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Maximal induced matchings of minimum/maximum size

by

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ABSTRACT

Let IMatch(G) be the set of all maximal induced matchings of a graph G. We define $\sigma(G) = \min\{|M| : M \in \text{IMatch}(G)\}$ and $\Sigma(G) = \max\{|M| : M \in \text{IMatch}(G)\}$. Stockmeyer and Vazirani [16] and Cameron [3] independently proved that the decision problem corresponding to the problem of computing $\Sigma(G)$ is NP-complete. We improve this result. Then we show NP-completeness for $\sigma(G)$. We prove that WEIGHTED INDUCED MATCHING Problem can be solved in polynomial time for $(G_1, G_2, \dots, G_{17})$-free graphs, see Figure 3. Also, INDUCED MATCHING Problem can be solved in polynomial time for $(H_1, H_2, \dots, H_{30})$-free graphs, see Figure 4. The latter result extends results of Kobler and Rotics [9], and Boliac and Lozin [2]. We give corollaries that generalize a result of Lozin [11] and prove a conjecture of Lozin [12] in a generalized form.
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1 Introduction

We use standard graph-theoretic terminology, see for example Melnikov, Sarvanov, Tyshkevich, Yemelichev, and Zverovich [13].

An *induced matching* in a graph G is edge-set of a 1-regular induced subgraph. An induced matching M is *maximal* if $M \not\subseteq M'$ for any other induced matching M' in G. Let $\mathrm{IMatch}(G)$ be the set of all maximal induced matchings of a graph G. We define $\sigma(G) = \min\{|M| : M \in \mathrm{IMatch}(G)\}$ and $\Sigma(G) = \max\{|M| : M \in \mathrm{IMatch}(G)\}$.

Stockmeyer and Vazirani [16] and Cameron [3] independently proved that the decision problem corresponding to the problem of computing $\Sigma(G)$ is NP-complete. We improve this result. Then we show NP-completeness for $\sigma(G)$.

We prove that WEIGHTED INDUCED MATCHING Problem can be solved in polynomial time for $(G_1, G_2, \ldots, G_{17})$-free graphs, see Figure 3. Also, INDUCED MATCHING Problem can be solved in polynomial time for $(H_1, H_2, \ldots, H_{30})$-free graphs, see Figure 4. The latter result extends results of Kobler and Rotics [9], and Boliac and Lozin [2]. We give corollaries that generalize a result of Lozin [11] and prove a conjecture of Lozin [12] in a generalized form.

For a graph $G = (V, E)$, we use notation $V = V(G)$ and $E = E(G)$. The subgraph of G induced by a set $X \subseteq V$ is denoted by $G(X)$. As usual, $N_G(x)$ is the neighborhood of a vertex $x \in V(G)$. The complete graph of order n is denoted by K_n. Also, K_3 is called a *triangle*. A *clique* in a graph is a complete subgraph.

2 Maximal induced matchings of minimum size

It is well-known that the decision problem corresponding to the problem of computing $\Sigma(G)$ is NP-complete, see Stockmeyer and Vazirani [16] and Cameron [3]. In this section we show that the analogous result holds for $\sigma(G)$ as well.

Recall that a *clause* over a set X of $0-1$ variables is a conjunction of some literals, a *literal* being either a variable $x_i \in X$ or a negation of a variable $x_j \in X$, denoted by $\overline{x}_j$. A truth assignment $\mathbf{x}^0$ *satisfies* a clause c_i if $c_i(\mathbf{x}^0) = 1$. The following satisfiability problem or, simply, SAT is well known (Cook [5], see also Garey and Johnson [7]).

Decision Problem 1 (SAT)

Instance: *A collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \ldots, x_n\}$ of $0-1$ variables.*

Question: *Is there a truth assignment for X that satisfies all the clauses in C?*

Given an instance $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ to SAT, we may assume without loss of generality that no clause in C contains a variable x_i and its negation $\overline{x}_i$

simultaneously, since such a clause is satisfied by any truth assignment and therefore can be eliminated.

The *satgraph* $S_{C,X}$ corresponding to (C, X) is defined by the following:

- $V(S_{C,X}) = L \cup C$, where $L = \{x_i, \overline{x}_i : i = 1, 2, \ldots, n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ are disjoint sets,
- the set C induces a clique,
- the set L induces a matching $M = \{x_i\overline{x}_i : i = 1, 2, \ldots, n\}$, and
- a vertex $l \in L$ is adjacent to a vertex $c_j \in C$ if and only if the clause c_j contains the literal l,

see Chvátal and Slater [4] and Zverovich [17]. We refer to M as the *literal matching* of $S_{C,X}$.

Definition 1 *We subdivide each edge $x_i\overline{x}_i$ of the literal matching of $S_{C,X}$ with new vertices y_i and $\overline{y}_i$ so that the resulting 4-path P^i contains edges x_iy_i, $y_i\overline{y}_i$ and $\overline{y}_i\overline{x}_i$. The resulting graph is called* msatgraph $M_{C,X}$.

We are now ready to prove the main result of this section.

Theorem 1 *The decision problem corresponding to the problem of computing $\sigma(G)$ is NP-complete within msatgraphs.*

Proof. It is clear that the problem is in NP. We construct a polynomial-time reduction from SAT. For that we consider the msatgraph $G = M_{C,X}$ corresponding to an instance $(C = \{c_1, c_2, \ldots, c_m\}, X = \{x_1, x_2, \ldots, x_n\})$ to SAT. Without loss of generality we may assume that $m = |C| \geq 2$.

For a set $U \subseteq V(G)$, let $EN_G(U)$ be the set of all edges in G that have at least one end-vertex in U. We denote $X_i = \{x_i, y_i, \overline{y}_i, \overline{x}_i\}$ for $i = 1, 2, \ldots, n$.

Claim 1 *For an arbitrary maximal induced matching M in G,*

(i) $|EN_G(C) \cap M| \leq 1$,

(ii) $EN_G(X_i) \cap M \neq \emptyset$ *for each* $i = 1, 2, \ldots, n$, *and*

(iii) $|M| \in \{n, n + 1\}$.

Proof. (i) Since C induces a clique, each induced matching in $EN_G(C)$ has at most one edge, i.e., $|EN_G(C) \cap M| \leq 1$.

(ii) If M is disjoint from $EN_G(X_i)$ for some i, then we can add the edge $y_i\overline{y}_i$ to M, contradicting to maximality of M.

(iii) By (i), $|EN_G(C) \cap M| \leq 1$. Also, M contains at most n edges of $G(L)$. Thus, $|M| \leq n+1$. As it follows from (ii), $|M| \geq n$. □

Note that G always has a maximum induced matching of size $n+1$, for example $\{c_1c_2\} \cup \{y_i\overline{y}_i : 1, 2, \ldots, n\}$. Recall that $m = |C| \geq 2$.

We show that C is satisfiable if and only if $\sigma(G) = n$. First, suppose that there exists truth assignment ϕ satisfying C. We construct an induced matching $M \subseteq \{x_iy_i, \overline{y}_i\overline{x}_i : i = 1, 2, \ldots, n\}$ choosing the n edges that correspond to true literals under ϕ. That is, if $\phi(x_i) = 1$, we include x_iy_i into M, otherwise $\overline{y}_i\overline{x}_i \in M$. Since ϕ satisfies C, each vertex of C is incident to an end-vertex of an edge in M. It implies that we cannot extend M by adding an edge $e \notin E(G(L))$. Indeed, such an edge is always incident to a vertex of C. Thus, M is a maximal induced matching, and $\sigma(G) \leq |M| = n$. Then Claim 1 implies that $\sigma(G) = n$.

Conversely, suppose that $\sigma(G) = n$. Let M be a maximal induced matching in G of size n.

Claim 2 *$M \subseteq E(G(L))$.*

Proof. Claim 1(ii) and $|M| = n$ imply that M contains exactly one edge from each set $EN_G(X_i)$. Moreover, M cannot have an edge connecting a vertex of L with a vertex of C. To see that, suppose that an edge x_ic_j is in M. By Claim 1(i), $M \setminus \{x_ic_j\} \subseteq E(G(L))$. The vertex $\overline{x}_i$ is non-adjacent to both x_i and c_j, since the clause cannot contain both x_i and $\overline{x}_i$. Also, $\overline{y}_i$ is non-adjacent to both x_i and c_j. It follows that $M \cup \{\overline{y}_i\overline{x}_i\}$ is an induced matching, a contradiction to maximality of M. □

We say that a vertex $c_j \in C$ is *dominated* by M if c_j is adjacent to an end-vertex of an edge in M.

Claim 3 *At most one vertex in C is not dominated by M.*

Proof. Suppose that c_1, c_2 are not dominated by M. Then $M \cup \{c_1c_2\}$ is an induced matching, contradicting to maximality of M. □

Claim 4 *There exists a maximal induced matching $M' \subseteq E(G(L))$ that dominates all vertices in C.*

Proof. If M dominates all vertices in C, then we may set $M' = M$, since $M \subseteq E(G(L))$ according to Claim 2. Otherwise, by Claim 3, we may assume that c_1 is the only vertex in C that is not dominated by M. Without loss of generality, let $c_1 = x_1 \vee x_2 \vee \cdots \vee x_k$. By maximality of M, we cannot add the edge $\overline{x}_i\overline{y}_i$ to M for each $i = 1, 2, \ldots, k$. Since $M \subseteq E(G(L))$, the edges $y_i\overline{y}_i$ are in M, $i = 1, 2, \ldots, k$. Now we define $M' = (M \setminus \{y_1\overline{y}_1\}) \cup \{x_1y_1\}$. Since both y_1 and $\overline{y}_1$ do not dominate any vertices in C, the induced matching M' dominates all vertices in C. In particular, M' is maximal. □

The matching M' of Claim 4 covers at most one of the vertices x_i, $\overline{x}_i$ for each $i = 1, 2, \ldots, n$. Hence we can define a partial truth assignment ϕ' satisfying C. It remains to extend ϕ' to a full assignment. This completes the proof of Theorem 1. $\square$

A graph G is *bi-size matched* if there exists $n \geq 1$ such that $|M| \in \{n, n+1\}$ for every maximal induced matching M in G. Theorem 1 implies that it is NP-hard to compute $\sigma(G)$ even for bi-size matched graphs.

3 Maximal induced matchings of maximum size

Computing Σ is hard for some narrow classes of graphs. First Stockmeyer and Vazirani [16] and Cameron [3] proved that it is NP-hard to compute Σ for bipartite graphs. This result was essentially improved by Ko and Shepherd [8].

Theorem 2 (Ko and Shepherd [8]) *For each $k \geq 1$, Σ is NP-hard to compute for the class of planar, bipartite graphs of maximum degree 4 with one side of the bipartition consisting only of degree two vertices and with each cycle having length $\equiv 0 \pmod{2^k}$.*

Recall that a *dominating set* D is a subset of $V(G)$ such that each vertex in $V(G) \setminus D$ is adjacent to a vertex of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set. Note that Theorem 2 is also valid for the problem of computing the domination number $\gamma(G)$. A 3-regular graph is called *cubic*.

Theorem 3 (Duckworth, Manlove, and Zito [6]) *Computing $\Sigma(G)$ in the class of planar cubic graphs is NP-hard.*

We give a common generalization of both Theorem 2 and Theorem 3.

Definition 2 (Korobitsyn [10]) *A* Korobitsyn transformation *of a graph G is any graph G' that can be obtained from G with the following procedure:*

(KOR1) *choose a vertex $u \in V(G)$,*

(KOR2) *arbitrarily subdivide the neighborhood of u: $N_1 \cup N_2 = N_G(u)$,*

(KOR3) *delete u from G, add a 4-path (u_1, v_1, v_2, u_2), and*

(KOR4) *connect u_i with all vertices in N_i for $i = 1, 2$.*

It is easy to show that the Korobitsyn transformation always produces a graph G' with $\gamma(G') = \gamma(G) + 1$, see Korobitsyn [10]. Lozin [11] noted that the Korobitsyn transformation also has the property that $\Sigma(G') = \Sigma(G) + 1$. In particular, if G' is obtained from G with a 3-subdivision of an edge, then $\Sigma(G') = \Sigma(G)+1$. A 3-*subdivision* of an edge uv is the result of replacing uv with a path of length four (u, x, y, z, v). Indeed, the Korobitsyn transformation with $|N_1| = 1$ is just a 3-subdivision of an edge.

Let $\mathrm{Sub}(G)$ be the set of all subgraphs of a graph G. A class of graphs $\mathcal{P}$ is called *strongly hereditary* if $\mathrm{Sub}(G) \subseteq \mathcal{P}$ for each graph $G \in \mathcal{P}$. A graph F is a *minimal forbidden subgraph* for a strongly hereditary class $\mathcal{P}$ if $\mathrm{Sub}(F)\backslash\mathcal{P} = \{F\}$. Let $Z(\mathcal{P})$ denote the set of all minimal forbidden subgraphs for a strongly hereditary class of graphs $\mathcal{P}$. A strongly hereditary class $\mathcal{P}$ is *finitely generated* if $Z(\mathcal{P})$ is a finite set.

Let $S_{i,j,k}$ be the graph shown in Figure 1, where $i \geq 0$, $j \geq 0$ and $k \geq 0$.

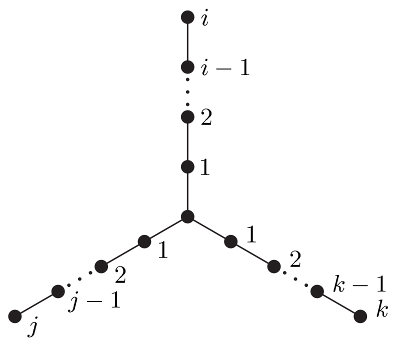

Figure 1: The graph $S_{i,j,k}$.

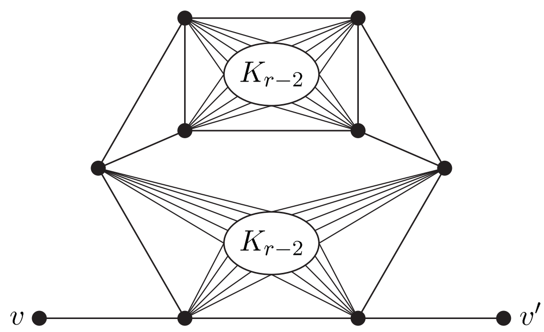

Figure 2: The graph H_v.

The class $\mathcal{S}$ consists of all graphs G such that each connected component in G is of the form $S_{i,j,k}$, where $i \geq 0$, $j \geq 0$ and $k \geq 0$ depend on a component.

Theorem 4 *Let $\mathcal{P}$ be a finitely generated strongly hereditary class of graphs. If $Z(\mathcal{P}) \cap \mathcal{S} = \emptyset$, then computing Σ is an NP-hard problem for graphs in $\mathcal{P}$ satisfying the following conditions:*

(SIGMA1) *G is planar,*

(SIGMA2) *G is bipartite graphs with parts A and B,*

(SIGMA3) $\deg_G a = 2$ *for each $a \in A$, and* $\deg_G b \in \{2, 3\}$ *for each $b \in B$, and*

(SIGMA4) *each cycle in G has length* $\equiv 0 \pmod{4^k}$.

Proof. Let G be a planar cubic graph with n vertices and m edges. We apply a 3-subdivision to every edge of G to produce a graph G_1 satisfying the conditions (SIGMA1), (SIGMA2) and

(SIGMA3). We have $|V(G_1)| = n + 3m$ and $\Sigma(G_1) = \Sigma(G) + m$. According to Theorem 3, computing $\Sigma(G)$ is NP-hard for planar cubic graphs, then so is computing $\Sigma(G_1)$. Applying 3-subdivisions to all edges of G repeatedly k times produces a graph G_k that also satisfies (SIGMA4).

Let $Z(\mathcal{P}) = \{F_1, F_2, \ldots, F_t\}$. We denote by c the length of the longest cycle in graphs F_i, $i = 1, 2, \ldots, t$. Also, p is the length of the longest path connecting two vertices of degrees at least three in graphs F_i, $i = 1, 2, \ldots, t$. The condition $Z(\mathcal{P}) \cap \mathcal{S} = \emptyset$ implies that each graph $F_i \in Z(\mathcal{P})$ either has a cycle, or a path connecting two vertices of degrees at least three, or a vertex of degree at least four. We consider the graph G_k defined above with $k > \max\{\log_4(c/3), \log_4 p\}$. It is easy to see that G_k is a $Z(\mathcal{P})$-free graph, i.e., $G_k \in \mathcal{P}$, and the theorem follows. $\Box$

Kobler and Rotics [9] proved that computing $\Sigma(G)$ is NP-hard on r-regular graphs for each fixed $r \geq 5$. Their construction cannot be extended to $r = 4$. The following theorem implies that the problem is NP-hard for any fixed $r \geq 3$.

Theorem 5 *Computing $\Sigma(G)$ is NP-hard on r-regular graphs for each fixed $r \geq 3$.*

Proof. By Theorem 3, the result holds for $r = 3$. Let G be an r-regular graph for some $r \geq 3$. The following polynomial-time transformation produces an $(r+1)$-regular graph H. Let G' be the copy of the graph G with $V(G') = \{v' : v \in V(G)\}$. For each vertex v of G, the pairs of vertices v and v' are connected by a component H_v shown in Figure 2.

It is not difficult to see that the resulting graph H is $(r+1)$-regular and $\Sigma(H) = 2\Sigma(G) + 2|V(G)|$. $\Box$

4 Maximum induced matchings and square-line graphs

Recall that an independent set in a graph G is a set of pairwise non-adjacent vertices. The INDEPENDENT SET Problem is to find an independent set of G with maximum cardinality. The INDUCED MATCHING Problem is that of finding an induced matching in G with $\Sigma(G)$ edges.

As usual, $L(G)$ is the line graph of a graph G. For a graph H, the *square-graph* H^2 is defined by $V(H^2) = V(H)$, and distinct vertices u and v are adjacent in H^2 if and only if $d_H(u, v) \leq 2$, where $d_H(*, *)$ is the distance function of H. We denote $(L(G))^2$ by $L^2(G)$, the *square-line graph* of G.

Cameron [3] observed that a set M is an induced matching in a graph G if and only if M is an independent set in the graph $L^2(G)$.

Let Z be a set of graphs. A graph G is called *Z-free* if G does not contain any member of Z as an induced subgraph. A *Claw* is the star $K_{1,3} = S_{1,1,1}$.

Theorem 6 (Minty [14], and Nakamura and Tamura [15]) WEIGHTED INDEPENDENT SET Problem *can be solved in polynomial time for* Claw*-free graphs.*

Note that the class of all Claw-free graphs is not closed under the transformation $G \to L^2(G)$, and therefore we cannot apply known algorithms for INDEPENDENT SET Problem within Claw-free graphs to finding maximum induced matchings from this class.

Now we consider maximum weight induced matchings in $S_{2,2,2}$-free graph G, assuming that a non-negative weight function is given on $E(G)$.

Theorem 7 WEIGHTED INDUCED MATCHING Problem *can be solved in polynomial time for* $(G_1, G_2, \ldots, G_{17})$*-free graphs, see Figure 3.*

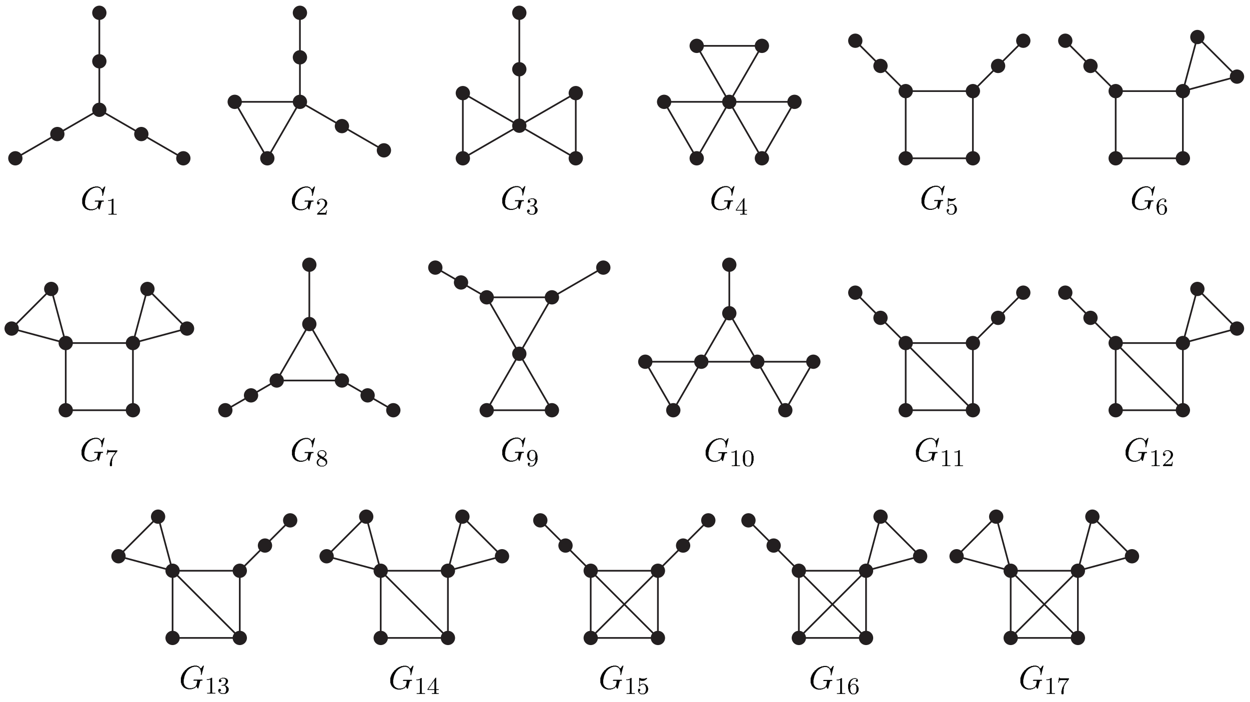

Figure 3: Graphs $G_1, G_2, \ldots, G_{17}$.

Proof. Let G be a $(G_1, G_2, \ldots, G_{17})$-free graph. We show that $H = L^2(G)$ is a Claw-free graph. Suppose that a set $\{a, b_1, b_2, b_3\} \subseteq V(H)$ induces in H a Claw centered at a. In G, there are edges a, b_1, b_2 and b_3 such that $\{b_1, b_2, b_3\}$ is an induced matching, but $\{a, b_i\}$ is not an induced matching for each $i = 1, 2, 3$. Let $b_i = u_i v_i$, $i = 1, 2, 3$. We distinguish between two cases.

Case 1. The edge a is adjacent to some edge b_i.

By symmetry, we may assume that $a = v_1 w$. Since $\{a, b_2\}$ and $\{a, b_3\}$ are not induced matchings, we may assume that the vertex w is adjacent to both v_2 and v_3. It follows that the subgraph of G induced by $\{u_1, u_2, u_3, v_1, v_2, v_3, w\}$ is one of G_1, G_2, G_3 or G_4, see Figure 3, a contradiction.

Case 2. The edge a is non-adjacent to b_1, b_2 and b_3.

Let $a = wx$, where $w, x \notin \{u_1, u_2, u_3, v_1, v_2, v_3\}$. If w is adjacent to a vertex of $\{u_i, v_i\}$ for each $i = 1, 2, 3$, then we essentially have Case 1, which was show to be impossible. Therefore we may assume that w is non-adjacent to both u_3 and v_3. Since $\{a, b_3\}$ is not an induced matchings, we may assume that x is adjacent to v_3. Similarly, x is non-adjacent to both u_1 and v_1, while w is adjacent to v_1. Finally, there is an edge between $\{u_2, v_2\}$ and $\{w, x\}$, say v_2 is adjacent to w.

Table 1 shows all possible variants for the undefined edges u_2w, u_2x and v_2x. We denote $U = \{u_1, u_2, u_3, v_1, v_2, v_3, w, x\}$.

Table 1

u_2w	u_2x	v_2x	Forbidden induced subgraphs
No	No	No	$U \setminus \{u_3\}$ induces G_1 or G_2
No	No	Yes	U induces G_8, G_9 or G_{10}
No	Yes	No	U induces G_5, G_6 or G_7
No	Yes	Yes	U induces G_{11}, G_{12}, G_{13} or G_{14}
Yes	No	No	$U \setminus \{u_3\}$ induces G_2 or G_3
Yes	No	Yes	U induces G_{11}, G_{12}, G_{13} or G_{14}
Yes	Yes	No	U induces G_{11}, G_{12}, G_{13} or G_{14}
Yes	Yes	Yes	U induces G_{15}, G_{16} or G_{17}

Since we have contradictions in all cases, the result follows. □

The only triangle-free graphs in Figure 3 are G_1 and G_5.

Corollary 1 WEIGHTED INDUCED MATCHING Problem *can be solved in polynomial time for* (G_1, G_5, K_3)*-free graphs, see Figure 3.*

In particular, WEIGHTED INDUCED MATCHING Problem can be solved in polynomial time for $(S_{1,2,2}, K_3)$-free graphs. A *Chair* is the graph $S_{1,1,2}$.

Theorem 8 (Alekseev [1]) INDEPENDENT SET Problem *can be solved in polynomial time for* Chair*-free graphs.*

Note that Theorem 8 was proved for unweighted graphs.

Theorem 9 INDUCED MATCHING Problem *can be solved in polynomial time for* $(H_1, H_2, \ldots, H_{30})$*-free graphs, see Figure 4.*

Proof. Let G be an $(H_1, H_2, \ldots, H_{30})$-free graphs, see Figure 4. We show that $H = L^2(G)$ is a Chair-free graph. Suppose that a set $\{a, b_1, b_2, b_3, c\} \subseteq V(H)$ induces a Chair in H, see Figure 5.

The graph G has edges $a = wx, b_1 = u_1v_1, b_2 = u_2v_2, b_3 = u_3v_3$ and $c = yz$. We denote by F the subgraph of G induced by $\{u_1, v_1, u_2, v_2, u_3, v_3, w, x, y, z\}$. The sets $\{b_1, b_2, b_3\}$, $\{b_1, b_2, c\}$ and $\{a, c\}$ are maximal induced matchings in F.

Claim 5 *An end-vertex of the edge $a \in E(F)$ cannot be adjacent to three pairwise non-adjacent vertices in $\{u_1, v_1, u_2, v_2, u_3, v_3\}$.*

Proof. Suppose w is adjacent to v_1, v_2 and v_3. The edge c cannot be incident to each of the vertices u_1, v_1, u_2, v_2, v_3 or w, since the sets $\{b_1, b_2, c\}$ and $\{a, c\}$ are induced matchings in F. Therefore either $y = u_3$ [i.e., $c = u_3z$] or $c = yz$ with $u_3 \notin \{y, z\}$. In any case, y and z are non-adjacent to u_1, v_1, u_2, v_2 and w.

If $c = u_3z$ then w is non-adjacent to u_3, and $F - x \in \{H_1, H_2, \ldots, H_6\}$, a contradiction. Hence $c = yz$ with $u_3 \notin \{y, z\}$. Since the vertices b_3 and c are adjacent in Chair, there is an edge connecting $\{u_3, v_3\}$ and $\{y, z\}$. Suppose that y and w are adjacent to the same vertex of $\{u_3, v_3\}$, say to v_3. Then we delete u_3 from $F - x$, and obtain one of the graphs $H_1, H_2, \ldots, H_6$, a contradiction. Therefore we may assume that u_3 is adjacent to y, and u_3 is non-adjacent to w. Then we delete z from $F - x$, and obtain one of the graphs H_1, H_2 or H_3, a contradiction. $\Box$

Claim 5 implies that the end-vertices of a are not in the set $\{u_1, v_1, u_2, v_2, u_3, v_3\}$. Indeed, if $w = v_1$, then the vertex x must be adjacent to a vertex of $\{u_2, v_2\}$ and to a vertex of $\{u_3, v_3\}$. As a result the vertex x is adjacent to three pairwise non-adjacent vertices in $\{u_1, v_1, u_2, v_2, u_3, v_3\}$ contrary to Claim 5.

Without loss of generality we may assume that x is adjacent to v_3. According to Claim 5, x is non-adjacent to both u_i and v_i for some $i \in \{1, 2\}$. By symmetry, let $i = 1$. Then w is adjacent to a vertex of $\{u_1, v_1\}$, say to v_1. If x is non-adjacent to both u_2 and v_2, then we may assume that w is adjacent to v_2. By Claim 5, w is non-adjacent to both u_3 and v_3. We see that the set $\{u_1, v_1, u_2, v_2, u_3, v_3, w, x\}$ induces H_1, H_2, H_3, H_4, H_5 or H_6, a contradiction. Thus, we may assume that x is adjacent to v_2.

Claim 6 $\{y, z\} \cap \{u_3, v_3\} = \emptyset$.

Proof. Suppose that $\{y, z\} \cap \{u_3, v_3\} \neq \emptyset$, say $z \in \{u_3, v_3\}$. Since vertices c and a are non-adjacent in Chair, $z = u_3$, and u_3 is non-adjacent to both w and x.

If w is adjacent to v_3, then Claim 5 implies that w is non-adjacent to both u_2 and v_2. Then $F \in \{H_7, H_8, H_9, H_{10}\}$, a contradiction. Thus, w is non-adjacent to v_3, and either

$F \in \{H_{11}, H_{12}, \ldots, H_{30}\}$ or F contains one of H_1, H_2, H_4, or H_5 as an induced subgraph, a contradiction. □

By Claim 6, $\{y, z\} \cap \{u_3, v_3\} = \emptyset$. Since the vertices b_3 and c are adjacent in Chair, there is an edge e connecting $\{u_3, v_3\}$ and $\{y, z\}$. If $e = yv_3$, then we delete the vertex u_3. If $e = yu_3$ and u_3 is non-adjacent to both w and x, then we delete the vertex z. If $e = yu_3$ and u_3 is adjacent to a vertex in $\{w, x\}$, then we delete the vertex v_3. It is not hard to see that the resulting graph contains one of $H_1, H_2, \ldots, H_{30}$ as an induced subgraph, a contradiction. □

Theorem 9 extends a result of Kobler and Rotics [9] that the INDUCED MATCHING problem on (Bull, Chair)-free graphs can be solved in polynomial time, see Bull in Figure 5. Indeed, each graph in Figure 4 has either a Chair or a Bull as an induced subgraph.

Boliac and Lozin [2] proved that given a $(S_{2,2,2}, A)$-free bipartite graph G with n vertices, one can find a maximum induced matching in G in time $O(n^2)$. The graph A is shown in Figure 5. Theorem 7 and Theorem 9 imply a more general result.

The only triangle-free graphs in Figure 4 are $H_1 = S_{2,2,3}$ and H_{11}. The graph H_{11} contains $S_{1,3,3}$ as an induced subgraph.

Corollary 2 INDUCED MATCHING Problem *can be solved in polynomial time for* $(S_{2,2,3}, S_{1,3,3}, K_3)$-*free graphs.*

Also, $S_{1,2,4}$ is an induced subgraph of H_{11}.

Corollary 3 INDUCED MATCHING Problem *can be solved in polynomial time for* $(S_{2,2,3}, S_{1,2,4}, K_3)$-*free graphs.*

Corollary 3 generalizes a result of Lozin [11] that INDUCED MATCHING is a polynomial-time solvable problem for $(S_{1,2,3}, \mathrm{Sun}_4)$-free bipartite graphs. Here Sun_4 is a particular bipartite graph of order eight, see Figure 5. In fact, Corollary 3 implies that this graph can be omitted and we may extend bipartite graphs to triangle-free graphs, i.e., the problem can be solved in polynomial time within $S_{1,2,3}$-free triangle-free graphs.

Lozin [12] conjectured that his result can be extended to the entire class of $S_{1,2,3}$-free bipartite graphs. Corollary 3 implies that this conjecture is true even for wider classes.

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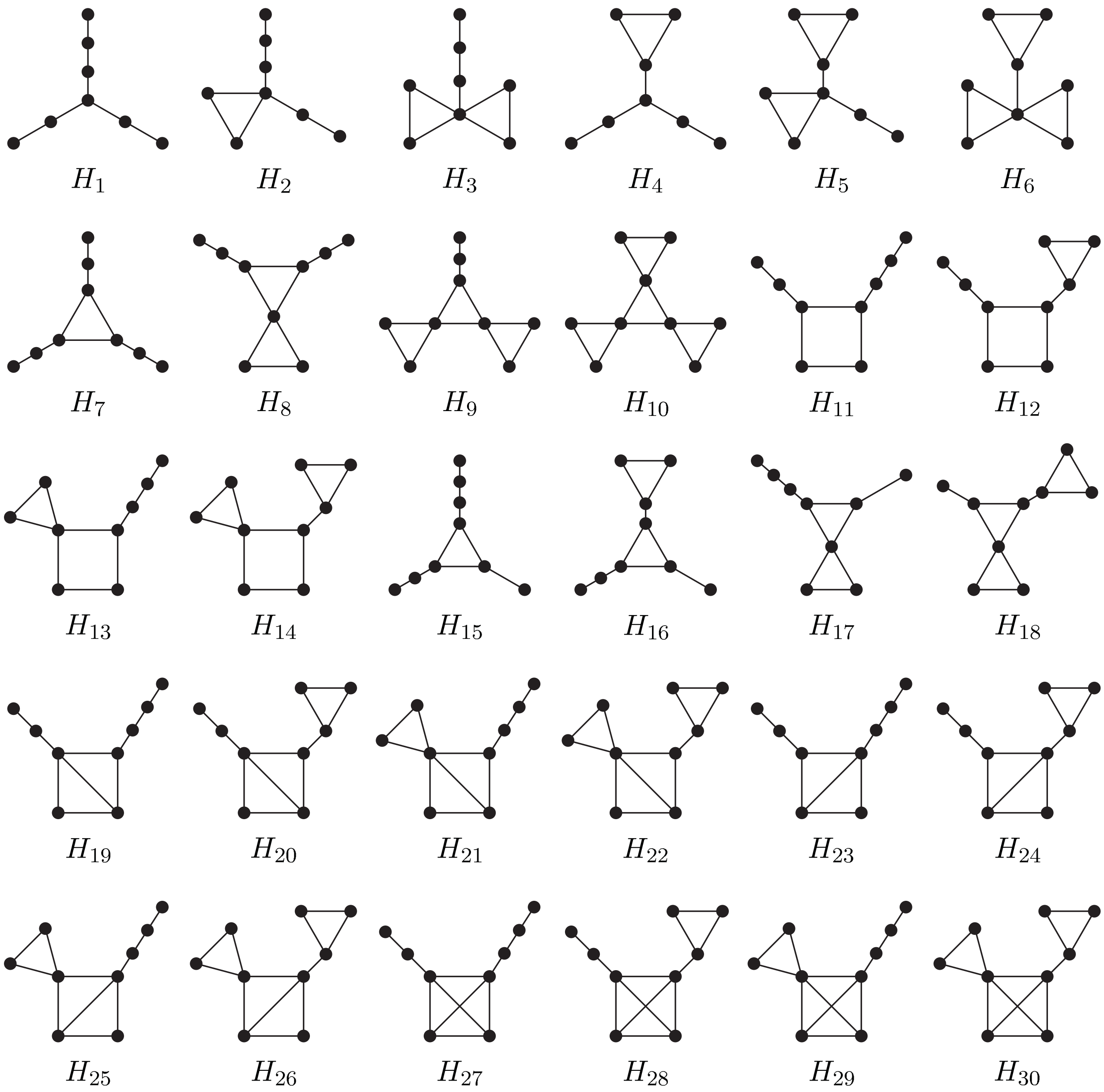

Figure 4: Graphs $H_1, H_2, \ldots, H_{30}$.

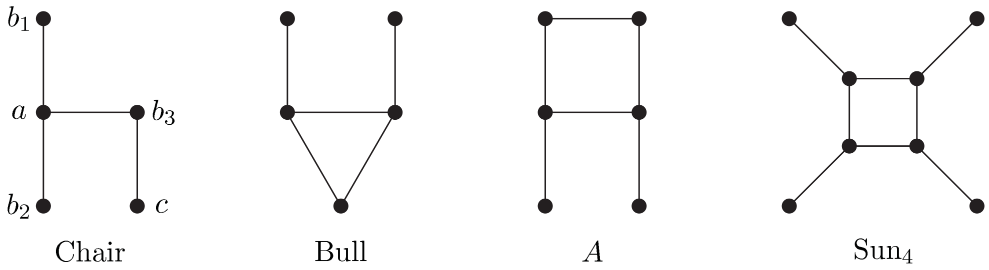

Figure 5: The graphs Chair, Bull, A and Sun_4.