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A Degree Constraint for Uniquely Hamiltonian Graphs

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ABSTRACT

A graph, G, is called uniquely Hamiltonian if it contains exactly one Hamilton cycle. We show that if $G = (V, E)$ is uniquely Hamiltonian then

$$\sum_{v \in V} \left(\frac{2}{3}\right)^{d(v)-\#(G)} \geq 1.$$

Where $\#(G) = 1$ if G has even number of vertices and 2 if G has odd number of vertices. It follows that every n-vertex uniquely Hamiltonian graph contains a vertex whose degree is at most $c \log_2 n + 2$ where $c = (\log_2 3 - 1)^{-1} \approx 1.71$ thereby improving a bound given by Bondy and Jackson [3].

1 Introduction

We only deal with simple graphs without loops or multiple edges. Most of the notation is standard and is borrowed from Bollobás' monograph on graph theory[2]. For a graph $G = (V, E)$, and a vertex $v \in V$ we say $v \sim_G w$ if $\{v, w\} \in E$ and let $N_G(v)$, the neighborhood of v, be the set of all vertices w such that $v \sim_G w$. $d_G(v)$ denotes the degree of a vertex v which is defined to be the size of it's neighborhood. The subscript G is ignored when there is no danger of confusion. A cycle in G is a sequence $v_1, \ldots, v_k$ of distinct vertices such that $v_i \sim v_{i+1}$ for all $i = 1, \ldots, k-1$ and $v_k \sim v_1$. A cycle is a Hamilton cycle if it contains all the vertices of G. Two Hamilton cycles are considered distinct if one is not a cyclic permutation of the other.

This paper is concerned with uniquely Hamiltonian graphs. As the name suggests, a graph is called uniquely Hamiltonian if it contains exactly one Hamilton cycle. Sheehan[8] has asked if every uniquely Hamiltonian graph contains a vertex of low degree. Entringer and Swart[6] construct infinitely many uniquely Hamiltonian graphs of minimum degree three. It is not known if there are any uniquely Hamiltonian graphs with minimum degree four. This is an intriguing open question.

In the other direction, Bondy and Jackson[3] have proved that every uniquely Hamiltonian graph contains a vertex of degree at most $c' \log_2 4n + 3$ where $c' = (2 - \log_2 3)^{-1} \approx 2.41$. This result improves the naïve argument based on Dirac's Theorem[5] that uniquely Hamiltonian graphs must have a vertex of degree at most $\frac{n}{2} + 1$ and an earlier observation of Jackson and Whitty[7] that uniquely Hamiltonian graphs must have a vertex of degree at most $\frac{n+9}{4}$. We further improve this result and show that every uniquely Hamiltonian graph contains a vertex of degree at most $c \log_2 n + 2$ where $c = (\log_2 3 - 1)^{-1} \approx 1.71$.

In fact, we can show that every uniquely Hamiltonian graph contains two vertices whose degrees do not exceed $c \log_2 n + 4$. These results are a corollary of the following degree constraint:

Theorem 1.1 *If $G = (V, E)$ is a uniquely Hamiltonian graph then*

$$\sum_{v \in V} \left(\frac{2}{3}\right)^{d(v) - \#(G)} \geq 1,$$

where $\#(G) = 1$ if $|V|$ is even and 2 otherwise.

This degree constraint says that if a uniquely Hamiltonian graph has many vertices of large degree then this must be compensated by having some vertices of reasonably small degree.

Our proof uses the probabilistic method[1]. The rest of the paper is organized as follows: In the next section we describe Thomassen's Lemma that states a sufficient condition for a graph to contain two distinct Hamilton cycles. Section 3 is devoted to proving Theorem 1.1 in the case when the number of vertices in G is even. In the last section we comment on how to deal with graphs with odd number of vertices and give some concluding remarks.

2 Thomassen's Lemma

Throughout this paper we will be working with a graph $G = (V, E)$ that has a Hamilton cycle $H = v_0, v_1, \ldots, v_{n-1}$. When there is no danger of confusion we will also denote the set of edges in the Hamilton cycle by H.

The main tool for dealing with uniquely Hamiltonian graphs is Thomassen's Lemma. Thomassen has applied this Lemma to obtain many interesting results about uniquely Hamiltonian graphs[4]. For example, he proves that there is no uniquely Hamiltonian regular graph of 300 or more vertices. In order to state this Lemma we assume that the edges, E, of G have been classified as follows: the ones that belong to the Hamilton cycle have been colored red and the remaining ones have been colored green. Let $S \subseteq V$, we say that S is *red-independent* if $\{v_i, v_{i+1}\} \not\subseteq S$; that is, if no edge of H has both its endpoints in S. We say that S green-dominates a vertex $v_i \in V$ if either v_i is in S or some green neighbor of v_i is in S; more formally:

$$S \cap ((N(v_i) \cup \{v_i\}) \setminus \{v_{i-1}, v_{i+1}\}) \neq \emptyset.$$

Furthermore, S is called a *green-dominating* set for G if it green-dominates all the vertices in G. Thomassen's Lemma gives a sufficient condition for the existence of another Hamilton cycle H' in G that is different from H.

Theorem 2.1 (Thomassen's Lemma) *If there exists a set $S \subseteq V$ that is red-independent and green-dominating then G is not uniquely Hamiltonian.*

Another stronger version of Thomassen's Lemma is also available. However, since we only use the above version we refer the interested reader to [4].

3 Proof of the main result

Let $G = (V, E)$ be a uniquely Hamiltonian graph on n vertices. We will assume that $n = 2m$ is an even number. At the end of the paper we comment on how our proofs can be adapted to the case when G has an odd number of vertices. Let H be the unique Hamilton cycle in G. We label the vertices of H as $x_0, y_0, x_1, y_1, \ldots, x_{m-1}, y_{m-1}$ and define

$$X = \{x_0, \ldots, x_{m-1}\} \text{ and } Y = \{y_0, \ldots, y_{m-1}\}.$$

We sometimes refer to the vertices of G that belong to X (respectively Y) as X-vertices (respectively Y-vertices). For a given vertex $v \in V$ let $N_H(v)$ denote the set of red neighbors of v; that is, the neighbors of v in the Hamilton cycle. Formally,

$$N_H(v) = \begin{cases} \{y_{i-1}, y_i\} \text{ if } v = x_i \text{ for some } i \\ \{x_i, x_{i+1}\} \text{ if } v = y_i \text{ for some } i. \end{cases}$$

Also, let

$$\hat{N}(v) = (N(v) \setminus N_H(v)) \cup \{v\}.$$

Trivially

$$|\hat{N}(v)| = d(v) - 1.$$

Since G is uniquely Hamiltonian we know that G does not satisfy the hypothesis of Thomassen's Lemma. Hence, if we choose a subset T of V according to *any* probability distribution then it is guaranteed that either T is not red-independent or it is not green-dominating. If the probability of this event can be estimated in terms of the degrees of vertices of G then one can obtain a degree constraint.

Towards this end we describe a probability distribution on the subsets of V. Our probability distribution is quite different from the ones used by Bondy and Jackson[3] and seems more natural. In not only yields a better bound, but it also seems to simplify the calculations.

To describe the probability distribution we describe how to construct a subset T with the following simple randomized procedure:

1. Construct a random subset, A, of X-vertices by choosing each $x_i \in X$ randomly and independently with probability $p = \frac{1}{3}$ (The choice $p = \frac{1}{3}$ will be justified by subsequent calculations).

2. Let B be the set of all Y-vertices that do not have a red neighbor in A. Formally let

$$B = Y \setminus N_H(A) = \{y_i \in Y : x_i \notin A \text{ and } x_{i+1} \notin A\}.$$

3. Output $T = A \cup B$.

This procedure clearly outputs a maximal red-independent set. Now, since G is uniquely Hamiltonian we know that T cannot be green-dominating. More precisely, by Theorem 2.1 we have:

$$\Pr[T \text{ is not a green-dominating set}] = 1.$$

In what follows we estimate $\Pr[T$ is not a green-dominating set$]$ in terms of degrees of vertices of G. Comparing this estimate with the above equation would yield our main theorem.

A vertex v is green-dominated by T if

$$\hat{N}(v) \cap T \neq \emptyset.$$

Let us define,

$$R(v) = \hat{N}(v) \cap X \text{ and } S(v) = \hat{N}(v) \cap Y.$$

We have,

$$\begin{aligned}
\Pr[v \text{ is not green-dominated by } T] &= \Pr[T \cap \hat{N}(v) = \emptyset] \\
&= \Pr[T \cap R(v) = \emptyset \text{ and } T \cap S(v) = \emptyset] \\
&= \Pr[A \cap R(v) = \emptyset \text{ and } B \cap S(v) = \emptyset].
\end{aligned}$$

The obvious thing to do is to estimate the probabilities of the two events; namely, $A \cap R(v) = \emptyset$ and $B \cap S(v) = \emptyset$ separately. Intuition tells us that these events should be negatively correlated for selecting more vertices from X reduces the number of vertices that are selected from Y. The following correlation inequality asserts that our intuition is indeed correct.

Lemma 3.1 *The events $A \cap R(v) = \emptyset$ and $B \cap S(v) = \emptyset$ are negatively correlated; that is,*

$$\Pr[A \cap R(v) = \emptyset \textit{ and } B \cap S(v) = \emptyset] \leq \Pr[A \cap R(v) = \emptyset] \Pr[B \cap S(v) = \emptyset].$$

Proof. Define

$$\mathcal{F}(v) = \{A \subseteq X : A \cap R(v) = \emptyset\}$$

and notice that $\mathcal{F}(v)$ is a monotonically decreasing subfamily of 2^X; that is, $A \in \mathcal{F}(v)$ and $A' \subseteq A$ implies that $A' \in \mathcal{F}(v)$.

Next consider a vertex y_i in Y. If the vertex y_i is *not* included in B then at least one red neighbor of y_i is included in A. Hence if we define,

$$\mathcal{G}_i = \{A \subseteq X : \{x_i, x_{i+1}\} \cap A \neq \emptyset\}.$$

Hence the event $y_i \in T$ is same as the event $A \in \mathcal{G}_i$. $\mathcal{G}_i$ is a monotonically increasing subfamily of 2^X; that is, if $A \in \mathcal{G}_i$ and $A' \supseteq A$ then $A' \in \mathcal{G}_i$. Next let us define

$$\mathcal{G}(v) = \bigcup_{i:\, y_i \in S(v)} \mathcal{G}_i$$

then $\mathcal{G}(v)$ being a union of monotonically increasing subfamilies of 2^X is itself a monotonically increasing subfamily of 2^X. The event $B \cap S = \emptyset$ is the same as the event

$$A \in \mathcal{G}(v).$$

As $\mathcal{F}(v)$ is monotonically decreasing and $\mathcal{G}(v)$ is monotonically increasing hence by the FKG-inequality (See [1]) we have

$$\Pr[A \in \mathcal{F}(v) \cap \mathcal{G}(v)] \leq \Pr[A \in \mathcal{F}(v)] \Pr[A \in \mathcal{G}(v)]. \quad \square$$

The following observation is immediate.

Lemma 3.2

$$\Pr[A \cap R(v) = \emptyset] = (1-p)^{|R(v)|}.$$

$\square$

Now we estimate the $\Pr[B \cap S(v) = \emptyset]$.

Lemma 3.3 *For any vertex* v

$$\Pr[B \cap S(v) = \emptyset] \le \alpha^{|S(v)|},$$

where $\alpha = \frac{p+\sqrt{4p-3p^2}}{2}$.

Proof. For $0 \le i \le j < m-1$, consider an interval, $I[i,j] = \{y_i, \ldots, y_j\}$, of consecutive Y-vertices. Note that the set of red neighbors of $I[i,j]$ forms an interval $J[i,j+1] = \{x_i, x_{i+1}, \ldots, x_{j+1}\}$ of X-vertices. Hence B contains a vertex from $I[i,j]$ if A does not does not "miss" two consecutive vertices from the interval $J[i,j+1] = \{x_i, \ldots, x_{j+1}\}$. Let $\mathrm{Con}(i,j)$ denote the event that A does not "miss" any two consecutive vertices from the interval $J[i,j]$; that is,

$$A \cap \{x_s, x_{s+1}\} \ne \emptyset \text{ for all } s = i, \ldots, j-1$$

and let $\mathrm{Inter}(i,j)$ denote the event that B intersects the interval $I[i,j]$; that is $B \cap I[i,j] \ne \emptyset$. We have

$$\mathrm{Inter}(i,j) \text{ if and only if } \mathrm{Con}(i,j+1).$$

It is readily seen that $\Pr[\mathrm{Con}(i,j)]$ and $\Pr[\mathrm{Inter}(i,j)]$ depend only on the length of the intervals; that is, $j-i+1$. Hence for $k = j-i+1$ if we define, $P_k = \Pr[\mathrm{Inter}(i,j)]$ and $Q_k = \Pr[\mathrm{Con}(i,j)]$ then we have

$$P_k = Q_{k+1}.$$

Furthermore, it is easy to see that Q_k satisfies the following Fibonacci-type recurrence:

$$Q_k = pQ_{k-1} + p(1-p)Q_{k-2}.$$

To see this consider $J[0,k-1]$. In order to construct a set does not miss any consecutive vertices $J[0,k-1]$ we can choose x_{k-1} then construct a set that does not miss any consecutive vertices from $J[0,k-2]$ or we can not choose x_{k-1}, choose x_{k-2} and then construct a set that does not miss any vertices from $J[0,k-3]$. Careful examination for the case $k=1$ and $k=2$ shows that Q_k is the solution to the recurrence

$$\begin{aligned} Q_0 &= 1, \\ Q_1 &= 1, \\ Q_k &= pQ_{k-1} + p(1-p)Q_{k-2} \text{ for } k \ge 2. \end{aligned}$$

The solution is given by:

$$Q_k = w\alpha^k + (1-w)\overline{\alpha}^k$$

where $\alpha = \frac{p+\sqrt{4p-3p^2}}{2}$, $\overline{\alpha} = \frac{p-\sqrt{4p-3p^2}}{2}$ and $w = \frac{1}{2} + \frac{2-p}{2\sqrt{4p-3p^2}}$. As $\overline{\alpha} < \alpha$ and $0 < w < 1$ we have

$$Q_k \le \alpha^k.$$

So far we have shown that

$$\Pr[\mathrm{Con}(i,j)] \le \alpha^{j-i+1}.$$

To complete the proof of our lemma we observe that every $S(v)$ that is a proper subset of Y, after appropriate cyclic relabelling of Y-vertices, is a union of disjoint intervals; that is,

$$S(v) = I[i_1, j_1] \cup I[i_2, j_2] \cup \cdots \cup I[i_r, j_r]$$

where

$$i_s \leq j_s \text{ and } j_{s+1} > i_s + 1 \text{ for } s = 1, \ldots, r-1.$$

Furthermore, $0 \leq i_1$ and $j_r < m - 1$. The size of $S(v)$ is given by:

$$|S(v)| = \sum_{s=1}^{r} (j_s - i_s + 1).$$

Now $B \cap S(v) = \emptyset$ if and only if

$$\bigwedge_{s=1}^{r} \mathrm{Inter}(i_s, j_s).$$

We now have,

$$\begin{aligned}
\Pr[B \cap S(v) = \emptyset] &= \Pr[\bigwedge_{s=1}^{r} \mathrm{Inter}(i_s, j_s)] \\
&= \Pr[\bigwedge_{s=1}^{r} \mathrm{Con}(i_s, j_s + 1)] \\
&= \prod_{s=1}^{r} \Pr[\mathrm{Con}(i_s, j_s + 1)] \\
&\leq \prod_{s=1}^{r} \alpha^{j_s - i_s + 2} \\
&= \alpha^{|S(v)| + r} \\
&\leq \alpha^{|S(v)| + 1}
\end{aligned}$$

Where the third equality follows from the fact that the X-vertices are chosen independently with probability p and hence the events in question are all independent.

The last annoying detail of the proof is the case when $S(v) = Y$. In this case, we take any subset S' of $S(v)$ such that $|S'| = |S(v)| - 1$ and observe that

$$\Pr[B \cap S(v) = \emptyset] \leq \Pr[B \cap S' = \emptyset] \leq \alpha^{|S'|+1} \leq \alpha^{|S(v)|}.$$

□

When we combine bounds of Lemma 3.2 and 3.3 and use our correlation inequality we realize that we have shown that

$$\Pr[v \text{ is not green dominated }] \leq (1-p)^{|R(v)|} \alpha^{|S(v)|},$$

where α is an increasing function of p. Thus if we want the best bound in terms of $d(v) = |S(v)| + |R(v)| + 1$, we should choose p so that the bounds in both Lemma 3.2 and 3.3 are simultaneously small. Thus we should choose $\alpha = 1 - p$. This dictates our choice of p to be $\frac{1}{3}$ which gives us the following neater result:

Corollary 3.1 *If $p = \frac{1}{3}$ then for any vertex*

$$\Pr[v \textit{ is not green-dominated }] \leq \left(\frac{2}{3}\right)^{d(v)-1}.$$

$\square$

Now it is a simple matter to obtain our main result.

Proof. (of Theorem 1.1) If we take $p = \frac{1}{3}$ then we have

$$\begin{aligned} 1 &= \Pr[T \text{ is not a green-dominating set}] \\ &\leq \sum_{v \in V} \Pr[v \text{ is not green-dominated}] \\ &\leq \sum_{v \in V} \left(\frac{2}{3}\right)^{d(v)-1}. \quad \square \end{aligned}$$

4 Dealing with odd graphs

Now let us comment on how to deal with graphs with an odd number of vertices. Let $n = 2m + 1$ be the number of vertices in G. We label a vertex z and label the remaining vertices alternatively as $x_0, y_0, x_1, y_1 \ldots, x_{m-1}, y_{m-1}$. One can now generate the maximal red-independent set T as before from X- and Y-vertices. z is never included in the red-independent set. The same analysis goes through but the degrees of the vertices that were connected to z have to be adjusted by 1. We leave the details to the reader. This degree constraint gives us the following corollaries.

Corollary 4.1 *Every n-vertex uniquely Hamiltonian graph contains a vertex with degree at most $c \log n + 2$ where $c = (\log_2 3 - 1)^{-1}$.* $\square$

Corollary 4.2 *Every n-vertex uniquely Hamiltonian graph contains two vertices of degree at most $c \log n + 4$.* $\square$

Corollary 4.2 does not follow from Theorem 1.1 directly. The argument fails when an odd graph contains a vertex of degree exactly 2 and Theorem 1.1 becomes useless. However, we observe that if G has a vertex of minimum degree 2 then we can label that vertex z and use the procedure described above and obtain a degree constraint that would give a result that is slightly stronger than this corollary.

This method also gives some useful information about uniquely Hamiltonian graphs. We can also say something about the location of the small degree vertices. For example, one can show that every uniquely Hamiltonian graph either contains two vertices of small degree that adjacent in the Hamilton cycle or it contains reasonable number of vertices that have small degree.

Theorem 4.1 *Let $b > c$ and $G = (V, E)$ be an n-vertex uniquely Hamiltonian graph and H be the unique Hamilton cycle. Either G contains two vertices v, w that are consecutive in H with*

$$d(v), d(w) \leq b \log n$$

or there are at least $\frac{(b-c)\log n}{2} - 2$ vertices of degree at most $b \log n$.

Proof. (Outline) Suppose some graph contains $k < \frac{(b-c)\log n}{2} - 2$ vertices of degree at most $b \log n$ no two of which are consecutive. Let W be this set and let us delete these vertices from the graph along with all their red neighbors. Now the red edges of G form a union of paths. Once again we can label the vertices as X-vertices and Y-vertices and generate a red-independent set as before. However, this time we add W to our independent set T. Similar analysis as before shows that

$$\sum_{v \in V \setminus W} \left(\frac{2}{3}\right)^{d(v)-2-2k} \geq 1.$$

This leads to a contradiction. $\square$

Our method is based on the probabilistic method and Thomassen's Lemma. Once can ask how far can one push this method. Bondy and Jackson[3] have constructed uniquely Hamiltonian graphs that do not satisfy the hypothesis of Thomassen's Lemma and have minimum degree $\lfloor \log_2 n \rfloor$. Hence, one cannot hope to prove a bound that is better than $\lfloor \log_2 n \rfloor$ using this method. Although, to prove that every uniquely Hamiltonian graph must have a vertex of constant degree asks us to abandon Thomassen's Lemma, we still think it would be interesting to see a new probabilistic proof based on Thomassen's Lemma that improves our result.

References

[1] N. Alon and J. Spencer. *The Probablistic Method.* John Wiley and Sons, New York, 2nd edition, 2000.

[2] Béla Bollobás. *Modern Graph Theory.* Springer-Verlag, New York, 1998.

[3] J. A. Bondy and B. Jackson. Vertices of small degree in uniquely hamiltonian graphs. *Journal of Combinatorial Theory (B)*, 74:265–275, 1998.

[4] Thomassen C. Chords of longest cycles in cubic graphs. *Journal of Combinatorial Theory (B)*, 71(2):211–214, 1997.

[5] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.*, 2:69–81, 1952.

[6] R. C. Entringer and H. Swart. Spanning cycles of nearly cubic graphs. *Journal of Combinatorial Theory (B)*, 29:303–309, 1980.

[7] B. Jakson and R. W. Whitty. A note concerning graphs with unique f-factors. *Journal of Graph Theory*, 13:577–580, 1989.

[8] J. Sheehan. Problem section. In C. St. J.A. Nash-Williams and J. Sheehan, editors, *Fifth British Combinatorial Conference, Congressus Numerantium XV, Utilitas Mathematica*, page 691, Winnipeg, 1975. Publ. Corp.