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The Relative Clique-Width of a Graph

by

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ABSTRACT

The tree-width of graphs is a well-studied notion the importance of which is partly due to the fact that many hard algorithmic problems can be solved efficiently when restricted to graphs of bounded tree-width. The same is true for the clique-width which is a relatively young notion generalizing tree-width in the sense that graphs of bounded tree-width have bounded clique-width.

Whereas tree-decompositions that are used to define tree-width are a very intuitive and easily visualizable way to represent the global structure of a graph, the clique-width is much harder to grasp intuitively. To better understand the nature of the clique-width, we introduce the notion of relative clique-width and study two algorithmical problems related to it. In conjunction, these problems would allow to determine the clique-width.

For one of the problems, which is to determine the relative clique-width, we propose a polynomial-time factor 2 approximation algorithm and also show an exact solution in a natural special case. The study of the other problem, which is left open in the paper, has brought us to an alternative and transparent proof of the known fact that graphs of bounded tree-width have bounded clique-width.

1 Introduction

The clique-width and the tree-width of graphs are two well-known complexity measures in the sense that many hard algorithmical problems can be solved efficiently for classes of graphs where one of these two parameters is bounded [2, 4, 6, 7, 10].

The tree-width of a graph G is the minimum width of a tree-decomposition of G. Whereas a tree-decomposition is a very intuitive and easily visualizable way to represent the global structure of a graph, the clique-width is much harder to grasp intuitively[1]: It relates to operations constructing labeled versions of graphs using as few labels as possible. Whenever two vertices carry the same label, they are treated equally by all subsequent operations where the term 'subsequent' indicates that there is some order in which the operations are executed. Therefore, the minimum number of labels seems to measure some kind of similarity of vertices with respect to a certain ordering.

The attempt to make these vague statements precise was the starting point of the present paper. Our main idea is to relate the clique-width to colorings of graphs induced by some similarity relations. Before we proceed, we collect some notation and terminology.

We consider finite, simple graphs $G = (V, E)$ with vertex set V and edge set E. The neighbourhood of a vertex $u \in V$ in the graph G is denoted by $N_G(u)$. For a set $U \subseteq V$ the subgraph of G induced by U is denoted by $G[U]$. The chromatic number of a graph G is denoted by $\chi(G)$.

For two finite sets C and V let $\mathcal{T}(C, V)$ denote the set of all well-formed terms t written with the nullary symbols $i(v)$ for $i \in C$ and $v \in V$, the unary symbols $\eta_{i,j}$ and $\rho_{i \to j}$ for $i, j \in C$ with $i \neq j$ and the commutative binary symbol $\oplus$ such that for every $v \in V$ the term t contains exactly one symbol of the form $i(v)$ (cf. [5]).

If the symbol $i(v)$ corresponds to the creation of a vertex v with label i, the symbol $\eta_{i,j}$ corresponds to the creation of edges between every pair of vertices such that one has label i and one has label j, the symbol $\rho_{i \to j}$ corresponds to changing the label of all vertices with label i to label j and $\oplus$ corresponds to the disjoint union, then every term $t \in \mathcal{T}(C, V)$ creates a labeled graph $val(t) = (G, \gamma)$ with $G = (V, E)$ and $\gamma : V \to C$. Note that, in view of this interpretation, it makes sense to assume $\oplus$ to be commutative.

The clique-width $\mathrm{cw}(G)$ of a graph $G = (V, E)$ is the minimum cardinality of a set C such that there exists a $t \in \mathcal{T}(C, V)$ creating G, i.e.

$$\mathrm{cw}(G) = \min\{|C| \mid \exists t \in \mathcal{T}(C, V) \text{ and } \gamma : V \to C \text{ with } val(t) = (G, \gamma)\}.$$

Similarly, let $\mathcal{R}(V)$ denote the set of all well-formed terms r written with the nullary symbols v for $v \in V$ and the commutative binary symbol $\oplus$ such that for every $v \in V$ the term r contains exactly one symbol of the form v. Note that every term in $\mathcal{R}(V)$ encodes a unique rooted binary tree whose leaves are the elements of V. Since $\oplus$ is commutative, the choice which of the two children of a non-leaf of this tree is the left child and which is the right child is arbitrary.

[1] This subjective statement is at least true for the authors.

We shall now relate $\mathcal{T}(C,V)$ to $\mathcal{R}(V)$. Therefore, let $t \in \mathcal{T}(C,V)$. If we replace in t every symbol $i(v)$ with the symbol v and delete all symbols $\eta_{i,j}$ and $\rho_{i\to j}$, then we obtain the *reduced term* $red(t) \in \mathcal{R}(V)$ of t. Intuitively, $red(t)$ arises from t by forgetting all information about the labeling, the creation of edges and the relabeling of vertices. The term $red(t)$ merely retains the information about the relative order in which the vertices are created by t. The definition is best clarified with the help of an example. For the term $t \in \mathcal{T}(\{1,2,3\},\{v_1,v_2,...,v_7\})$ with

$$t = \eta_{2,3}(((\rho_{2\to 1}(\eta_{2,3}((\eta_{1,2}(1(v_1)\oplus 2(v_2)))\oplus 3(v_3))))\oplus 2(v_4))\oplus(\eta_{1,3}(\eta_{1,2}(1(v_5)\oplus 2(v_6)))\oplus 3(v_7)))$$

we obtain

$$\begin{aligned} red(t) &= ((((((((v_1 \oplus v_2)) \oplus v_3))) \oplus v_4) \oplus (((v_5 \oplus v_6)) \oplus v_7)) \\ &= (((v_1 \oplus v_2) \oplus v_3) \oplus v_4) \oplus ((v_5 \oplus v_6) \oplus v_7). \end{aligned}$$

The graph created by t and the rooted binary tree encoded by $red(t)$ are shown in the left and right of Figure 1, respectively.

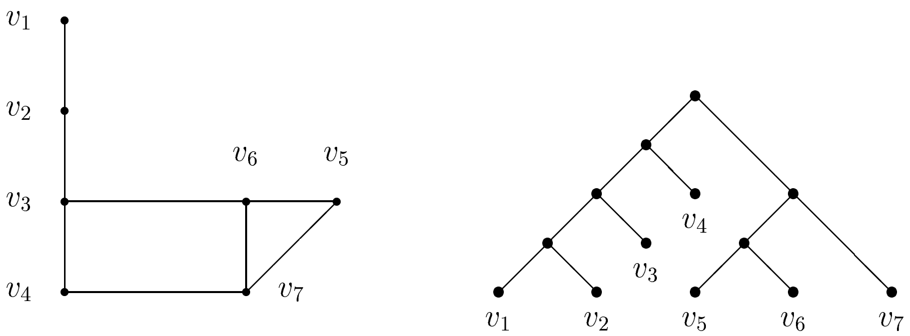

Figure 1

For a graph $G=(V,E)$ and $r \in \mathcal{R}(V)$ we define *the relative clique-width of* G *with respect to* r as

$$\mathrm{cw}(G,r) = \min\{|C| \mid \exists t \in \mathcal{T}(C,V) \text{ and } \gamma : V \to C \text{ with } val(t) = (G,\gamma) \text{ and } red(t) = r\},$$

i.e. $\mathrm{cw}(G,r)$ is the minimum number of labels used by a term t with $val(t) = (G,\gamma)$ which creates the vertices of G as specified by r. The following result is straightforward from the definitions.

Proposition 1 *If* $G=(V,E)$ *is a graph and* $r \in \mathcal{R}(V)$, *then*

(i) $\mathrm{cw}(G) \le \mathrm{cw}(G,r) \le |V|$ *and*

(ii) $\mathrm{cw}(G) = \min\{\mathrm{cw}(G,s) \mid s \in \mathcal{R}(V)\}$.

In view of Proposition 1 we are naturally led to the following two optimization problems.

Determining the relative clique-width
Input: A graph $G = (V, E)$ and a term $r \in \mathcal{R}(V)$.
Problem: Find a term $t \in \mathcal{T}(C, V)$ for some set C with $val(t) = (G, \gamma)$ and $red(t) = r$ that minimizes $|C|$.

Finding a good reduced term
Input: A graph $G = (V, E)$.
Problem: Find a term $r \in \mathcal{R}(V)$ that minimizes $\mathrm{cw}(G, r)$.

By Proposition 1 and the definitions, the optimum value of the problem Determining the relative clique-width is $\mathrm{cw}(G, r)$ and the optimum value of the problem Finding a good reduced term is $\mathrm{cw}(G)$.

In the next section we prove lower bounds on the relative clique-width in terms of the chromatic number. In Section 3 we use these bounds and prove — among other results — that Determining the relative clique-width can be approximately solved in polynomial time within a factor of 2. In Section 4 we consider terms of a special structure for which relative clique-width can be determined exactly in polynomial time. Finally, in Section 5, we use tree-decompositions to construct terms of bounded relative clique-width.

2 Lower Bounds on $cw(G, r)$

Throughout this section we assume a graph $G = (V, E)$ and a term $r \in \mathcal{R}(V)$ to be fixed. We introduce some notation that will be used throughout the rest of the paper.

For a subterm s of r let V_s denote the set of vertices that appear in s. Let the graph

$$H_s = (V(H_s), E(H_s))$$

be such that $V(H_s) = V_s$ and $uv \in E(H_s)$ for $u, v \in V_s$ if and only if $N_G(u) \setminus V_s \neq N_G(v) \setminus V_s$.

Lemma 1 *If s is a subterm of r, then H_s is a complete multipartite graph and $cw(G, r) \geq \chi(H_s)$.*

Proof: If $u, v, w \in V_s$ with $N_G(u) \setminus V_s = N_G(v) \setminus V_s$ and $N_G(v) \setminus V_s = N_G(w) \setminus V_s$, then $N_G(u) \setminus V_s = N_G(w) \setminus V_s$. This implies that the complement of H_s is P_3-free and thus a union of cliques. Therefore, H_s is a complete multipartite graph.

Let $t \in \mathcal{T}(C, V)$ be such that $val(t) = (G, \gamma)$ and $red(t) = r$. Let t' be the smallest subterm of t such that $red(t') = s$. If $|C| < \chi(H_s)$, then there are two vertices $u, v \in V_s$ with $uv \in E(H_s)$ that will permanently bear the same label after the operations prescribed by t' have been carried out. This immediately implies the contradiction $N_G(u) \setminus V_s = N_G(v) \setminus V_s$ and the proof is complete. $\Box$

By Lemma 1, the chromatic number of H_s equals just the number of partite sets of H_s. We denote these partite sets by

$$U(H_s,1),U(H_s,2),...,U(H_s,\chi(H_s)).$$

Lemma 2 *If $s=s_1\oplus s_2$ is a subterm of r, then every partite set of H_s is the union of partite sets of H_{s_1} and H_{s_2}.*

Proof: This follows immediately from the observation that all these graphs are complete multipartite graphs and H_s arises from the disjoint union $H_{s_1}\cup H_{s_2}$ by deleting edges totally within V_{s_1} and V_{s_2} and adding new edges between V_{s_1} and V_{s_2}. $\Box$

Let $s=s_1\oplus s_2$ be a subterm of r. We define the graph

$$\tilde{H}_s=(V(\tilde{H}_s),E(\tilde{H}_s))$$

such that $V(\tilde{H}_s)=V_s$ and $uv\in E(\tilde{H}_s)$ for $u,v\in V_s$ with $u\neq v$ if and only if

(i) either $u\in U(H_{s'},i)$ and $v\in U(H_{s'},j)$ for some $s'\in\{s,s_1,s_2\}$ and $1\leq i,j\leq\chi(H_{s'})$ with $i\neq j$

(ii) or $u\in U(H_{s_1},j)$, $v\in U(H_{s_2},k)$, $U(H_{s_1},j)\cup U(H_{s_2},k)\subseteq U(H_s,i)$ for some $1\leq i\leq\chi(H_s)$, $1\leq j\leq\chi(H_{s_1})$ and $1\leq k\leq\chi(H_{s_2})$ and

$$((N_G(u)\cap V_{s_2})\setminus N_G(v))\cup((N_G(v)\cap V_{s_1})\setminus N_G(u))\neq\emptyset.$$

Lemma 3 *If $s=s_1\oplus s_2$ is a subterm of r, then $cw(G,r)\geq\chi(\tilde{H}_s)$ and*

$$\max\{\chi(H_s),\chi(H_{s_1}),\chi(H_{s_2})\}\leq\chi(\tilde{H}_s)\leq\chi(H_{s_1})+\chi(H_{s_2}).$$

Proof: Let $t\in\mathcal{T}(C,V)$ be such that $val(t)=(G,\gamma)$ and $red(t)=r$. Let t' be the smallest subterm of t such that $red(t')=s_1\oplus s_2$.

For contradiction, we assume that $|C|<\chi(\tilde{H}_s)$. This implies that there are two vertices $u,v\in V_s$ with $uv\in E(\tilde{H}_s)$ that will permanently bear the same label after the operations prescribed by t' have been carried out. Note that, by the choice of t', none of the edge between V_{s_1} and V_{s_2} has been created by the operations prescribed by t'.

If uv corresponds to an edge as in (i) of the definition of $\tilde{H}_s$, then we obtain a similar contradiction as in the proof of Lemma 1. If uv corresponds to an edge as in (ii) of the definition of $\tilde{H}_s$, then the operations following those prescribed by t' cannot create the correct adjacencies between V_{s_1} and V_{s_2}. This contradiction completes the proof of the first statement.

The edges as in (i) of the definition of $\tilde{H}_s$ imply $\chi(\tilde{H}_s)\geq\max\{\chi(H_s),\chi(H_{s_1}),\chi(H_{s_2})\}$. The inequality $\chi(\tilde{H}_s)\leq\chi(H_{s_1})+\chi(H_{s_2})$ follows by combining optimal colorings of H_{s_1} and H_{s_2} that use disjoint sets of colors. This completes the proof. $\Box$

3 Determining the Relative Clique-Width

The following is the main result of this section.

Theorem 1 *If $G = (V, E)$ is a graph and $r \in \mathcal{R}(V)$, then there exists a term $t(r) \in \mathcal{T}(C, V)$ for some set C with*

$$|C| \quad = \quad \max\{\chi(H_{s_1}) + \chi(H_{s_2}) \mid s_1 \oplus s_2 \textit{ is a subterm of } r\} \tag{1}$$

such that $val(t(r)) = (G, \gamma)$ and $red(t(r)) = r$.

Furthermore, $t(r)$ can be constructed in polynomial time given the graph $G = (V, E)$ and the term r as input.

Proof: Let $G = (V, E)$ and r be as in the statement of the theorem. We describe a recursive construction for $t(r)$. Therefore, let s be an arbitrary subterm of r. As in the last section, let V_s denote the set of vertices that appear in s. We assume that for every proper subterm s' of s we are given a term $t(s') \in \mathcal{T}(C, V_{s'})$ such that

(i) $val(t(s')) = (G[V_{s'}], \gamma_{t(s')})$,

(ii) $red(t(s')) = s'$ and

(iii) $\gamma_{t(s')}$ assigns the same label to two vertices in $V_{s'}$ if and only if they lie in the same partite set of $H_{s'}$, i.e. $|\gamma_{t(s')}(V_{s'})| = \chi(H_{s'})$.

Now, we construct a term $t(s) \in \mathcal{T}(C, V_s)$ satisfying the conditions (i) to (iii) as follows.

If $s = v$ for some $v \in V$, let $t(s) = i(v)$ for some $i \in C$.

If $s = s_1 \oplus s_2$, then, by (1), we may assume that $t(s_1) \in \mathcal{T}(C, V_{s_1})$ and $t(s_2) \in \mathcal{T}(C, V_{s_2})$ are such that — additionally to conditions (i) to (iii) —

$$\gamma_{t(s_1)}(V_{s_1}) \cap \gamma_{t(s_2)}(V_{s_2}) \quad = \quad \emptyset. \tag{2}$$

The term $t(s)$ will be of the form

$$t(s) = \beta_l(\beta_{l-1}(...\beta_1(\alpha_k(\alpha_{k-1}(...\alpha_1(t(s_1) \oplus t(s_2)))))))$$

where $\alpha_1, \alpha_2, ..., \alpha_k$ are of the form $\eta_{i,j}$ for $i, j \in C$ with $i \neq j$ and $\beta_1, \beta_2, ..., \beta_l$ are of the form $\rho_{i \to j}$ for $i, j \in C$ with $i \neq j$.

The operations $\alpha_1, \alpha_2, ..., \alpha_k$ will create all edges with one endpoint in V_{s_1} and one endpoint in V_{s_2}. Note that this is clearly possible in view of condition (iii), (2) and the definition of the graphs H_{s_1} and H_{s_2}.

After all such edges have been created, the operations $\beta_1, \beta_2, ..., \beta_l$ will ensure condition (iii) for $t(s)$. Note that this is possible in view of condition (iii), (2) and Lemma 2. Conditions (i) and (ii) for $t(s)$ follow immediately.

The construction of the graphs H_s can clearly be done in polynomial time. Furthermore, by Lemma 1, their partite sets — and thus optimal colorings — can easily be determined in polynomial time. Since r has exactly $(2|V|-1)$ subterms, we obtain an overall polynomial running time and the proof is complete. $\Box$

Let $G=(V,E)$ and $r \in \mathcal{R}(V)$ be an instance of Determining the relative clique-width. Lemma 1 and Theorem 1 imply

$$\begin{aligned} \mathrm{cw}(G,r) &\leq \max\{\chi(H_{s_1})+\chi(H_{s_2}) \mid s_1 \oplus s_2 \text{ is a subterm of } r\} \\ &\leq 2\max\{\chi(H_s) \mid s \text{ is a subterm of } r\} \\ &\leq 2\mathrm{cw}(G,r). \end{aligned}$$

Thus Theorem 1 immediately implies the following.

Corollary 1 *There is a polynomial-time factor 2 approximation algorithm for* Determining the relative clique-width.

4 A Special Case: Caterpillar Terms

A term $r \in \mathcal{R}(V)$ is called a *caterpillar term* if $\min\{|V_{s_1}|,|V_{s_2}|\}=1$ for every subterm $s_1 \oplus s_2$ of r. The term '*caterpillar*' clearly refers to the structure of the binary tree encoded by r. Intuitively, a term $t \in \mathcal{T}(C,V)$ with $val(t)=(G,\gamma)$ for which $red(t)$ is a caterpillar term adds the vertices of G one by one to the part of G that has been constructed so far.

Lemma 4 *Let $G=(V,E)$ be a graph and let $r \in \mathcal{R}(V)$ be a term. If $s=s_1 \oplus s_2$ is a subterm of r with $V_{s_2}=\{v\}$, then*

(i) $\chi(\tilde{H}_s) \in \{\chi(H_{s_1}), \chi(H_{s_1})+1\}$.

(ii) There is a coloring ϕ of $\tilde{H}_s$ using $\chi(\tilde{H}_s)$ colors that assigns the same color to two vertices in V_{s_1} if and only if they lie in the same partite set of H_{s_1}.

(iii) If $\chi(\tilde{H}_s)=\chi(H_{s_1})+1$, then ϕ assigns a color to v not used in V_{s_1}, i.e. $\phi(v) \not\in \phi(V_{s_1})$, and if $\chi(\tilde{H}_s)=\chi(H_{s_1})$, then ϕ assigns the same color to v as to the vertices in the unique partite set $U(H_{s_1},j)$, $1 \leq j \leq \chi(H_{s_1})$ of H_{s_1} with $U(H_{s_1},j) \cup \{v\} \subseteq U(H_s,i)$ for some $1 \leq i \leq \chi(H_s)$ and $U(H_{s_1},j) \cap N_G(v)=\emptyset$.

Proof: By Lemma 3, $\chi(H_{s_1}) \leq \chi(\tilde{H}_s) \leq \chi(H_{s_1})+\chi(H_{s_2})=\chi(H_{s_1})+1$ which implies (i).

If $\chi(\tilde{H}_s)=\chi(H_{s_1})+1$, then the statement is obvious. Therefore, we assume $\chi(\tilde{H}_s)=\chi(H_{s_1})$. Let $\phi: V_s \to \{1,2,...,\chi(\tilde{H}_s)\}$ be an optimal coloring of $\tilde{H}_s$.

By the definition of $\tilde{H}_s$ and Lemma 1, $|\{\phi(u) \mid u \in V_{s_1}\}| \geq \chi(H_{s_1})$ and ϕ assigns the same color to two vertices in V_{s_1} if and only if they lie in the same partite set of H_{s_1}. Thus $\phi(v) \in \{\phi(u) \mid u \in V_{s_1}\}$. Let $1 \leq i \leq \chi(H_s)$ and $1 \leq j \leq \chi(H_{s_1})$ be such that $\phi(v)=\phi(u)$ for all $u \in U(H_{s_1},j)$ and $U(H_{s_1},j) \cup \{v\} \subseteq U(H_s,i)$.

By the definition of $\tilde{H}_s$, $U(H_{s_1},j) \cap N_G(v)=\emptyset$. If there is some $1 \leq j' \leq \chi(H_{s_1})$ such that $U(H_{s_1},j') \subseteq U(H_s,i)$, then the definition of H_{s_1} implies that $U(H_{s_1},j') \subseteq N_G(v)$. Therefore, the index j defined above is uniquely determined and the proof is complete. $\Box$

Now, let $G = (V, E)$ be a graph and let $r \in \mathcal{R}(V)$ be a caterpillar term. In view of the simple structure of the optimal coloring of $\tilde{H}_s$ for every subterm $s = s_1 \oplus s_2$ of r whose existence is guaranteed by Lemma 4, one can clearly find such a coloring in polynomial time.

Furthermore, some small obvious modifications of the construction in the proof of Theorem 1 yield a term $t(r) \in \mathcal{T}(C, V)$ for some C with

$$|C| = \max\{\chi(\tilde{H}_s) \mid s = s_1 \oplus s_2 \text{ is a subterm of } r\}.$$

such that $val(t(r)) = (G, \gamma)$ and $red(t(r)) = r$. Altogether, using Lemma 3, we obtain the following result.

Theorem 2 *If $G = (V, E)$ and $r \in \mathcal{R}(V)$ is an instance of* DETERMINING THE RELATIVE CLIQUE-WIDTH *such that r is a caterpillar term, then*

$$\text{cw}(G, r) = \max\{\chi(\tilde{H}_s) \mid s = s_1 \oplus s_2 \textit{ is a subterm of } r\} \tag{3}$$

and DETERMINING THE RELATIVE CLIQUE-WIDTH *can be solved exactly in polynomial time.*

We do not believe that (3) can hold without any further restrictions such as the condition $\min\{|V_{s_1}|, |V_{s_2}|\} = 1$. The main problem for a construction of a term $t(r)$ defining G using this smaller number of labels is that for every subterm $s = s_1 \oplus s_2$ of r one has to find optimal colorings of $\tilde{H}_{s_1}$ and $\tilde{H}_{s_2}$ that may be transformed into an optimal coloring of $\tilde{H}_s$ by operations of the form $\rho_{i \to j}$.

For the rest of the section we investigate how much the smallest possible value of $\text{cw}(G, r)$ for a caterpillar term r can differ from $\text{cw}(G)$. We need the following technical lemma about binary trees.

Lemma 5 *Let $T = (V(T), E(T))$ be a binary tree and let $n(u)$ denote the number of leaves of the subtree of T rooted in some vertex $u \in V(T)$. For every non-leaf $u \in V(T)$ with left child u_l and right child u_r let $n(u_l) \geq n(u_r)$.*

If $P : v_0 v_1 v_2 ... v_l$ is a path in T from the root $r = v_0 \in V(T)$ to a vertex $v = v_l \in V(T)$ and

$$\nu = |\{i \mid 1 \leq i \leq l,\ v_i \textit{ is the right child of } v_{i-1}\}|,$$

then

$$n(r) \geq 2^{\nu} n(v) + (l - \nu).$$

Proof: Let $u \in V(T)$ be a non-leaf and let u_l and u_r be the left and right child of u, respectively. Since $1 \leq n(u_r) \leq n(u_l)$, we have

$$n(u) = n(u_l) + n(u_r) \geq \max\{2n(u_r), n(u_l) + 1\}.$$

Hence $n(r)$ is at least a number that arises from $n(v)$ by ν multiplications with 2 and $(l - \nu)$ additions of 1 which easily implies the desired result. $\square$

Theorem 3 *If $G = (V, E)$ is a graph and $r \in \mathcal{R}(V)$, then there is a caterpillar term $\tilde{r} \in \mathcal{R}(V)$ such that*

$$\max\{\chi(H_{\tilde{s}}) \mid \tilde{s} \textit{ is a subterm of } \tilde{r}\} \leq (\lfloor \log_2(|V|) \rfloor + 1) \max\{\chi(H_s) \mid s \textit{ is a subterm of } r\}.$$

Proof: Let $T = (V(T), E(T))$ be the binary tree encoded by r. Let $V(u)$ denote the set of leaves of the subtree of T rooted in some vertex $u \in V(T)$. Let $n(u) = |V(u)|$ for $u \in V(T)$. Clearly, if $n = |V|$ and r denotes the root of T, then $n(r) = n$.

We may assume that $n \geq 4$. Furthermore, since $\oplus$ is commutative, we may assume that $n(u_l) \geq n(u_r)$ for every non-leaf $u \in V(T)$ with left child u_l and right child u_r.

Let

$$v_1, v_2, ..., v_n$$

be the ordering of the leaves of T (= the vertices of G) '*from left to right*', i.e. for $1 \leq i < j \leq n$ there is a vertex $u \in V(T)$ such that v_i lies in the subtree rooted in the left child of u and v_j lies in the subtree rooted in the right child of u.

Claim. For every $1 \leq i \leq n$ there are $l \leq \lfloor \log_2(n) \rfloor + 1$ vertices $u_1, u_2, ..., u_l \in V(T)$ such that

$$\{v_1, v_2, ..., v_i\} = \bigcup_{j=1}^{l} V(u_j).$$

Proof of the claim: Let $P : w_0 w_1 w_2 ... w_k$ be the path in T from the root $r = w_0 \in V(T)$ to the vertex $v_i = w_k \in V(T)$. Let $\nu = |\{i \mid 1 \leq i \leq k, \ w_i \text{ is the right child of } w_{i-1}\}|$. By Lemma 5, $n \geq 2^\nu n(v_i) + (k - \nu) \geq 2^\nu$. It is easy to see that there are $l \leq \nu + 1$ vertices $u_1, u_2, ..., u_l \in V(T)$ such that $\{v_1, v_2, ..., v_i\} = \bigcup_{j=1}^{l} V(u_j)$.

In Figure 2 we indicate how such vertices can be chosen. The encircled vertices form the set $\{u_1, u_2, ..., u_l\}$. Since $l \leq \nu + 1 \leq \lfloor \log_2(n) \rfloor + 1$ the proof of the claim is complete. $\square$

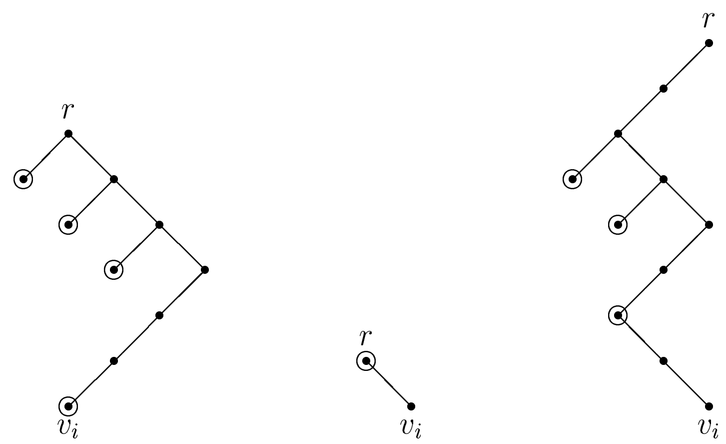

Figure 2 Examples for $\nu = 3, 1$ and 4.

Let $\tilde{r} = (...((v_1 \oplus v_2) \oplus v_3)...) \oplus v_n$. If $\tilde{s}$ is a subterm of $\tilde{r}$, then the above claim and the definition of the graphs H_s in Section 2 imply that the graph $H_{\tilde{s}}$ is a subgraph of the disjoint union of at most $\lfloor \log_2(n) \rfloor + 1$ graphs H_s where s is a subterm of r. This immediately implies the desired result and the proof is complete. $\square$

Combining previous results implies the following.

Corollary 2 *If $G = (V, E)$ is a graph, then*

$$\min\{\mathrm{cw}(G,t) \mid t \in \mathcal{R}(V) \textit{ is a caterpillar term}\} \leq 2(\lfloor \log_2(|V|) \rfloor + 1)\mathrm{cw}(G).$$

Proof: By Proposition 1, there is some $r \in \mathcal{R}(V)$ with $\mathrm{cw}(G) = \mathrm{cw}(G, r)$. By Theorem 3, there is a caterpillar term $\tilde{r} \in \mathcal{R}(V)$ with

$$\max\{\chi(H_{\tilde{s}}) \mid \tilde{s} \text{ is a subterm of } \tilde{r}\} \leq (\lfloor \log_2(|V|) \rfloor + 1) \max\{\chi(H_s) \mid s \text{ is a subterm of } r\}.$$

By Lemma 1 and Theorem 1, we obtain

$$\begin{aligned}
& \min\{\mathrm{cw}(G,t) \mid t \in \mathcal{R}(V) \text{ is a caterpillar term}\} \\
\leq\; & \mathrm{cw}(G, \tilde{r}) \\
\leq\; & 2 \max\{\chi(H_{\tilde{s}}) \mid \tilde{s} \text{ is a subterm of } \tilde{r}\} \\
\leq\; & 2\,(\lfloor \log_2(|V|) \rfloor + 1) \max\{\chi(H_s) \mid s \text{ is a subterm of } r\} \\
\leq\; & 2\,(\lfloor \log_2(|V|) \rfloor + 1)\, \mathrm{cw}(G, r) \\
=\; & 2\,(\lfloor \log_2(|V|) \rfloor + 1)\, \mathrm{cw}(G)
\end{aligned}$$

and the proof is complete. $\square$

5 Finding a Good Reduced Term

A polynomial time approximation algorithm for FINDING A GOOD REDUCED TERM and Corollary 1 would yield a polynomial time approximation algorithm for the clique-width. Unfortunately, we don't know such an algorithm.

In this final section we show that a tree-decomposition leads to a 'good' reduced term r in the sense that $\mathrm{cw}(G, r)$ is at least bounded by a function of the width of the tree-decomposition. This is interesting in view of the fact that tree-decompositions of small width can be determined efficiently [1, 8, 9] but it clearly does not imply that $\mathrm{cw}(G, r)$ is somewhere close to $\mathrm{cw}(G)$.

We need the exact definitions of a *tree-decomposition* and of the *tree-width* of a graph. A tree-decomposition of a graph $G = (V, E)$ is a pair $(T, \mathcal{W})$ such that $T = (V(T), E(T))$ is a tree and $\mathcal{W} = (W_t \mid t \in V(T))$ is a collection of sets $W_t \subseteq V$ of vertices of G such that

(i) $V = \bigcup\limits_{t \in V(T)} W_t$,

(ii) for every edge $uv \in E$ of G there exists a $t \in V(T)$ such that $u, v \in W_t$ and

(iii) for every vertex $u \in V$ of G the set $\{t \in V(T) \mid u \in W_t\}$ induces a subtree of T.

The width of the tree-decomposition $(T, \mathcal{W})$ is $\max\{|W_t|-1 \mid t \in V(T)\}$ and the tree-width $\mathrm{tw}(G)$ of G is the minimum width of a tree-decomposition of G.

For simplicity we will assume that all tree-decompositions $(T, \mathcal{W})$ additionally satisfy that

(iv) for every edge $st \in E(T)$ of T we have $W_t \neq W_s$

which can always be ensured without increasing the width of $(T, \mathcal{W})$ by contracting edges of T.

The following simple lemma plays a key-role for us.

Lemma 6 *Let $G = (V, E)$ be a graph and let $(T, \mathcal{W})$ be a tree-decomposition of G. Let T be rooted at some vertex $r \in V(T)$. For some $t_0 \in V(T)$ and some children $t_1, t_2, ..., t_l$ of t — not necessarily all — let*

$$\tilde{V} = \left(\bigcup_{i=1}^{l} \bigcup_{s \leq t_i} W_s \right) \setminus W_{t_0}$$

where "$s \leq t_i$" means that either $s = t_i$ or s is a descendant of t_i.

Then $N_G(u) \setminus \tilde{V} \subseteq W_{t_0}$ for all $u \in \tilde{V}$ and thus there are at most $2^{|W_{t_0}|}$ different sets in

$$\{N_G(u) \setminus \tilde{V} \mid u \in \tilde{V}\}.$$

Proof: This follows immediately from conditions (ii) and (iii) in the definition of a tree-decomposition and the fact that every path in T from a vertex in $\{s \mid s \leq t_i \text{ for some } 1 \leq i \leq l\}$ to a vertex in $V(T) \setminus \{s \mid s \leq t_i \text{ for some } 1 \leq i \leq l\}$ includes t_0. □

We proceed to the main result of this section.

Theorem 4 *If $G = (V, E)$ is a graph and $(T, \mathcal{W})$ is a tree-decomposition of G of width k, then there exists a term $r \in \mathcal{R}(V)$ with $\chi(H_s) \leq 2^{k+1}$ for every subterm s of r.*

Furthermore, r can be constructed in polynomial time given the graph $G = (V, E)$ and the tree-decomposition $(T, \mathcal{W})$ as input.

Proof: Let $G = (V, E)$, $(T, \mathcal{W})$ and k be as in the statement of the theorem.

Let T be rooted at some vertex $r \in V(T)$. Let $U_r = W_r$ and let $U_t = W_t \setminus W_s$ for every vertex $t \in V(T) \setminus \{r\}$ with parent s. It follows immediately from the definition of a tree-decomposition that the sets U_t for $t \in V(T)$ form a partition of V.

Let the term $r \in \mathcal{R}(V)$ be such that for every vertex $t_0 \in V(T)$ there is a subterm r_0 of r such that

$$V_{r_0} = \bigcup_{s \leq t_0} U_s. \tag{4}$$

(*Remember that* V_{r_0} *denotes the set of all vertices that appear in the term* r_0.)

Furthermore, if $t_1, t_2, ..., t_{l-1}$ and t_l are the children of t_0 and $l \geq 1$, then we require that there are subterms r_1 and r_2 of r such that

$$V_{r_1} \;=\; U_{t_0} \text{ and } V_{r_2} = \left(\bigcup_{s \leq t_0} U_s\right) \setminus U_{t_0} = \bigcup_{i=1}^{l} \bigcup_{s \leq t_i} U_s. \tag{5}$$

It is easy to see that such a term $r \in \mathcal{R}(V)$ exists and that it can be constructed in polynomial time in a bottom-up fashion starting at the leaves of T. (*See Figure 3 which shows a rooted tree* T *and the overall structure of a rooted binary tree encoded by a possible term* r *where a leaf labeled* U_i *corresponds to an arbitrary term in* $\mathcal{R}(U_i)$.) We leave the notationally tedious but very straightforward details of this claim as an exercise to the reader.

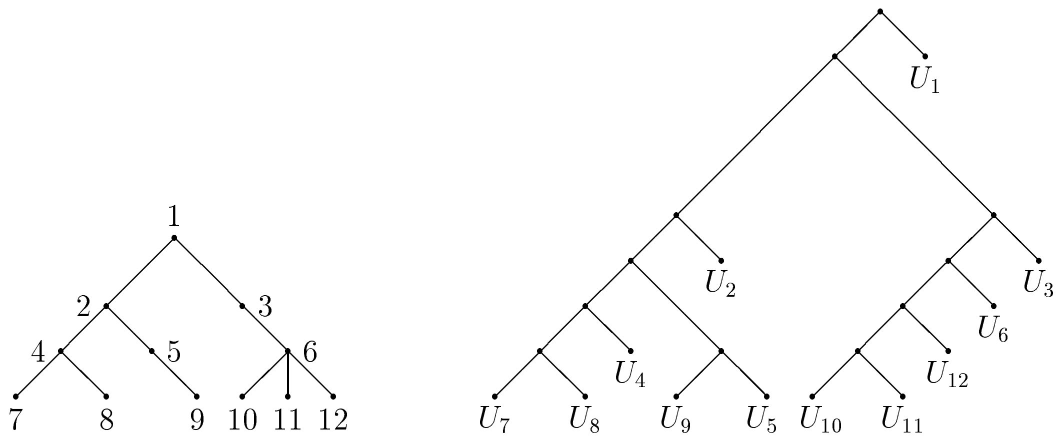

Figure 3

Now let s be a subterm of r. It follows from (4) and (5) that the set V_s is either V or a subset of a set U_t for some $t \in V(T)$ or has the form of the set $\tilde{V}$ as considered in Lemma 6. In the first case $\chi(H_s) \leq 1$, in the second case $|V_s| \leq k+1$ which implies $\chi(H_s) \leq k+1$ and in the third case Lemma 6 implies $\chi(H_s) \leq 2^{k+1}$. This completes the proof. □

As an immediate consequence of Theorem 1 and Theorem 4 we obtain the following corollary.

Corollary 3 *If* $G = (V, E)$ *is a graph, then* $\mathrm{cw}(G) \leq 2^{\mathrm{tw}(G)+2}$.

Note that Courcelle and Olariu [5] proved $\mathrm{cw}(G) \leq 2^{\mathrm{tw}(G)+1} + 1$ and that Corneil and Rotics [3] proved $\mathrm{cw}(G) \leq 3 \cdot 2^{\mathrm{tw}(G)-1}$.

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