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ON HAMILTONICITY OF CLAW- AND NET-FREE GRAPHS

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ABSTRACT

The results of this paper include necessary and sufficient conditions for a claw- and net-free graph G with $s, t \in V(G)$ and $e \in E(G)$ to have $(a1)$ a Hamiltonian path and Hamiltonian s–path, $(a2)$ a Hamiltonian st–path and Hamiltonian s- and st–paths containing e if G has connectivity one, and $(a3)$ a Hamiltonian cycle containing e if G is 2–connected. For example, we show that

if G is a 2–connected claw- and net-free graph and $e = xy \in E(G)$ then e belongs to a Hamiltonian cycle of G if and only if $G - \{x, y\}$ is connected.

These results imply that a connected claw- and net-free graph has a Hamiltonian path and a 2–connected claw- and net-free graph has a Hamiltonian cycle [3], and are used in [5] to obtain more general Hamiltonicity results for k–connected graphs. Our proofs of $(a1) - (a3)$ are shorter, than the proofs of their corollaries in [3], and provide polynomial time algorithms for solving the corresponding Hamiltonicity problems.

Keywords: claw, net, graph, Hamiltonian path, Hamiltonian cycle, polynomial time algorithm.

1 Introduction

We consider simple undirected graphs. All notions on graphs that are not defined here can be found in [1].

A *claw* is a graph having exactly three edges that are incident to a (the) common vertex. A claw looks like the letter Y. A *net* is a graph obtained from a triangle by attaching to each vertex a new dangling edge. A graph is called *claw- and net-free* if it has no induced subgraph isomorphic to either a claw or a net.

There are many papers devoted to the study of Hamiltonicity of claw-free graphs, and, in particular, claw- and net-free graphs (e.g. [3, 6] and survey papers [2, 4]).

In this paper we establish some new Hamiltonicity results on claw- and net-free graphs.

An *st–path* is a path with the end–vertices s and t. An *s–path* is a path with an end–vertex s. Let G be a claw- and net-free graph, $s, t \in V(G)$, $s \neq t$, and $e \in E(G)$.

The results of this paper include necessary and sufficient conditions for G to have:

($c1$) a Hamiltonian path and a Hamiltonian s–path,

($c2$) a Hamiltonian st–path if G has connectivity one (see **4.3** below)

($c3$) a Hamiltonian st–paths containing e if G has connectivity one (**4.7**),

($c4$) a Hamiltonian s–path containing e if G has connectivity one, (**4.8**), and

($c5$) a Hamiltonian cycle containing e if G is 2–connected (**4.11**).

From the above mentioned results we have the following corollaries.

1.1 [3] (Corollary of **4.3**) *Every connected claw- and net-free graph has a Hamiltonian path.*

1.2 [3] (Corollary of **4.11**) *Every 2–connected claw- and net-free graph has a Hamiltonian cycle.*

Our proofs of **4.3** and **4.11** are shorter then the proofs of their corollaries **1.1** and **1.2** in [3]. They also provide polynomial time algorithms for solving the corresponding Hamiltonian problems for claw- and net-free graphs.

Section 2 provides the main notions and notation. In Section 3 we prove the key lemma which is used to prove important results in this paper and in [5]. The main Hamiltonicity results and their proves are given in Section 4.

The results of this paper were presented at the Discrete Mathematics Seminar at the University of Puerto Rico in November 1998 (see also [6]).

2 Main notions and notation

We use the following notation:

$V(G)$ and $E(G)$ are the sets of vertices and edges of a graph G, respectively,
$v(G) = |V(G)|$ and $e(G) = |E(G)|$,
$cn(G)$ is the vertex connectivity of G, and
AvB is the union of two graphs A and B having exactly one vertex v in common.

A *uv–path* (*v–path*) is a path with the end–vertices u and v (an end–vertex v, respectively). A uv–path P is sometimes denoted by uPv. If a and b are vertices of P then aPb denote the subpath of P with the end–vertices a and b.

Let $X, Y \subseteq V(G)$. We say that G is *Hamiltonian* (X, Y)*–connected* if G has a Hamiltonian xy–path for every distinct vertices $x \in X$ and $y \in Y$. If G is Hamiltonian $(V(G), V(G))$–connected then G is called *Hamiltonian connected*.

Let H be a subgraph of G. A vertex x of H is called an *inner vertex of* H if x is adjacent to no vertices in $G - H$, and a *boundary vertex of* H, otherwise. An edge e of H is called an *inner edge of* H if e is incident to an inner vertex of H.

A *non-trivial* (*or 2–connected*) *block* of G is a maximal 2–connected subgraph of G. A *trivial block* of G is a subgraph of G induced by an edge–cut.

A block B of G is called an *end–block* of G if B has exactly one boundary vertex, and an *inner block*, otherwise.

We write $G = AXB$ if A and B are induced subgraphs of G, $V(A) \cap V(B) = X$, $E(A) \cap E(B) = \emptyset$, and so $E(A) \cup E(B) = E(G)$. If $X = \{x\}$, we write AxB instead of AXB. If B is an edge xy, we write simply Axy instead of AxB.

A graph G is called a *chain* if G is connected and has either exactly one edge or exactly two end–blocks.

3 The key lemma

First we observe the following.

3.1 *Let G be a graph. The following are equivalent:*

($c1$) *G has no induced subgraph isomorphic to a claw or a net and*

($c2$) *G has no connected induced subgraph with at least three end–blocks.*

Proof Obviously ($c2$) implies ($c1$). Clearly if G is claw- and net-free then $G - x$ is also claw- and net-free for every $x \in V(G)$. By using this observation, one can easily prove by induction on $v(G)$ that ($c1$) implies ($c2$).

From **3.1** we have:

3.2 *If G is a claw- and net-free graph and $cn(G) = 1$ then G is a chain.*

Our results on Hamiltonicity of claw- and net-free graph in this paper and in [5] are based essentially on the following key lemma.

3.3 *Let G be a connected claw- and net-free graph and $z \in V(G)$. Suppose that $G - z$ has a Hamiltonian xy–path P and there exists $e_z = zp \in E(G)$ (and so G is connected). Let e_x and e_y be the end-edges of P. Then G has a Hamiltonian ab–path Q such that $\{a, b\} \subset \{x, y, z\}$, $e_z \in E(Q)$ and $\{e_x, e_y\} \cap E(Q) \neq \emptyset$.*

Proof (uses **3.1**). For short, we will call a Hamiltonian xy–path in a graph an *xy–trace*. Let $A = \{e_x, e_y, e_z\}$.

(**p1**) If $p \in \{x, y\}$ then clearly G has an az–path where $a = \{x, y\} \setminus p$. Therefore let $p \notin \{x, y\}$. Let $xPy = xXpYy$ where pXx is a path $p = x_0x_1 \dots x_{k-1}x_k = x$ and pYy is a path $p = y_0y_1 \dots y_t = y$. Let $M_{r,s} = x_rPy_s$. Let $\dot{M}_{r,s}$ denote the subgraph of G induced by $V(M_{r,s})$ and $\bar{M}_{r,s} = \dot{M}_{r,s} \cup \{x_rx_{r+1}, y_sy_{s+1}, zp\}$.

A subpath $M_{r,s}$ is called *good* if

(x1) $\dot{M}_{r,s}$ has a py_s-trace containing $x_{r-1}x_r$,

(y1) $\dot{M}_{r,s}$ has a px_r-trace containing $y_{s-1}y_s$,

(x2) if $x_r \neq x$ then the graph $\dot{M}_{r,s} \cup \{x_rx_{r+1}, x_{r+1}v\}$, obtained from $\dot{M}_{r,s}$ by adding the edge x_rx_{r+1} and a new edge $x_{r+1}v$, has a py_s-trace containing the path $x_rx_{r+1}v$ for every $v \in V(M_{r,s}) \setminus x_r$,

(y2) if $y_s \neq y$ then the graph $\dot{M}_{r,s} \cup \{y_sy_{s+1}, y_{s+1}v\}$, obtained from $\dot{M}_{r,s}$ by adding the edge y_sy_{s+1} and a new edge $y_{s+1}v$, has a px_r-trace containing the path $y_sy_{s+1}v$ for every $v \in V(M_{r,s}) \setminus y_s$,

(z) the graph $\dot{M}_{r,s} \cup \{zp, zv\}$, obtained from $\dot{M}_{r,s}$ by adding the edge zp and a new edge zv, has an x_ry_s-trace for every $v \in V(M_{r,s}) \setminus p$ (which obviously contains $e_z = zp$ and zv), and

(**p2**) It is easy to see that $M_{1,1}$ is a good path.

Let $M_{r,s}$ be a maximal good path.

Suppose $x_r = x$. By **(x1)**, $\dot{M}_{r,s}$ has a py_s-trace pHy_s containing $x_{r-1}, x_r = e_x$. Then $zpHy_sPy$ is a yz-trace in G containing A. Similarly, if $y_s = y$ then G has an xz-trace containing A.

Now suppose that $x_r \neq x$ and $y_s \neq y$. Then the subgraph $\bar{M}_{r,s}$ of G has at least three end-blocks. Therefore by **3.1**, there exists an edge ab in G such that $a \in \{x_{r+1}, y_{s+1}, z\}$ and $b \in V(\bar{M}_{r,s} - a)$.

(**p2.1**) Suppose that $a = z$ and $b \in V(M_{r,s})$. By **(z)**, $\bar{M}_{r,s} \cup zb$, has an $x_{r+1}y_{s+1}$-trace $x_{r+1}Hy_{s+1}$ containing e_z. Then $xPx_{r+1}Hy_{s+1}Py$ is an xy-trace in G containing A.

(**p2.2**) Suppose that $a = z$ and $b \in \{x_{r+1}, y_{s+1}\}$. By symmetry, we can assume that $b = x_{r+1}$. By **(x1)**, $\dot{M}_{r,s}$ has a py_s-trace pHy_s. Then $xP'y = xPx_{r+1}zpHy_sPy$ is an xy-trace in G. If $x \neq x_{r+1}$ then $xP'y$ contains A. If $x = x_{r+1}$ then $xP'y$ contains $A \setminus e_x$.

(**p2.3**) Now suppose that $a = \{x_{r+1}, y_{s+1}\}$ and $b \neq z$. By symmetry, we can assume that $a = x_{r+1}$. Then $b \in V(M_{r,s} - x_r) \cup y_{s+1}$.

(**p2.3.1**) Suppose that $x_{r+1} = x$.

Suppose that $b \neq y_{s+1}$. By **(x2)**, $M_{r,s} \cup xb$ has a zy_s-trace pHy_s containing $e_x = x_r x_{r+1}$. Then $zpHy_s y_{s+1} Py$ is a yz-trace in G containing A.

Now suppose that $b = y_{s+1}$. By **(y1)**, $\dot{M}_{r,s}$ has a $\{p, x_r\}$-trace pHx_r. Then $zP'y = zpHx_r x_{r+1} y_{s+1} Py$ is a yz-trace in G. If $y_{s+1} \neq y$ then $zP'y$ contains A.

Suppose that $y_{s+1} = y$. Then $zP'y$ contains $A - e_y$. By symmetry, there also exists a zx-trace containing $A - e_x$.

(**p2.3.2**) Now suppose that $a = x_{r+1} \neq x$. Our goal is to show that

(s1) if $b \neq y_{s+1}$ then $M' = M_{r+1,s}$ is a good path and

(s2) if $b = y_{s+1}$ (i.e. $x_{r+1}y_{s+1} \in E(G)$) then $M' = M_{r+1,s+1}$ is a good path.

This will lead to a contradiction because $M_{r,s} \subset M'$, and therefore a good path $M_{r,s}$ is not maximal. We recall that we consider the case when $x_r \neq x$ and $y_s \neq y$.

(**p2.3.2.1**) Suppose that $b \neq y_{s+1}$. We want to show that $M_{r+1,s}$ is a good path.

(**p2.3.2.1.xy1**) Let us show that $M_{r+1,s}$ satisfies **(x1)**. By **(x2)** for $M_{r,s}$, the graph $\dot{M}_{r,s} \cup \{x_r x_{r+1}, x_{r+1}b\}$ has a py_s-trace $zpHy_{s+1}$ containing the path $x_r x_{r+1} v$. Then pHy_s is a py_s-trace in $\dot{M}_{r+1,s}$ containing the path $x_r x_{r+1}$.

Let us show that $M_{r+1,s}$ satisfies **(y1)**. By **(y1)** for $M_{r,s}$, the graph $\dot{M}_{r+1,s}$ has a px_{r+1}-trace $x_{r+1}Hp$ containing $y_{s-1}y_s$. Then $x_{r+2}x_{r+1}Hp$ is a px_{r+2}-trace in $\dot{M}_{r+2,s}$ containing $y_{s-1}y_s$.

(**p2.3.2.1.x2**) Let us show that $M_{r+1,s}$ satisfies **(x2)**.

Let us consider $\bar{M}_{r+1,s} \cup x_{r+2}v$ where $v \in V(M_{r+1,s}) \setminus x_{r+1}$.

Suppose $v \neq x_r$. By **(x2)** for $M_{r,s}$, the graph $\bar{M}_{r,s} \cup x_{r+1}v$ has a zy_{s+1}-trace $y_{s+1}Hz$ containing the path $x_r x_{r+1} v$. Then $y_{s+1}Hz - x_{r+1}v \cup x_{r+1}x_{r+2}v$ is a zy_{s+1}-trace containing the path $x_{r+1}x_{r+2}v$.

Now suppose that $v = x_r$. By (**p2.3.2.1.xy1**), $M_{r+1,s}$ satisfies **(x1)**, i.e. the graph $\dot{M}_{r+1,s+1}$ has a py_{s+1}-trace $y_{s+1}Hp$ containing $x_r x_{r+1}$. Then $y_{s+1}Hpz - x_r x_{r+1} \cup x_{r+1}xr + 2x_r$ is a zy_{s+1}-trace containing the path $x_{r+1}x_{r+2}v$.

(**p2.3.2.1.y2**) Let us show that $M_{r+1,s}$ satisfies **(y2)**.

Let us consider $\bar{M}_{r+1,s} \cup y_{s+1}v$ where $v \in (V(M_{r+1,s}) \cup x_{r+2}) \setminus y_s$.

Suppose that $v \neq x_{r+2}$. By **(y2)** for $M_{r,s}$, the graph $\bar{M}_{r,s} \cup y_{s+1}v$ has a zx_{r+1}-trace $x_{r+1}Hz$ containing the path $y_s y_{s+1} v$. Then $x_{r+2}x_{r+1}Hz$ is a $\{z, x_{r+2}\}$-trace containing the path $y_s y_{s+1} v$.

(**p2.3.2.1.z**) Let us show that $M_{r+1,s}$ satisfies **(z)**.

Let us consider a graph $M_{r+1,s} \cup \{zp, zv\}$ where $v \in V(M_{r+1,s}) \setminus p$.

Suppose that $v \in V(M_{r,s}) \setminus p$. By **(z)** for $M_{r,s}$, the graph $M_{rs} \cup \{zp, zv\}$ has an $x_r y_s$-trace $x_r Hy_s$. Then $x_{r+1}x_r Hy_s$ is an $x_{r+1}y_s$-trace in $M_{r+1,s} \cup \{zp, zv\}$.

Now suppose that $v = x_{r+1}$. By **(x1)** for $M_{r,s}$, the graph $\dot{M}_{r,s}$ has a py_s-trace pHy_s. Then $x_{r+1}zpHy_s$ is an $x_{r+1}y_s$-trace in $M_{r+1,s} \cup \{zp, zv\}$.

(**p2.3.2.2**) Now suppose that $b = y_{s+1}$. We want to show that $M_{r+1,s+1}$ is a good path.

Let us show that $M_{r+1,s+1}$ satisfies **(x1)**. As in (**p2.3.1**), by **(y1)** for $M_{r,s}$, the graph

$\dot{M}_{r,s}$ has a px_r-trace pHx_r. Then $pHx_rx_{r+1}y_{s+1}$ is an $y_{s+1}p$-trace in $\dot{M}_{r+1,s+1}$ containing x_rx_{r+1}. Therefore $M_{r+1,s+1}$ satisfies **(x1)**. Similarly by using **(x1)** for $M_{r,s}$, one can show that $M_{r+1,s+1}$ satisfies **(y1)**. The proof that $M_{r+1,s+1}$ satisfies the remaining conditions of a good path is similar to that for $M_{r+1,s}$ in (**p2.3.2.1**).

4 More on claw- and net-free graph Hamiltonicity

Lemma **3.3** allows to give an easy proof of the following strengthening of **1.1**.

4.1 *Let G be a connected claw- and net-free graph. Then*

($a1$) *G has a Hamiltonian path and*

($a2$) *if $sz \in E(G)$ and $G - z$ is connected then sz belongs to a Hamiltonian path of G.*

Proof (uses **3.3**). We prove our claim by induction on $v(G)$. The claim holds if $v(G) = 1$. Since G is connected, there exists $z \in V(G)$ such that $G-z$ is also connected. Let $sz \in E(G)$. Since G is claw- and net-free, clearly $G - z$ is also claw- and net-free. Therefore by the induction hypothesis, $G - z$ has a Hamiltonian path. Then by **3.3**, G has a Hamiltonian path containing sz.

Here is another strengthening of **1.1** for graphs of connectivity one.

4.2 *Let G be a connected claw- and net-free graph, $G = AaHbB$ where A and B are end–blocks of G. Let $a' \in A - a$, $b' \in B - b$, and $a'x$ is an edge of A such that if $v(A) \geq 3$ then x is an inner vertex of an end-block of $A - a'$. Then*

($a1$) *there exists a Hamiltonian $a'b'$–path in G (and so G is Hamiltonian $(V(A - a), V(B - b))$–connected) and*

($a2$) *there exists a Hamiltonian $a'b'$–path in G containing e.*

Proof (uses **1.1** and **3.2**). We prove our claim by induction on $v(G)$.

If $v(A \cup B - \{a, b\}) = 2$ (and if, in particular, $v(G) = 3$) then our claim follows from **1.1**. Therefore let $v(A \cup B - \{a, b\}) \geq 3$, say $v(A - a) \geq 2$, and so A is 2–connected. Then A is 2–connected. Let $A' = A - a'$ and $G' = G - a'$, and so $G' = A'aHbB$. Since A is 2–connected, clearly A' and G' are connected. Since G is claw- and net-free, G' is also claw- and net-free. By **3.2**, G' is a chain. Therefore A' is also a chain and a belong to an end-block of A'.

Since A is 2–connected, and so a' is adjacent to an inner vertex in the end-block of A' avoiding a. Therefore there exists an edge $e = a'x$ satisfying our assumption.

Since G' satisfies our assumptions and $v(G') < v(G)$, by the induction hypothesis, there exists a Hamiltonian xb'–path P in G'. Then $a'xPb'$ is a Hamiltonian $a'b'$–path in G containing e.

From **4.2** we have, in particular:

4.3 *Let G be a claw- and net-free graph, $v(G) \geq 3$, $cn(G) = 1$, and $x, y \in V(G)$. Then G has a Hamiltonian xy–path if and only if x and y are inner vertices of different end–blocks of G.*

From **4.1** and **4.2** it is easy to obtain the following stronger result.

4.4 *Let G be a connected claw- and net-free graph having $k \geq 2$ blocks. Let A_j, $j = 1, 2$, be end–blocks of G, a'_j the boundary vertex of A_j, $a_j \in A_j - a'_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \ldots, k-2\}$. Suppose that*

(h1) $\alpha_j = a_j x_j$ is such that if $v(A) \geq 3$ then x_j is an inner vertex of an end–block of $A_j - a'_j$, $j \in \{1, 2\}$, and

(h2) β_i is an inner edge of B_i if $v(B_i) \geq 3$, $i \in \{1, \ldots, k-2\}$.

Then G has a Hamiltonian $a_1 a_2$–path containing U.

Proof (uses **4.1** and **4.2**). Since G is connected, for every end-block A_j of G there is an edge $a'_j p_j \in E(G) \setminus E(A_j)$. Similarly for every inner block B_i of G there are edges $b_i q_j, b'_i q'_j \in E(G) \setminus E(B_i)$ where b_i and b'_i are the boundary vertices of B_i. Let $\bar{A}_j = A_j a'_j p_j$ and $\bar{B}_i = q_i b_i B_i b'_i q'_j$. Then all $\bar{A}_j$'s and $\bar{B}_i$'s are induced subgraphs of G and, therefore, are claw- and net-free. By **4.1**, each $\bar{B}_i$ has a Hamiltonian path $q_i b_i Q_i b'_i q'_j$ containing β_i. By **4.2**, each $\bar{A}_j$ has a Hamiltonian path $a_j P_j a'_j p_j$ containing α_j. Then $a_1 H a_2 = P_1 \cup Q_1 \ldots Q_{k-2} \cup P_2$ is a Hamiltonian $a_1 a_2$–path containing U.

Given a connected graph G and $x, y \in V(G)$, $x \neq y$, let $H(G, xy)$ denote the set of edges ab of G such that $\{a, b\} \neq \{x, y\}$ unless $V(G) = \{x, y\}$ and if $G - \{a, b\}$ is not connected then no component of $G - \{a, b\}$ avoids $\{x, y\}$.

Obviously the following is true for every graph.

4.5 *Let G be a graph and $x, y \in V(G)$, $x \neq y$. If e belongs to a Hamiltonian xy–path of G, then $e \in H(G, xy)$.*

We will see that for claw- and net-free graphs of connectivity one the converse holds.

4.6 *Let G be a connected graph and Gyy' be a claw- and net-free graph. Let C be the end–block of Gyy', distinct from yy', and let c be the boundery vertex of C. Let $x \in V(C - c)$ and $e \in E(G)$. Then Gyy' has a Hamiltonian xy'–path containing an edge e if and only if $e \in H(G, xy)$.*

Proof (uses **4.2**, **4.4**, and **4.5**). Let $e = ab$. As we have mentioned in **4.5**, if G has a Hamiltonian xy–path containing ab then $ab \in H(G, xy)$. Now suppose that $ab \in H(G, xy)$. If $ab \notin E(C)$ then our claim follows from **4.4**. Therefore let $ab \in E(C)$. We prove our claim by induction on $v(G)$. If $v(C) = 2$ our claim is obviously true. Therefore let $v(C) \geq 3$, and so C is 2–connected and $v(G) \geq 3$.

(**p1**) Suppose that $G - \{a, b\}$ is not connected. Since $ab \in H(G, xy)$, vertices x and y belong in $G - \{a, b\}$ to different components, say X and Y, respectively. Since C is 2–connected, the subgraph $\bar{X}$, induced in G by $X \cup \{a, b\}$, is also 2–connected.

(**p1.1**) Suppose that $|X| = 1$, i.e. $X = \{x\}$. Then xa is an end–block of $G - b$. Since Gyy' is claw- and net-free, by **4.2**, $Gyy' - b$ has a Hamiltonian xy'–path $xaPy'$. Then $xbaPy'$ is

a Hamiltonian xy'–path containing ab.

(**p1**.**2**) Now suppose that $|X| \geq 2$ and x is adjacent to a vertex z in X. Since $\bar{X}$ is 2–connected, either $\bar{X} - x$ is 2–connected or x is adjacent in G to an inner vertex z of a (the) end-block of $\bar{X} - x$ that does not contain ab. In both cases, by the induction hypothesis, $Gyy' - x$ has a Hamiltonian zy'–path zPy' containing ab. Then $xzPy'$ is a Hamiltonian xy'–path containing ab.

(**p2**) Now suppose that $G - \{a, b\}$ is connected. Since $ab \in H(G, xy)$ and $v(G) \geq 3$, we have: $\{a, b\} \neq \{x, y\}$. Since C is 2–connected, x is adjacent to a vertex z in $C - c$. Then by the induction hypothesis, $Gyy' - x$ has a Hamiltonian zy'–path zPy' containing ab. Then $xzPy'$ is a Hamiltonian xy'–path containing ab.

From **4.4** and **4.6** we have:

4.7 *Let G be a claw- and net-free graph, $v(G) \geq 3$, $cn(G) = 1$, and $x, y \in V(G)$, $x \neq y$ and $e \in E(G)$. Then G has a Hamiltonian xy–path containing e if and only if x and y are inner vertices of different end-blocks of G and $e \in H(G, xy)$.*

From **4.7** we have:

4.8 *Let G be a claw- and net-free graph, $v(G) \geq 3$, $cn(G) = 1$, and $s \in V(G)$, and $e \in E(G)$. Then G has a Hamiltonian s–path containing e if and only if s is an inner vertex of an end-block in G and $e \in H(G, sb)$ where b is the boundary vertex of the end–block B avoiding s.*

From **4.4** and **4.7** we have the following strengthening of **4.4**.

4.9 *Let G be a connected claw- and net-free graph having $k \geq 2$ blocks. Let A_j, $j = 1, 2$, be end–blocks of G, a'_j the boundary vertex of A_j, $a_j \in A_j - a'_j$, and $\alpha_j \in E(A_j)$. Let B_i be an inner block of G and $\beta_i \in E(B_i)$. Let $U = \{\alpha_1, \alpha_2\} \cup \{\beta_i : i = 1, \ldots, k-2\}$. Then G has a Hamiltonian a_1a_2–path containing U if and only if*
(*c*1) *$\alpha_j \in H(A_j, a_ja'_j)$, $j \in \{1, 2\}$ and*
(*c*2) *β_i is an inner edge of B_i if $v(B_i) \geq 3$, $i \in \{1, \ldots, k-2\}$.*

Given a 2–connected graph G, let $H(G)$ denote the set of edges x_1x_2 of G such that if for some x_i, $G - x_i$ is not 2–connected, then x_j is an inner vertex of an end-block of $G - x_i$, where $\{i, j\} = \{1, 2\}$. It is easy to see that if a 2–connected graph G is claw-free, then an edge $x_1x_2 \in H(G)$ if and only if $G - \{x_1, x_2\}$ is connected.

Obviously the following is true for every graph.

4.10 *Let G be a graph and $e \in E(G)$. If e belongs to a Hamiltonian cycle of G, then $e \in H(G)$.*

Now we give the following strengthening of **1.2**. This theorem shows, in particular, that for 2–connected claw- and net-free graphs the converse of **4.10** is also true.

4.11 *Let G be a 2–connected claw- and net-free graph and $e \in E(G)$. Then*

($a1$) *G has a Hamiltonian cycle,*

($a2$) *the following are equivalent:*

($c1$) *e belongs to a Hamiltonian cycle of G,*

($c2$) *$e \in H(G)$, and*

($a3$) *if $e = sz$ does not satisfies ($c2$), say $G - z = AaDdF$ where $s = a$ and A, D are blocks of $G - z$, then for every $a' \in V(A - a)$ and $d' \in V(D - \{a, d\}$ there exists a Hamiltonian $a'd'$–path of G containing e.*

Proof (uses **3.3** and **4.2** ($a1$)). As we mentioned above, ($c1$) implies ($c2$). We prove ($a1$) and ($c2$) $\Rightarrow$ ($c1$) by induction on $v(G)$. The claim holds if $v(G) = 3$. If G is a cycle, we are done. Therefore we can assume that G is not a cycle.

(**p1**) Suppose that $G - z$ is 2–connected. Since G is claw- and net-free, clearly $G - z$ is also claw- and net-free. Therefore by the induction hypothesis, $G - z$ has a Hamiltonian cycle C and $s \in V(C)$. Since G is 2–connected, there is a vertex t in C distinct from s and adjacent to z. Let x and y be the two vertices adjacent to t in C. Then $G' = G - t$ satisfies the assumptions of **3.3**, namely, G' is connected and $xPy = C - t$ is a Hamiltonian x, y–path of $G' - z$. By **3.3**, G' has a Hamiltonian path aLb such that $e \in E(L)$ and $\{a, b\} \subset \{x, y, z\}$. Since t is adjacent to x, y, and z, clearly $taLbt$ is a Hamiltonian cycle of G containing e.

(**p2**) Suppose that $G - z$ is not 2–connected. Let $G - z = AaHbB$ where A and B are end-blocks of G. Suppose that ($c2$) is satisfied. Then s is an inner vertex of an end-block, say $s \in V(A - a)$. Since G is 2–connected, $zb' \in E(G)$ for some $b' \in V(B - b)$. By **4.2** ($a1$), $G - z$ has a Hamiltonian sb'-path P. Then $zsPb'z$ is a Hamiltonian cycle in G containing $e = sz$.

(**p3**) Suppose that $e = sz$ does not satisfy ($c2$), say $s = a$ in (**p2**). Let $a' \in V(A - a)$ and let D be the block of $G - z$ containing a and distinct from A, $d' \in D - a$ and d' is an inner vertex of D if $v(D) \geq 3$. By **4.2** ($a1$), A has a Hamiltonian $a'a$–path P_1 and $G - A$ has a Hamiltonian zd'–path P_2. Then $a'P_1azP_2d'$ is a Hamiltonian $\{a', d'\}$–path of G containing e.

From **4.11** we have, in particular:

4.12 *Let G be a 2–connected claw- and net-free graph. Then every edge in G belongs to a Hamiltonian path of G.*

From **4.11** we also have:

4.13 *Let G be a 3–connected claw- and net-free graph. Then every edge in G belongs to a Hamiltonian cycle of G.*

In [5] we use the results of this paper to give more general results on k–connected claw- and net-free graphs. One of them concerns 3–connected graphs and shows, in particular, that 3–connected claw- and net-free graphs are Hamiltonian connected.

4.14 *Let G be a 3–connected claw- and net-free graph, $e = uv \in E(G)$, and $s, t \in V(G)$, $s \neq t$. Then G has a Hamiltonian st–path containing e if and only if either $\{s,t\} \cap \{u,v\} = \emptyset$ or $\{s,t\} \setminus \{u,v\} = z \in V(G)$ and $G - \{z,u,v\}$ is connected.*

From **4.14** we have:

4.15 *Let G be a claw- and net-free graph. If G is 3–connected then every two non-adjacent edges in G belong to a Hamiltonian cycle. If G is 4–connected then every two edges in G belong to a Hamiltonian cycle.*

Another result in [5] concerns k–connected claw- and net-free graphs for $k \geq 3$.

4.16 *Let G be a k–connected graph, $k \geq 3$, L_1 and L_2 two disjoint paths in G, $v(L_1) + v(L_2) \leq k$, and x_1, x_2 the end-vertices of L_1, L_2, respectively. Then the following are equivalent:*

($a1$) G has a Hamiltonian x_1x_2–path containing L_1 and L_2,

($a2$) G has a Hamiltonian z_1z_2–path containing L_1 and L_2 for every end–vertices z_1, z_2 of L_1, L_2, respectively, and

($a3$) $G - (L_1 \cup L_2)$ is connected.

From **4.11** and **4.16** we have the following generalization of **4.11** for k–connected graphs.

4.17 *Let G be a k–connected graph, $k \geq 2$, L a path in G, and $v(L) \leq k$. Then G has a Hamiltonian cycle containing L if and only if $G - L$ is connected.*

In [5] (see also [**?**]) we gave a structure characterization of so-called closed the claw- and net-free graphs. This structure theorem together with the known properties of the graph closure can be used to provides alternative proofs for some of the above Hamiltonicity results.

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