. Thus G_u is the ring $x_2 D_1 d_1 \dots d_{s-1} D_s x'_2 B_2 x_2$. The arguments similar to those in (**p2.2.1.1**) show that there is $z \in N(u, G)$ such that $G_u; v, z)$ satisfies $(a2.1)$. By the induction hypothesis, G_u has a Hamiltonian vz–path P. Then $uzPv$ is a Hamiltonian uv–path in G.

(**p2.2.2**) Suppose that G_u is a 2–string. We remind that v is an inner vertex of a brick of G. If there is $z \in N(u, G)$ such that $(G_u; v, z)$ satisfies $(a2.2)$ (i.e. $G_u - \{v, z\}$ is connected) then by the induction hypothesis, G_u has a Hamiltonian vz–path P and then $uzPv$ is a Hamiltonian uv–path in G. So our goal is to show that there exists a vertex $z \in N(u, G)$ such that $G_u - \{v, z\}$ is connected.

If $\bar{B}_i - u$ is 3–connected, then obviously there is $z \in N(u, G)$ such that $G_u - \{v, z\}$ is connected. So we assume that $\bar{B}_i - u$ is not 3–connected.

If B_i is an end-brick then (**p2.1**), B_i is not a square.

Suppose that $\bar{B}_i$ is 3–connected. Since $\bar{B}_i - u$ is not 3–connected and G_u is a 2–string, $\bar{B}_i - u = D_1 Y D_2$ where $Y = \{y, y'\}$, $x_i x'_{i-1} \in E(D_1)$ and $x_i x'_i \in E(D_2)$, if B_i is an inner brick of G, and $x_1 x'_1 \in E(D_2)$ if $B_i = B_1$. Since $\bar{B}_i$ is 3–connected and G has no induced claw centered at u, each D_i has an inner vertex in $N(u, G)$. Therefore there is $z \in N(u, G)$ such that $G_u - \{v, z\}$ is connected.

(**p2.2.3**) Now suppose that G_u is 3–connected. Then B_i is an end–brick of G and B_i is a triangle. Thus $G = aubBa$ where $B = G - u$ is 3–connected and $v \in B - \{a, b\}$. Since G is claw- and net-free, $N(p, B - b)$ induces a complete subgraph Q in $B - b$. Since B is 3–connected, $v(Q) \geq 2$. By **5.6**, there is an edge e in Q such that $G - b$ has a Hamiltonian av–path $uPxyPv$ containing $e = xy$. Then $uPxbyPv$ is a Hamiltonian uv–path in G.

Using **3.7** and **5.11**, it is easy to prove the following:

5.12 *Let $G = B_1 x_1 \dots x_{k-1} B_k$ be a chain, G claw- and net-free, and p_1, p_2, q_1, q_2 different vertices in G. Suppose that*

$(h1)$ $p_1 \in In(B_1)$, $q_1 \in In(B_k)$, $p_2 \in B_i$, $q_2 \in B_j$, $i \leq j$, and

$(h2)$ $\beta_G(p_2, q_2) \leq 1$ or $\beta_G(p_2, q_2) = 2$ and there is a β_G–shortest $p_2 q_2$–path containing a block–edge of G.

Then G has two disjoint paths $p_1 P p_2$ and $q_1 Q q_2$ such that $V(P \cup Q) = V(G)$.

Let G be a 2–connected graph.

Let $W_1(G)$ denote the set of pairs $\{x, y\}$ of different vertices in G such that $G - \{x, y\}$ is not connected.

Let $W_2(G)$ denote the set of pairs $\{x, y\}$ of different vertices in G such that $y = z_0$ and $x \in V(Z_2 - \{z_1, z_2\})$ where $G = z_0 Z_1 z_1 Z_2 z_2 Z_3 z_0$ and each $v(Z_i) \geq 3$.

Let $W_3(G)$ denote the set of pairs $\{x, y\}$ of different vertices in G such that $x \in V(Z_1 - \{z_0, z_1\})$ and $y \in V(Z_3 - \{z_2, z_3\})$ where $G = z_0 Z_1 z_1 Z_2 z_2 Z_3 z_3 Z_4 z_0$ and each $v(Z_i) \geq 3$.

Let $W(G) = W_1(G) \cup W_2(G) \cup W_3(G)$.

Obviously

5.13 *Let G be a connected graph. If $\{x, y\} \in W(G)$ then G has no Hamiltonian xy–path.*

Obviously theorem **5.11** can be reformulated as follows. This reformulation shows that the converse of **5.13** is true for a claw- and net-free graph that is either a ring or a 2–string.

5.14 *Let G be a claw- and net-free graph–ring and $c(G) = 2$. Let $u, v \in V(G)$ and $u \neq v$. Then the following are equivalent:*

$(a1)$ *G has a Hamiltonian uv–path and*

$(a2)$ *$\{u, v\} \notin W(G)$, namely:*

$(a2.1)$ *if G is a ring then $\{u, v\} \notin W(G)$ and*

$(a2.2)$ *if G is a 2–string then $\{u, v\} \notin W_1(G)$.*

It turns out that the converse of **5.13** is true for any 2–connected, claw- and net-free graph.

5.15 *Let G be a 2–connected claw- and net-free graph and $x, y \in V(G)$, $x \neq y$. Then G has a Hamiltonian xy–path if and only if $\{x, y\} \notin W(G)$.*

Proof (uses **4.1**, **4.2**, **4.13**, and **5.11**). We know that if G has a Hamiltonian xy–path then $\{x, y\} \notin W(G)$.

We prove by induction on $v(G)$ that if $\{x, y\} \notin W(G)$ then G has a Hamiltonian xy–path. Our claim is obviously true if G is a cycle. So we assume that $v(G) \geq 4$ and G is not a cycle. Let $G_x = G - x$. Since G is 2–connected and claw- and net-free, clearly G is connected and claw- and net-free, respectively. Obviously if P is a Hamiltonian yz–path for some vertex z in G_x adjacent to x then $xzPy$ is a Hamiltonian xy–path in G. Thus, we assume that G_x has no Hamiltonian yz–path for every $z \in N(x, G)$. Then, by the induction hypothesis, $yz \in W(G_x)$ for every $z \in N(x, G)$.

Suppose that G_x is not 2–connected. Then $c(G) = 2$, and so by **4.13**, G is either a ring or a 2–string. By **5.11**, $\{x, y\} \notin W(G)$ implies that G has an xy–path.

Now we suppose that G_x is 2–connected. Since G is connected, there is $z \in N(x, G)$. Since $yz \in W(G_x)$, clearly $c(G_x) = 2$, and so by **4.13**, G_x is either a ring or a 2–string.

(**p1**) Suppose that G_x is a ring, and so G_x has at least three bricks.

Let $z \in N(x, G)$. Since $\{y, z\} \in W(G_x)$, by **5.11** $(a2.1)$, $\beta(y, z) \geq 2$ and if $\beta(y, z) = 2$ then no β–shortest yz–path contains a brick–edge of G_x (here $\beta(y, z)$ is the β–distance between y and z in the ring G_x). Then there exists a minimal connected subgraph C of G_x such that $N(x, G) \subseteq V(C)$ and each block of C is a brick of G_x. (possibly, C consists of one vertex). Obviously such a subgraph C is unique, $Bnd(C) \subseteq S(G_x)$, and if $v(C) \geq 2$ then $|Bnd(C)| = 2$, say $Bnd(C) = \{z_1, z_2\}$.

(**p1**.**1**) Suppose that $xy \in E(G)$. Then $G_x - C = Fx_0x$, and so $G_x - C$ has three end–blocks. Obviously $G_x - C$ is an induced subgraph of G. By **4.2**, $G_x - C$ has at most two end–blocks, a contradiction.

(**p1.2**) Suppose that $v(C) = 1$, i.e. C consists of one vertex, say c. Then G_x has exactly three bricks each having at least three vertices, $c \in S(G_x)$, y is an inner vertex of a brick B avoiding c, and $N(x, G) = \{c, x\}$. Then $G_x - In(B)$ has three end–blocks where $In(B)$ is the set of inner vertices of B. Obviously $G_x - In(B)$ is an induced subgraph of G. By **4.2**, $G_x - In(B)$ has at most two end–blocks, a contradiction.

(**p1.3**) Now suppose that $xy \notin E(G)$ and $v(C) \geq 2$.

Let Z denote the subgraph of G induced by $C \cup x$. Since $xy \notin E(G)$, clearly $Bnd(Z) = Bnd(C) = \{z_1, z_2\}$. Because of minimality of C, every end–block of C has a vertex adjacent to x. Since G is 2–connected, $|N(x, G)| \geq 2$. Since $N(x, G) \subseteq V(C)$, clearly Z is 2–connected.

Suppose that $y \in S(G_x)$. Obviously there is a unique minimal connected subgraph Z_i of the ring G_x such that $\{y, z_i\} \subseteq V(Z_i)$ and each block of Z_i is a brick of G_x, $i \in \{1, 2\}$. It follows from the properties of C that each Z_i is either 2–connected or a chain. Thus $G = z_1 Z z_2 Z_2 y Z_1 z_1$ where $x \in In(Z)$, and each of Z, Z_1, Z_2 has at least two vertices. Therefore $\{x, y\} \in W_2(G)$, a contradiction.

Now suppose that y is an inner vertex of a brick Z' in G_x. Obviously there is a unique minimal connected subgraph Z_i of the ring G_x such that $z_i \in V(Z_i)$, $|V(Z' \cap V(Z_i)| = 1$ and each block of Z_i is a brick of G_x, $i \in \{1, 2\}$. It follows from the properties of C that each Z_i is either 2–connected or a chain. Let $V(Z' \cap V(Z_i) = z'_i$. Thus $G = z_1 Z z_2 Z_2 z'_2 Z' z'_1 Z_1 z_1$ where $x \in In(Z)$, and each of Z, Z', Z_1, Z_2 has at least two vertices. Therefore $\{x, y\} \in W_3(G)$, a contradiction.

(**p2**) Now suppose that G_x is a 2–string. Then $\{a, b\} \in W(G_x)$ if and only if $\{a, b\} \in W_1(G_x)$, i.e. if and only if $\{a, b\}$ is a vertex 2–cut of G_x. Therefore $N(x, G) = \{y, z\}$ where $\{y, z\}$ is a vertex 2–cut of G_x. Then $G - \{y, z\}$ has three components and $\{y, z\}$ is a minimal vertex cut in G. Since G is claw- and net-free, clearly $G - \{y, z\}$ is also claw- and net-free. By **4.1**, $G - \{y, z\}$ has at most two components, a contradiction.

From **5.15** we have:

5.16 *Let G be a 3–connected claw- and net-free graph and L a 3–path in G. Then the following are equivalent:*

$(a1)$ *L belongs to a Hamiltonian cycle in G and*

$(a2)$ *$G - L$ is connected.*

Proof (uses **4.13** and **5.15**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$. We assume that $G - L$ is connected. Let $L = x_1 x_2 x_3$ and $G' = G - x_2$. Since G is 3–connected, obviously G' is 2–connected.

If G' has a Hamiltonian $x_1 x_3$–path P, then $x_1 P x_3 x_2 x_1$ is a Hamiltonian cycle of G containing L. So we assume that G' has no Hamiltonian $x_1 x_3$–path. Then by **5.15**, $\{x_1, x_3\} \in W(G')$, and so $c(G) = 2$. By **4.13**, G' is either a ring or a 2–string.

Suppose that G' is a ring. Since G is 3–connected and G has no induced claw centered at x_2, $r(G') \leq 5$ and, moreover,

if $r(G') = 3$ then G' is either a triangle or has exactly two bricks–hammocks,

if $r(G') = 4$ then G' has at most two bricks–hammocks and these hammocks are disjoint, and
if $r(G') = 5$ then G' is a 5-cycle.

Therefore $W_1(G') = W(G')$. Then $\{x_1, x_3\} \in W_1(G')$, and so $G - L$ is not connected, a contradiction.

Now suppose that G' is a 2–string. Then again $\{x_1, x_3\} \in W_1(G')$, and so $G - L$ is not connected, a contradiction.

From **5.16** we can obtain the following.

5.17 *Let G be a 3–connected claw- and net-free graph and L a 3–path in G. Then G has a Hamiltonian path containing L.*

Proof (uses **4.13** and **5.16**). Let $L = x_1x_3x_2$ be a 3–path in G and $G' = G - x_3$. Since G is 3–connected, obviously G' is 2–connected. If P belongs to a Hamiltonian cycle in G then our claim is obviously true. So suppose that P belongs to no Hamiltonian cycle in G. Then by **5.16**, $G' - \{x_1, x_2\}$ is not connected. Therefore $G' - x_i$ is a chain and x_j is a vertex cut of $G' - x_i$ where $\{i, j\} = \{1, 2\}$. By **4.13**, $G' - x_i$ has a Hamiltonian path $P_i = z_iP_ix_iP_iz'_i$ where $z_i \in G_i - \{x_1, x_2\}$, $i \in \{1, 2\}$. Then $z_1P_1x_1x_3x_2P_2x_2z_2$ is a Hamiltonian path of G containing L.

It turns out that for 3-connected claw- and net-free graphs the following strengthening of **3.3** holds.

5.18 *Let G be a 3–connected claw- and net-free graph and $x, y \in V(G)$, $x \neq y$. Then*

$(a1)$ *G has a Hamiltonian $\{x, y\}$–path* (*and so G is Hamiltonian connected*).

$(a2)$ *If $e = xz \in E(G)$ and $y \neq z$ then G has a Hamiltonian $\{x, y\}$–path containing e if and only if $\{x, y, z\}$ is not a 3–cut of G.*

Proof (uses **4.13** and **5.11**). As we mentioned above, if G has a Hamiltonian xy path containing xz then $\{x, y, z\}$ is not a 3–cut of G.

We prove by induction on $v(G)$ that if $\{x, y, z\}$ is not a 3–cut of G then G has a Hamiltonian xy path. It is easy to check that K_4 is a smallest 3–connected graph and K_4 is claw- and net-free and obviously, our claim is true for K_4. So we assume that $v(G) \geq 5$. Let $G_x = G - x$. Then $y, z \in V(G_x)$ and $y \neq z$. Since G is 3–connected and claw- and net-free, clearly G_x is 2–connected and claw- and net-free. If G_x has a Hamiltonian yz–path P then $xzPy$ is a Hamiltonian xy–path in G containing xz. Therefore it is sufficient to show that G_x has a Hamiltonian yz–path.

If G_x is 3–connected then by the induction hypothesis, G_x has a Hamiltonian yz–path. Therefore we assume that G_x is not 3–connected, and so $c(G_x) = 2$. By **4.13**, G_x is either a ring or a 2–string.

(**p1**) Suppose that G_x is a ring, and so $r(G) \geq 3$. Since G is 3–connected and has no induced claw centered at x, $r(G_x) = 3$ and G has exactly two hammocks. Therefore $G_x = x_0B_1x_1B_2x_2x_0$ and $N(x, G) \subseteq V(B_1 \cup V(B_2)$. By **5.11**, G_x has no Hamiltonian yz–path if and only if $\{y,z\} = \{x_1, v\}$ where $v \in \{x_0, x_2\}$. But then $\{x,y,z\}$ is a vertex 3–cut of G, a contradiction.

(**p2**) Now suppose that G_x is a 2–string. By **5.11**, G_x has no Hamiltonian yz–path only if $\{y,z\}$ is a vertex 2–cut of G_x. Then $\{x,y,z\}$ is a vertex 3–cut of G, a contradiction.

Let $T(G)$ denote the set of pairs (x, uv) such that $G = aAbBa$, where A and B are hammocks of G, $x \in V(A - \{a,b\})$, $uv \in E(B)$, and a, b belong to different components of $B - \{u,v\}$. Obviously $(x,uv) \in T(G)$ if and only if $\{x,u\} \in W_2(G-v)$ and $\{x,v\} \in W_2(G-u)$.

5.19 *Let $G = B_1 X_1 \ldots X_{k-1} B_k$ be a claw- and net-free 2–string where $X_i = \{x_i, x_i'\}$. Suppose that $(G; s,t,uv)$ satisfies the following condition:*

(h) $e = uv \in E(B_r)$, $s \in V(B_p - X_p)$, $t \in V(B_q - X_{q-1})$, and $p < r \le q$.

Then $(a1)$ and $(a2)$ are equivalent where

$(a1)$ exactly one of the following two claims holds:

$(a1.1)$ if $p < r-1$ or $p = r-1$ and $\bar{B}_p$ is 3–connected, then G has a Hamiltonian st–path containing e,

$(a1.2)$ if $p = r-1$ and $\bar{B}_p$ is a square then G has a Hamiltonian zt–path containing e for some $z \in X_{r-2}$ and

$(a2)$ $(G; s,t,uv)$ satisfies one of the following conditions:

$(a2.1)$ $\{s,t\} \cap \{u,v\} = \emptyset$,

$(a2.2)$ $\{s,t\} \setminus \{u,v\} = z \in V(G)$, $G - \{z,u,v\}$ is connected and $\{z, uv\} \notin T(G)$.

Proof (uses **3.5**, **3.6**, **4.13**, **5.2**, **5.5**, **5.6**, and **5.11**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$. If G is a diamond then our claim is obviously true. Therefore let $v(G) \ge 5$. Let $G' = G - s$ and $B_p' = B_p - s$. Since G is claw- and net-free, G' is also claw- and net-free.

Assume that $s \in \{u,v\}$, say $s = u$, and so $t \notin \{u,v\}$. By $(a2)$, $G' - \{t,u\}$ is connected. Hence by **5.11**, G' has a Hamiltonian vt–path P. Then $uvPt$ is a Hamiltonian uv–path in G containing $e = uv$.

Therefore we assume that $\{s,t\} \cap \{u,v\} = \emptyset$. Then it is sufficient to show that there is $ss' \in E(G)$ such that G' has a Hamiltonian $s't$–path containing e.

(**p1**) Suppose that $k = 2$, and so $G = x_1 B_1 x_2 B_2 x_1$, each $\bar{B}_i = B_i \cup x_0 x_2$ is either a triangle or 3–connected. Let $X = \{x_1, x_2\}$, $s \in B_1 - X$, and $t \in B_2 - X$.

Assume that $\{u,v\} = \{x_1, x_2\}$. Since $\bar{B}_2$ is 2–connected, there is $b_2 \in V(B_2 - X)$ such that $b_2 u \in E(G)$. Since G is claw- and net-free, $(B_1 \cup b_2 u) - v$ is also claw- and net-free. Since $\bar{B}_1$ is either a triangle or 3–connected, by **3.5**, $B_1 - v$ has a Hamiltonian su–path P_1. Similarly $B_2 - u$ has a Hamiltonian tv–path P_2. Then $P = sP_1 uv P_2 t$ is a Hamiltonian st–path in G.

Therefore we assume that $\{u,v\} \ne \{x_1, x_2\}$. We can also assume that $e = uv \in E(B_1)$. Since s is an inner vertex of B_1, clearly $v(B_1) \ge 4$ and $\bar{B}_1$ is 3–connected.

(**p1.1**) Suppose that $\bar{B}_2$ is 3–connected. Then $B_2 - x_i$ is 2–connected for every $i \in \{0, 1\}$. Since each B_i is 2–connected, there are $b_0, b_1 \in V(B_1 - \{x_1, x_2\})$ and $b_2 \in V(B_2 - \{x_1, x_2\})$ such that $b_0x_1, b_1x_2, b_2x_2 \in E(G)$. Since G is claw- and net-free, clearly $(B_1 - x_1) \cup x_2b_2)$, $(B_2 - x_1) \cup x_2b_1)$, and $(B_2 - x_2) \cup x_1b_0)$ are also claw- and net-free.

If $B_1 - \{u, v\}$ is not connected then since $\bar{B}_1$ is 3–connected, $B_1 - \{u, v\}$ has exactly two components C_1, C_2 and $x_1 \in C_1$, $x_2 \in C_2$. We can assume that $s \in V(C_1)$. Then $uv \notin Z(B_1 - x_1, sx_2)$.

If $B_1 - \{u, v\}$ is connected then $(B_1 - x_1) - \{u, v\}$ is also connected because $N(x_1, B_1)$ induces a complete subgraph in B_1. Therefore again $uv \notin Z(B_1 - x_1, sx_2)$.

Thus in both cases by **5.2**, $B_1 - x_1$ has a Hamiltonian sx_2–path P_1 containing uv. Since $(B_2 - x_2) \cup x_1b_0$ is claw- and net-free, $N(x_1, B_2 - x_2)$ induces a complete subgraph in $B_2 - x_2$. Since $\bar{B}_2$ is 3–connected, by **5.6** (applied to $B_2 \cup x_2b_1$), B_2 has a Hamiltonian tx_1–path P_2. Then $sP_1x_2P_2t$ is a Hamiltonian st–path in G containing e.

(**p1.2**) Now suppose that $\bar{B}_2$ is a triangle, and so $V(B_2) = \{t, x_1, x_2\}$. Since $v(G) \geq 5$, clearly $v(B_1) \geq 4$, and so $\bar{B}_1$ is 3–connected. If $x_1x_2 \in E(G)$ then $sP_1x_2x_1t$ is a Hamiltonian st–path in G containing e where P_1 is the path described in (**p1**). So we can assume that $x_1x_2 \notin E(G)$. Let $B'_1 = B_1 - s$ and, as above, $G' = G - s$. Since $B_1 \cup x_1x_2$ is 3–connected, $B'_1 - \{u, v\}$ has a vertex s' adjacent to s and G' is 2–connected.

(**p1.2.1**) Suppose that $B'_1 \cup x_1x_2$ is 3–connected.

If $s' \notin X$ then by the induction hypothesis, G' has a Hamiltonian $s't$–path P containing uv. Then $ss'Pt$ is a Hamiltonian st–path in G containing uv. Therefore let $s' \in X$, say $s' = x_1$.

If $N(s', B'_1) = \{u, v\}$ then by **5.5**, $B'_1 - \{s', u, v\}$ is connected. By **5.2**, $G' - s'$ has a Hamiltonian vt–path P containing uv. Then $ss'uvPt$ a Hamiltonian st–path in G containing uv.

If $N(s', B'_1) \neq \{u, v\}$ then there is $s'' \in N(s', B'_1) \setminus \{u, v\}$. By the arguments, similar to that in (**p1**), $uv \notin Z(B_1 - x_1, s''x_2)$. Therefore again by **5.2**, $G' - s'$ has a Hamiltonian $s''t$–path P containing e. Then $ss's''Pt$ a Hamiltonian st–path in G containing e.

(**p1.2.2**) Now suppose that $B'_1 \cup x_1x_2$ is not 3–connected. Since G' is 2–connected and $\{x_1, x_2\}$ is a 2–cut in G', clearly $c(G') = 2$. By **4.13**, G' is either a 2–string with at least three bricks or a ring.

(**p1.2.2.1**) Suppose that G' is a ring. Since $B_1 \cup x_1x_2$ is 3–connected and G has no induced claw centered at s, clearly $G' = x_1D_1xD_2x_2tx_1$, where each $D_i \cup xx_i$ is either a triangle or 3–connected, and there is $d_i \in V(D_i - \{x, x_i\})$ such that $sd_i \in E(G)$, $i \in \{1, 2\}$. We can assume that $e \in E(D_2)$.

Suppose that $\{u, v\} \neq \{x, x_2\}$. Then $G' - t$ is a chain with two hammocks. By **3.6**, $G' - t$ has a Hamiltonian d_1x_2 path P containing e. Then sd_1Px_2t is a Hamiltonian st–path in G containing e.

Suppose that $\{u, v\} = \{x, x_2\}$. Since $D_2 - x$ is 2–connected, by **3.5**, $D_2 - x$ has a Hamiltonian d_2x_2–path P_2. By the same reason, D_1 has a Hamiltonian xx_1–path P_1. Then $sd_2x_2xP_1x_1t$ is a Hamiltonian st–path in G containing $e = xx_2$.

(**p1.2.2.2**) Now suppose that G' is a 2–string with at least three bricks. Since G has no induced claw centered at s, clearly B_1' is a 2–string D_1YD_2, where $Y = \{y_1, y_2\}$ and $X \subseteq V(D_1)$. Since $B_1 \cup x_1x_2$ is 3–connected, there is $d_2 \in D_2 - \{y_1, y_2\}$ such that $sd_2 \in E(G)$. Then by the induction hypothesis, G' has a Hamiltonian d_2t–path P containing e. Then sd_2Pt is a Hamiltonian st–path in G containing e.

(**p2**) Now suppose that $k \geq 3$. We know that $s \in V(B_p - X_p)$. We want to find a vertex s' adjacent to s in G such that G' has a Hamiltonian $s't$–path containing e.

(**p2.1**) Suppose that $p = r-1$ and $\bar{B}_{r-1}$ is a triangle or a square, say $V(B_1) = \{s, y, x_{r-1}, x'_{r-1}\}$, and $sy, sx_{r-1} \in E(G)$ and either $yx'_{r-1} \in E(G)$ or , $y = x'_{r-1}$. We define s' as follows.

Suppose that the vertices of $\{x_{r-1}, x'_{r-1}\} \setminus \{u, v\}$ belong to the same component of $B_r - \{u, v\}$. Then let $s' = y$.

Now suppose that the vertices of $\{x_{r-1}, x'_{r-1}\} \setminus \{u, v\}$ do not belong to the same component of $B_r - \{u, v\}$ (and so $B_r - \{u, v\}$ is not connected). Then $\bar{B}_r$ is 3–connected, $\{x_{r-1}, x'_{r-1}\} \cap \{u, v\} = \emptyset$, and x_{r-1} and x'_{r-1} belong to different components of $B_r - \{u, v\}$.

Let $G'' = G - \{s, y\}$. Since $\bar{B}_r$ is 3–connected, $B_r - \{u, v\}$ has exactly two components, and so $G'' - \{u, v\}$ has at most two components. If $G'' - \{u, v\}$ is connected, then let $s' = y$. If $G'' - \{u, v\}$ is not connected and t is in the component containing x_{r-1}, then let $s' = y$. If $G'' - \{u, v\}$ is not connected and t is in the component avoiding x_{r-1}, then let $s' := s$ and $s := y$.

Obviously $G - s$ is a chain (with two blocks) and $uv \notin Z(G - s, s't)$. By **5.2** $G - s$ has a Hamiltonian $s't$–path P containing uv. Then $zs'Pt$ is a Hamiltonian st–path in G containing uv where $\{s', z\} = \{s, y\}$, and so either $(a1.1)$ or $(a1.2)$ holds.

Now we assume that if $p = r - 1$ then $\bar{B}_r$ is 3–connected.

(**p2.2**) Suppose that $p = 1$. If $r = 2$ then by (**p2.1**), our claim is true. Therefore we assume that $r \geq 3$. We define s' as follows.

Suppose that B_1 is a triangle. Then $V(B_1) = \{s, x_1, x_1'\} \cap X_1$. Since $k \geq 3$, clearly G' is also a 2–string. Since G is claw-free, s is adjacent to a vertex $s' \in V(B_2 - X_2)$.

Suppose that $\bar{B}_1$ is a square, say $V(B_1) = \{s, z, x_1, x_1'\}$, say $sx_1, zx_1' \in E(G)$, and so $sz \in E(G)$. Then let $s' = z$.

Suppose that $\bar{B}_1$ is 3–connected. Then obviously $B_1 - X_1$ has a vertex s' adjacent to s.

We will show that in all these cases G' has a Hamiltonian $s't$–path containing e.

If $\bar{B}_1$ is not a square, then $G' = G - s$ is a 2–string because $k \geq 3$. Obviously (G', s', t, e) satisfies (h) and $\{s', t\} \neq \{u, v\}$. Suppose that $\{s', t\} \setminus \{u, v\} = a \in V(G)$. Since G' is a 2–string, $G' - \{a, u, v\}$ is connected. Since $(s, e) \notin T(G)$, also $(s', uv) \notin T(G')$. Therefore (G', s', t, e) satisfies $(a2)$. By the induction hypothesis, G' has a Hamiltonian $s't$–path containing e, and so $(a2.1)$ holds.

If $\bar{B}_1$ is a square, then let s'' be the vertex in X_1 adjacent to z. By the arguments, similar to that above, (G', s'', t, e) satisfies (h) and $(a2)$. By the induction hypothesis, G' has a Hamiltonian $s''t$–path P containing e. Then $s's''Pt$ is a Hamiltonian $s't$–path in G' containing e.

(**p2.3**) Now suppose that $p \geq 2$, and so $\bar{B}_p$ is an inner brick of G.

Suppose that $s \in X_{p-1}$ then $G' = G - s$ is a chain with two blocks and there is a vertex s' in $B_{p-1} - X_{p-1}$ adjacent to s. Then $uv \in Z(G', st)$. By **5.2**, G' has a Hamiltonian $s't$–path in G' containing e.

Suppose that $s \notin X_{p-1}$, and so s is an inner vertex of B_p. Then $\bar{B}_p$ is 3–connected.

(**p2.3.1**) Suppose that G' is a ring. Then the 2–string G has exactly three bricks, i.e. $G = B_1X_1B_2X_2B_3$, $V(B_2) = Bnd(B_2) \cup s = X_1 \cup X_2 \cup s$, and $N(s, G) = Bnd(B_2)$. If $|Bnd(B_2)| = 3$, say $x'_1 = x'_2$, then $G' = x'_1B_1x_1x_2B_3x'_1$, and if $|Bnd(B_2)| = 4$, then $G' = x'_1B_1x_1x_2B_3x'_2x'_1$ where $X_i = \{x_i, x'_i\}$, $i \in \{1, 2\}$. We know that $e = uv \in E(B_3)$, $t \in In(B_3)$, and $t \notin \{u, v\}$.

Suppose that $\{u, v\} = X_2$, say $v = x_2$. By **5.6**, $B_3 - u$ has a Hamiltonian tv–path P_3. By **3.7**, $G - (B_3 - u)$ has a Hamiltonian x_1u–path P_1. Then $sx_1P_1uvP_3t$ is a Hamiltonian st–path in G containing $e = uv$.

Suppose that $\{u, v\} \neq X_2$ and $B_3 - \{u, v\}$ is connected. Then $uv \notin Z(B_3, tx'_2)$. By **5.2**, B_3 has a Hamiltonian tx'_2–path P_3 containing $e = uv$. By **3.7**, B_1 has a Hamiltonian $x_1x'_2$–path P_1. Then $sx_1P_1x'_2P_3t$ is a Hamiltonian st–path in G containing $e = uv$.

Now suppose that $\{u, v\} \neq X_2$ and $B_3 - \{u, v\}$ is not connected. Since $\bar{B}_3$ is 3–connected, $B_3 - \{u, v\}$ is not connected implies that no component of $B_3 - \{u, v\}$ avoids X_2.

Suppose that t belongs in $B_3 - \{u, v\}$ to the component containing x_2. Then $uv \notin Z(B_3, tx'_2)$. By **5.2**, B_3 has a Hamiltonian tx'_2–path P_3 containing $e = uv$. By **3.7**, B_1 has a Hamiltonian $x_1x'_2$–path P_1. Then $sx_1P_1x'_2P_3t$ is a Hamiltonian st–path in G containing $e = uv$.

Now suppose that t belongs in $B_3 - \{u, v\}$ to the component containing x'_2. If $x'_1 \neq x'_2$ then this case is similar to that in the previous paragraph. So we assume that $x'_1 = x'_2 = x'$. Since $\bar{B}_3$ is 3–connected, clearly $B_3 - x'$ is 2–connected and t belongs to the component of $B_3 - \{u, v, x'\}$ avoiding x_2, and so $uv \notin Z(B_3 - x', tx_2)$. By **5.2** (applied to $x_1x_2(B_3 - x')$) $B_3 - x'$ has a Hamiltonian tx_2–path P_3 containing $e = uv$. By **3.7**, $G - (B_3 - x'_2)$ has a Hamiltonian $x_2x'_1$–path P_1. Then $sx'_1P_1x_2P_3t$ is a Hamiltonian st–path in G containing $e = uv$.

(**p2.3.2**) Suppose that G' is a 2–string. Since $\bar{B}_p$ is 3–connected, there is a vertex s' in $B_p - X_p$ adjacent to s. Then $\{s', t\} \cap \{u, v\} = \emptyset$, and so $(G'; s', t, uv)$ satisfies $(a2.1)$. By the induction hypothesis, G' has a Hamiltonian $s't$–path in G' containing e.

5.20 *Let G be a 3–connected claw- and net-free graph, $e = uv \in E(G)$, and $s, t \in V(G)$, $s \neq t$. Then the following are equivalent:*

$(a1)$ *G has a Hamiltonian st–path containing e and*

$(a2)$ *either $\{s, t\} \cap \{u, v\} = \emptyset$ or $\{s, t\} \setminus \{u, v\} = z \in V(G)$ and $G - \{z, u, v\}$ is connected.*

Proof (uses **3.7**, **4.13**, **5.2**, **5.5**, and **5.11**). Obviously, $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$ by induction on $v(G)$. If $v(G) \leq 4$ then our claim is obviously true. So we assume that $v(G) \geq 5$. By $(a2)$, we can assume that $s \notin \{u, v\}$. Let $G' = G - s$. Since G is claw- and net-free, G' is also claw- and net-free. It is sufficient to show that there is a vertex s'

adjacent to s in G such that G' has a Hamiltonian $s't$–path containing e. By **4.13**, G' is either 3–connected or a ring or a 2–string.

(**p1**) Suppose that $G' = B_1X_1 \dots X_{k-1}B_k$ is a 2–string and $uv \in E(B_r)$. We can assume that $t \in V(B_q - X_{q-1})$ for $1 < r \le q$. Since G is 3–connected, s is adjacent to an inner vertex of each end–brick B of G' and if B is a square then s is adjacent to both vertices of degree two in B.

If $B_r - \{u, v\}$ is not connected then $\bar{B}_r$ is 3–connected, and so $B_r - \{u, v\}$ has exactly two components C, C' and $x_r \in C$, $x'_r \in C'$.

(**p1.1**) Suppose that $r = k$ (and so $q = k$), $t \in \{u, v\}$, say $t = u$, and $B_k - \{u, v\}$ is not connected. Then $\bar{B}_k$ is 3–connected. Therefore x_k and x'_k belong to different components, say C and C', of $B_k - \{u, v\}$, respectively. If $N(s, B_k1) \subseteq \{u, v\}$ then each vertex in $N(s, B_1)$ is the center of an induced claw in G, a contradiction. Therefore there is a vertex $s' \in V(B_1 - \{u, v\})$ adjacent to s. We can assume that $s' \in C - x_k$. Obviously $G' - u$ is a ring. Let D be the subgraph of B_k induced by $C \cup v$. If D is a chain then, since $\bar{B}_k$ is 3–connected, t is the center of an induced claw in B_k, a contradiction. Therefore $D \cup vx_k$ is a brick of the ring $G' - u$ $v \in D$, and s' is an inner vertex of D. By **5.11**, $G' - u$ has a Hamiltonian $s'v$–path P. Then $s'Pvt$ is a Hamiltonian $s't$–path in G' containing uv.

(**p1.2**) Suppose that $r = 2$ and $\bar{B}_1$ is a square. Since G is 3–connected, s is adjacent to both vertices in $B_1 - X_1$. By **5.11**, there is a vertex $s' \in B_1 - X_1$ such that G' has a Hamiltonian $s't$–path in G' containing uv.

(**p1.3**) Now suppose that the assumptions of neither (**p1.1**) nor (**p1.2**) holds. Let s' be an inner vertex of B_1 adjacent to s. By **5.11**, there is a vertex $s' \in B_1 - X_1$ such that G' has a Hamiltonian $s't$–path in G' containing uv.

(**p2**) Suppose that G' is a ring. Since G has no induced claw centered at s, $r(G') \le 5$.

(**p2.1**) Suppose that $r(G') = 3$. Since $v(G) \ge 5$, G' is not a triangle. Since G has no induced claw centered at s, clearly G' has exactly two hammocks, and so $G' = x_1B_1xB_2x_2x_1$ where each $\bar{B}_i$ is 3–connected. Since G is 3–connected, s is adjacent to an inner vertex b_i in B_i, $i \in \{1, 2\}$. We can assume that $t \in V(B_1)$.

If a, y, z are different vertices and e is an edge in B_i then let yP_iz denote a Hamiltonian yz–path in B_i, $yP_i(a)z$ a Hamiltonian yz–path in B_i, and $yP_i[e]z$ a Hamiltonian yz–path in B_i containing e. By **5.6**, B_i has a path $yP_i(a)z$ for $\{a, z\} = \{x, x_i\}$ and $y \notin \{x, x_i\}$. By **3.7**, B_i has a path yP_iz for $z \in \{x, x_i\}$ and $y \ne z$. If $e = uv$, $z \in \{x, x_i\}$, $y \ne z$, and either $B_i - \{u, v\}$ is connected or y, z are in different components of $B_i - \{u, v\}$, then $uv \notin Z(B_i, yz)$ and by **5.2**, B_i has a path $yP_i[e]z$.

(**p2.1.1**) Suppose that $\{u, v\} = \{x_1, x_2\}$. If $t \in \{x_1, x_2\}$, say $t = x_1$, then $sb_1P_1(x_1)xP_2x_2t$ is a Hamiltonian st–path containing uv. If $t \notin \{x_1, x_2\}$ then $sb_2P_2(x)x_2x_1P_1t$ is a Hamiltonian st–path containing uv.

(**p2.1.2**) Suppose that $uv \in E(B_1)$ and $\{u, v\} = \{x, x_1\}$. Then $G - \{u, v\}$ is not connected, and so if $t \in \{u, v\}$, then (G, uv, s, t) does not satisfy $(a2)$, a contradiction. Therefore

$t \notin \{u, v\}$. Then $tP_1(x_1)xx_1x_2P_2(x)s's$ is a Hamiltonian st–path containing uv.

(**p2.1.3**) Suppose that $uv \in E(B_1)$ and $\{u, v\} \neq \{x, x_1\}$.

(**p2.1.3.1**) Suppose that $B_1 - \{u, v\}$ is connected (which is the case if $\{u, v\} \cap \{x, x_1\} \neq \emptyset$ because $\bar{B}_1$ is 3–connected). If $x_1 \notin \{u, v\}$ then $sb_2P_2xP_1[e]t$ is a Hamiltonian st–path containing uv. If $x \notin \{u, v\}$ then $sb_2P_2(x)x_2x_1P_1[e]t$ is a Hamiltonian st–path containing uv.

(**p2.1.3.2**) Suppose that $uv \in E(B_1)$ and $B_1 - \{u, v\}$ is not connected. Since $\bar{B}_1$ is 3–connected, $B_1 - \{u, v\}$ has exactly two components C, C_1 and $x \in C$, $x' \in C'$.

Suppose that $t \in C$. Then t and x_1 are in different components of $B_1 - \{u, v\}$. Hence $sb_2P_2(x)x_2x_1P_1[e]t$ is a Hamiltonian st–path containing uv.

Suppose that $t \in C_1$. Then t and x are in different components of $B_1 - \{u, v\}$. Hence $sb_2P_2xP_1[e]t$ is a Hamiltonian st–path containing uv.

Suppose that $t \in \{u, v\}$, say $t = u$. Then $G' - t$ is a ring. If $N(s, B_1) \subseteq \{u, v\}$ or $N(s, B_1) = \{u, v, x\}$ and $C \neq x$ or $N(s, B_1) = \{u, v, x_1\}$ and $C_1 \neq x_1$ then u (and v) is the center of an induced claw in G, a contradiction. Therefore if $C \neq x$ and $C_1 \neq x_1$ then there is $b \in N(s, B_1) \setminus \{u, v, x, x_1\}$.

Suppose that $N(s, B_1) = \{u, v, x\}$ and $C = x$. Then vx is a brick of $G' - t$. By **5.11**, $G' - t\}$ has a Hamiltonian vx–path P. Then $sxPvt$ is a Hamiltonian st–path containing uv.

Suppose that $N(s, B_1) = \{u, v, x_1\}$ and $C_1 \neq x_1$. Then vx_1 is a brick of the ring $G' - t$. By **5.11**, $G - \{s, t\}$ has a Hamiltonian vx_1–path P. Then sx_1Pvt is a Hamiltonian st–path containing uv.

Now suppose that there is $b \in N(s, B_1) \setminus \{u, v, x, x_1\}$.

Suppose that $b \in C$. Let D be the subgraph of B_1 induced by $C \cup v$. If D is a chain then, since $\bar{B}_1$ is 3–connected, t is the center of an induced claw in B_1, a contradiction. Therefore $D \cup vx$ is a brick of the ring $G' - u$, $v \in D$, and b is an inner vertex of D. By **5.11**, $G' - u$ has a Hamiltonian bv–path P. Then $sbPvt$ is a Hamiltonian st–path in G containing uv.

The case $b \in C_1$ is similar.

(**p2.1.4**) Now suppose that $uv \in E(B_2)$ and $t \neq x$.

This case is similar the case (**p2.1.3**) if we put $t := b_2$ and $b_2 := t$.

(**p2.2**) Suppose that $r(G) \in \{4, 5\}$.

Since G has no induced claw centered at s, we have:
if $r(G') = 5$ then G' is a 5-cycle,
if $r(G') = 4$ then G' has at most two hammocks and these hammocks are disjoint.

If G' is a 4–cycle or 5–cycle, then our claim is obviously true. Therefore there are the following two cases to consider:

$G' = x_1B_1x_2x_3x_4x_1$ and $G' = x_1B_1x_2x_3B_3x_4x_1$,
where each $\bar{B}_i$ is either a triangle or 3–connected.

Now the arguments, similar to those in (**p2.1**), show that our claim is true.

(**p3**) Suppose that G' is 3–connected. If there is a vertex $s' \in N(s, G)$ such that (G', s', t, uv) satisfies $(a2)$, then by the induction hypothesis, G' has a Hamiltonian $s't$–path P containing

uv, and so $ss'Pt$ is Hamiltonian st–path in G containing uv. Therefore it is sufficient to show that there is a vertex $s' \in N(s, G)$ such that (G', s', t, uv) satisfies $(a2)$.

Since G is 3–connected, $N(s, G) - \{u, v\} \neq \emptyset$. If there is $y \in N(s, G) - \{u, v\}$ such that (G', y, t, uv) satisfies $(a2)$ then we are done. So suppose not. Then $t \in \{u, v\}$ and $G' - \{u, v, y\}$ is not connected for every $y \in N(s, G) - \{u, v\}$.

Since G' is claw-free and $\{u, v, y\}$ is a minimal vertex cut of G', by **4.1**, $G' - \{u, v, y\}$ has exactly two components, say A_1 and A_2. If $N(s, G) = \{u, v, y\}$ then $G - \{u, v, y\}$ has at least three components. Then by **4.1**, G has an induced claw, a contradiction. Therefore there is $a_1 \in V(A_1) \cap (N(s, G) \setminus \{u, v, y\})$, and so $a_1 \neq t$.

Since (G', a_1, t, uv) does not satisfy $(a2)$, $G' - \{a_1, u, v\}$ is not connected. Let B_1 be the component of $G' - \{a_1, u, v\}$ avoiding y. Since $a_1 \in A_1$, obviously B_1 is an induced subgraph of A_1.

Suppose that $G' - \{a_1, y, z\}$ is not connected for some $z \in \{u, v\}$, say for $z = u$. Let C be the component of $G' - \{a_1, y, u\}$ avoiding v. Then u is adjacent to vertices a, b, c in A_2, B_1, and C, respectively. Hence $\{u; a, b, c\}$ induces a claw in G centered at u, a contradiction. Therefore $G' - \{a_1, y, z\}$ is connected, and so $a_1 y \in E(G)$.

Suppose that there is $h \in N(s, G) \cap V(B_1)$. Since (G', h, t, uv) does not satisfy $(a2)$, $G' - \{h, u, v\}$ is not connected. Then $G' - \{h, y, u\}$ is also not connected, and so, as above, u is the center of an induced claw in G, a contradiction. Therefore $N(s, G) \cap V(A_1) = a_1$.

Suppose that $N(s, G) \cap V(A_2) \neq \emptyset$. Since $d(s, G) \geq 3$, there is $z \in N(s, G) \cap \{u, v\}$. Since $\{u, v, y\}$ is a minimal vertex cut in G', there is $z_i \in N(z, G) \cap V(A_i)$ for $i \in \{1, 2\}$. Then $\{z; , s, z_1, z_2\}$ induces a claw in G centered at z, a contradiction. Therefore there is $a_2 \in N(s, G) \cap V(A_2)$.

Now by the above arguments for (A_1, a_1), applied to (A_2, a_2), $a_2 y \in E(G)$ and $N(s, G) \cap V(A_2) = a_2$. Since G is 3–connected, $xy \in E(G)$ for some $x \in \{u, v\}$, say for $x = u$. Then $\{z; , x, z_1, z_2\}$ induces a claw in G centered in z, a contradiction.

6 On Hamiltonicity of k–connected cn-free graphs

In this section we give generalizations of some of the results in Section 5 for k–connected claw and net-free graphs.

6.1 *Let G be a k–connected graph, $k \geq 4$, C be a cycle in G, $v(C) \leq k - 1$, $e = uv \in E(G)$, and $s, t \in V(G)$, $s \neq t$. Then the following are equivalent:*

$(c1)$ G has a Hamiltonian st–path containing $C - e$ (and so, avoiding e) and

$(c2)$ (G, C, s, t, uv) satisfies exactly one of the following conditions:

$(c2.1)$ $\{s, t\} \cap V(C) = \emptyset$,

$(c2.2)$ $V(C) \cap \{s, t\} = x \in \{u, v\}$ and $G - (C \cup (\{s, t\} - z))$ is connected.

Proof (uses **5.20**). Obviously $(c1) \Rightarrow (c2)$. We prove $(c2) \Rightarrow (c1)$. Let $G' = G - (C - \{u, v\})$. Obviously G has a Hamiltonian st–path containing $C - e$ if and only if G' has a

Hamiltonian st–path containing $e = uv$. By **5.20**, G' has a Hamiltonian st–path containing $e = uv$ if and only if (G', s, t, uv) satisfies $(a2)$ in **5.20**. Obviously (G', s, t, uv) satisfies $(a2)$ if and only if (G, C, s, t, uv) satisfies $(c2)$.

It is easy to see that the following is true for every k–connected graph.

6.2 *Let G be a connected graph, L_1 and L_2 two disjoint paths in G, and x_1, x_2 the end–vertices of L_1, L_2, respectively. If $G - (L_1 \cup L_2)$ is not connected then G has no Hamiltonian x_1x_2–path containing L_1 and L_2.*

The following is a generalization of **5.18** for k–connected claw- and net-free graphs that shows, in particular, that the converse of **6.2** is also true if $v(L_1) + v(L_2) \leq k$.

6.3 *Let G be a k–connected graph, $k \geq 3$, L_1 and L_2 two disjoint paths in G, $v(L_1)+v(L_2) \leq k$, and x_1, x_2 the end-vertices of L_1, L_2, respectively. Then the following are equivalent:*

$(a1)$ G has a Hamiltonian x_1x_2–path containing L_1 and L_2,

$(a2)$ G has a Hamiltonian z_1z_2–path containing L_1 and L_2 for every end–vertices z_1, z_2 of L_1, L_2, respectively, and

$(a3)$ $G - (L_1 \cup L_2)$ is connected.

Proof (uses **5.18**). Obviously $(a1) \Leftrightarrow (a3)$ implies $(a1) \Leftrightarrow (a2)$. As we mentioned above, $(a1) \Rightarrow (a3)$. We prove $(a3) \Rightarrow (a1)$ by induction on k. Obviously it is sufficient to prove our claim when $v(L_1) + v(L_2) = k$. By **5.18**, our claim is true for $k = 3$. So we assume that $k \geq 4$ and, therefore, that $v(L_1) \geq 2$. Let $G_x = G - x$ and $x_1x_1' \in E(L_1)$. Since G is k–connected and claw- and net-free, clearly G_x is $(k-1)$–connected and claw- and net-free. By $(a2)$, $V(L_1 \cup L_2)$ is not a vertex cut of G. Therefore $V(L_1 \cup L_2) - x_1$ is not a vertex cut of G_x. By the induction hypothesis, G_x has a Hamiltonian $x_1'x_2$–path P containing $L_1 \cup L_2 - x_1$. Then $x_1x_1'Px_2$ is a Hamiltonian x_1x_2–path P containing $L_1 \cup L_2$.

Using **3.9** and **5.18**, it is easy to prove the following generalization of **5.16**.

6.4 *Let G be a k–connected graph, $k \geq 2$, L be a path in G, and $v(L) \leq k$. Then the following are equivalent:*

$(a1)$ G has a Hamiltonian cycle containing L and

$(a2)$ $G - L$ is connected.

6.5 *Let G be a connected claw- and net-free graph and L a path in G. Suppose that $V(L)$ is a minimal vertex cut of G. Then G has a Hamiltonian path containing L.*

Proof (uses **3.7** and **4.1**). Let x_1 and x_2 be the end–vertices of L. Since G is claw- and net-free, by **4.1**, $G - L$ has exactly two components, say C_1 and C_2. Let $G_i = G - (L - x_j)$ where $\{i, j\} = \{1, 2\}$. Since $V(L)$ is a minimal vertex cut of G, clearly each G_i is connected and x_i is a vertex cut of G_i. By **3.7**, G_i has a Hamiltonian path $c_1P_ic_2$ where $c_j \in C_j$. Then $c_1P_1x_1Lx_2P_2c_2$ is a Hamiltonian path in G containing L.

Now we can obtain the following generalization of **5.17** for k–connected graphs.

6.6 *Let G be a k–connected claw- and net-free graph, $k \geq 1$, and L be a path in G, and $v(L) \leq k$. Then G has a Hamiltonian path containing L.*

Proof (uses **3.7**, **4.1**, and **6.3**).

Suppose that $G - L$ is not connected. Since G is k–connected, clearly $v(L) = k$, and so $V(L)$ is a minimal vertex cut of G. Then by **6.5**, G has a Hamiltonian path containing L.

Now suppose that $G - L$ is connected. Then by **6.3**, L belongs to a Hamiltonian cycle of G, and therefore also to a Hamiltonian path of G.

There are infinitely many examples showing that the claim of **6.6** is not true if $v(L) = k + 1$.

The following is a strengthening of **6.4**.

6.7 *Let G be a k–connected claw- and net-free graph, $k \geq 3$, $e = uv \in E(G)$, and L a path in G with $v(L) \leq k - 1$. Then the following are equivalent:*

$(a1)$ *G has a Hamiltonian cycle containing e and L and*

$(a2)$ *the subgraph L' in G, induced by $E(L) \cup e$, is either a path or not connected* (*i.e. has two components L and uev*) *and if L' is a path then $G - L'$ is connected.*

Proof (**5.20** and **6.4**). Obviously $(a1) \Rightarrow (a2)$. We prove $(a2) \Rightarrow (a1)$. If L' is a path, then by **6.4**, $(a1)$ holds. Therefore we assume that L' is not connected.

Let x and y be the end–vertices of L and $G' = G - V(L - \{x, y\})$. Since $v(L) \leq k - 1$, clearly $|V(L) - \{x, y\}| = k - 3$ and G is k–connected, obviously G' is 3–connected. Clearly $\{u, v\} \cap V(L) = \emptyset$ implies $\{u, v\} \cap \{x, y\} = \emptyset$. Then by **5.20**, G' has a Hamiltonian xy–path P containing e. Therefore $P \cup L$ is a Hamiltonian cycle in G containing both e and L.

There are infinitely many $(k+1)$–connected claw- and net-free graphs G with two disjoint paths P and L such that $v(P)+v(L) = k+1$, $v(P) \geq 3$, $v(L) \geq 3$, and G has no Hamiltonian cycle containing $P \cup L$. Therefore the claims of **1.7** and **6.7** are not true if condition "*$e \in E(G)$ and L is a path in G with $v(L) \leq k - 1$*" is replaced by condition "*P and L are disjoint paths in G such that $v(P) + v(L) = k + 1$*".

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, MacMillan Co., New York, 1976.

[2] H.J. Broesma, On Some Intriguing Problems in Hamiltonian Graph Theory - a Survey, *Discrete Mathematics* **251** (2002) 47–69.

[3] D. Duffus, R.J. Gould, M.S. Jacobson, Forbidden Subgraphs and the Hamiltonian Theme, in *The Theory and Application of Graphs (Kalamazoo, Mich., 1980)*, Wiley, New York (1981) 297–316.

[4] R.J. Gould, Advances on the Hamiltonian Problems - a Survey, *Graphs and Combinatorics* **19** (2003) 7–52.

[5] A. Kelmans, On Graph Closures, *RUTCOR Research Report 11–99*, Rutgers University (1999) and *Discrete Mathematics* **271**, (2003) 141–168.

[6] A. Kelmans, Claw- and Net-Free Graphs, *RUTCOR Research Report 18–99*, Rutgers University (1999).

[7] A. Kelmans, On Hamiltonicity of Claw- and Net-Free Graphs, submitted.

[8] Z. Ryjaček, On a Closure Concept in Claw-Free Graphs, *J. Combinatorial Theory*, **B 70** (1997) 217–224.

[9] B. Shepherd, Hamiltonicity in Claw-Free Graphs, *J. Combinatorial Theory*, **B 53** (1991) 173–194.