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The number of spanning trees in circulant graphs with non–fixed jumps[1]

by

Yuanping Zhang,[2]
College of Mathematics and Computer Science
Hunan Normal University
Changsha, 410081, Hunan
P.R.China
ypzhang@hunnu.edu.cn

Zhiyong Zhang
Center of Networks
Hunan Normal University
Changsha, 410081, Hunan
P. R. China
zyzhang@hunnu.edu.cn

Xuerong Yong[3]
DIMACS
CoRE Building
Rutgers University
96 Frelinghuysen Road
Piscataway, New Jersey 08854

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[2]P. Zhang is the corresponding author.
[3]Xuerong Yong is also at the Dept. of Computer Science, The Hong Kong University of Science and Technology, Hong Kong

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ABSTRACT

Let $1 \le s_1 < s_2 < \cdots < s_k$, and $q < p$ be given integers. An *undirected circulant graph*, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices labelled $0,\ 1,\ 2,\ \cdots,\ n-1$, and for each s_i $(1 \le i \le k)$ and j $(0 \le j \le n-1)$ there is an edge between j and $j+s_i \mod n$. Let $T(C_n^{s_1,s_2,\cdots,s_k})$ be the number of spanning trees of $C_n^{s_1,s_2,\cdots,s_k}$. We proved in [16] that $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where a_n satisfies a linear recurrence relation of order 2^{s_k-1}. In this paper we show that $T(C_{pn}^{s_1,s_2,\cdots,s_k,qn})$ can be splitted into the products of $T(C_n^{s_1,s_2,\cdots,s_k})$ and $\frac{1}{p}c_{n,p,q}^2$, where $c_{n,p,q}$ satisfies a linear recurrence relation for some $p,\ q$. As examples of its use, we describe the method and, for certain p, q, we provide the recurrence relations and the asymptotics of $T(C_{pn}^{s_1,s_2,\cdots,s_k,qn})$.

Key Words: Circulant graphs; number of spanning trees; recurrence relations; asymptotics

1 Introduction

The study of the number of spanning trees in a graph G (network) has been very active due to the reason that it characterizes the reliability of a network in communications [6]. In this framework, a well-known and classic result is the so called *Matrix Tree Theorem* [8] which expresses the number of spanning trees $T(G)$ of G in terms of the determinant of a matrix that is constructed from G's incidence matrix. However, this method of counting spanning trees by calculating determinants is practically infeasible for large graphs.

Nevertheless, for some special classes of graphs it is possible to derive explicit, simple formulas for the number of spanning trees. If G is the complete graph K_n, then *Cayley's tree formula* [7] states that $T(K_n) = n^{n-2}$. Other special cases can be found in [4, 11, 12]. In this paper we restrict our problems to the class of circulant graphs with some non-fixed jumps.

Let $s_1,\ s_2,\ \cdots,\ s_k$ be integers with $1 \le s_1 < s_2 < \cdots < s_k$. The *undirected circulant graph*, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices labelled $0,\ 1,\ 2,\ \cdots,\ n-1$, and for each s_i $(1 \le i \le k)$ and j $(0 \le j \le n-1)$ there is an edge between j and $j + s_i \mod n$. So $C_n^{s_1,s_2,\cdots,s_k}$ is a $2k$-regular graph. Multiple edges and self-loops are permitted for the reasoning that the multiple connections between two work-stations are often used in real designing of networks and the self-loops have no contribution to the spanning trees.

Recently, the recurence formulas for $T(C_n^{s_1,s_2,\cdots,s_k})$ have been studied extensively. The formula $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, was originally conjectured by Bedrosian [2] and subsequently proven by Kleitman and Golden [9]. The same formula was also conjectured by Boesch and Wang [5] (without the knowledge of [9]). Different proofs can been found in [1, 4, 12]. The formulas for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ are provided in [11]. A general result is provided in [16], that is, if $\gcd(s_1, s_2, \cdots, s_k) = 1$ then $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where a_n satisfies a recurrence relation of order 2^{s_k-1}. The initial conditions of a_n can be obtained by the *Matrix Tree Theorem*. Note that $T(C_n^{s_1,s_2,\cdots,s_k}) = 0$ when $\gcd(s_1, s_2, \cdots, s_k) \ne 1$. But unfortunately, that method does not, though, provide a general recurrence relation or formula for the case $T(C_n^{s_1,s_2,\cdots,s_k})$ when some s_i are permitted to vary as n. This is an original motivation of this work.

In this paper, for given integers $s_1,\ s_2,\ \cdots,\ s_k$, $1 \le s_1 < s_2 < \cdots < s_k$, and $q < p$. we will provide a method for determining the recurrence relations for $T(C_{pn}^{s_1,s_2,\cdots,s_k,qn})$. Its asymptotic properties will also be investigated. Several recurrence relations will be given for applications of the method.

2 Basic lemmas

From Biggs [[3], Proposition 3.5 and Corollary 6.5], or [13], we know the following simpler product formula in the circulant graph case:

Lemma 1

$$
\begin{aligned}
T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(2k-\varepsilon^{-s_1j}-\varepsilon^{-s_2j}-\cdots-\varepsilon^{-s_kj} \\
&\qquad -\varepsilon^{s_1j}-\varepsilon^{s_2j}-\cdots-\varepsilon^{s_kj}) \\
&= \frac{1}{n}\prod_{j=1}^{n-1}\left(\sum_{i=1}^{k}(2-2\cos\frac{2js_i\pi}{n})\right),
\end{aligned}
$$

where ε^{-j} is the conjugate of ε^j, $\varepsilon=e^{\frac{2\pi i}{n}}$.

We should point out that even though Lemma 1 expresses the number of spanning trees for the circulant graphs exactly, it is really difficult to use it in practice because these $\cos\frac{2js_i\pi}{n}$ are involved in and the computation is ill-conditioned. Therefore we need to manipulate the above formula into a more convenient form.

Lemma 2 *Let*

$$g_{s_1,s_2,\cdots,s_k;\lambda}(x)=\lambda-x^{-s_1}-x^{-s_2}-\cdots-x^{-s_k}-x^{s_1}-x^{s_2}-\cdots-x^{s_k}, \tag{1}$$

and

$$f_{s_1,s_2,\cdots,s_k;\lambda}(x)=-x^{s_k}g_{s_1,s_2,\cdots,s_k;\lambda}(x). \tag{2}$$

If $\lambda>2k$, $f_{s_1,s_2,\cdots,s_k;\lambda}(\alpha)=0$, then $|\alpha|\neq 1$; even more, $f_{s_1,s_2,\cdots,s_k;\lambda}(1/\alpha)=0$ and, α and $1/\alpha$ have the same multiplicity as roots of $f_{s_1,s_2,\cdots,s_k;\lambda}(x)=0$.

Proof. For any α with $|\alpha|=1$, suppose that

$$\alpha=e^{i\phi}=\cos\phi+i\sin\phi$$

for some ϕ. Thus

$$
\begin{aligned}
\frac{f_{s_1,s_2,\cdots,s_k;\lambda}(\alpha)}{-\alpha^{s_k}} &= \sum_{i=1}^{k}(2-\alpha^{-s_i}-\alpha^{s_i})+(\lambda-2k) \\
&= \sum_{i=1}^{k}(2-2\cos s_i\phi)+(\lambda-2k).
\end{aligned}
$$

From this equation, if $\lambda>2k$, then $\frac{f_{s_1,s_2,\cdots,s_k;\lambda}(\alpha)}{-\alpha^{s_k}}>0$ always holds for any α with $|\alpha|=1$. So $f_{s_1,s_2,\cdots,s_k;\lambda}(x)$ has no roots with modulus 1. It is easy to see that

$$f_{s_1,s_2,\cdots,s_k;\lambda}(1/x)=\frac{1}{x^{2s_k}}f_{s_1,s_2,\cdots,s_k;\lambda}(x),$$

so for any α ($\neq 0$), α and $1/\alpha$ are roots of $f_{s_1,s_2,\cdots,s_k;\lambda}(x)$ simultaneously, or not. Going further, set

$$\overline{f}_{s_1,s_2,\cdots,s_k;\lambda}(x) = \frac{f_{s_1,s_2,\cdots,s_k;\lambda}(x)}{(x-\alpha)(x-1/\alpha)}.$$

Then

$$\overline{f}_{s_1,s_2,\cdots,s_k;\lambda}(1/x) = \frac{1}{x^{2s_k-2}}\overline{f}_{s_1,s_2,\cdots,s_k;\lambda}(x).$$

Continuing in this fashion shows that α and $1/\alpha$ always have same multiplicity as roots of $f_{s_1,s_2,\cdots,s_k;\lambda}(x)$. $\square$

From Lemma 2 we infer that, for any given $\lambda \neq 0$, $f_{s_1,s_2,\cdots,s_h;\lambda}(x)$ has roots $\alpha_1, \alpha_2, \cdots, \alpha_{s_h}$, $1/\alpha_1, 1/\alpha_2, \cdots, 1/\alpha_{s_h}$ with $|\alpha_i| \geq 1$ for each i, $1 \leq i \leq s_h$. It is possible that some α_i repeats.

Now we return to Lemma 1 to derive the following lemma.

Lemma 3 *For given integers $s_1, s_2, \cdots, s_k$, $1 \leq s_1 < s_2 < \cdots < s_k$, and $p > q$,*

$$T(C_{pn}^{s_1,s_2,\cdots,s_k,qn}) = \frac{1}{p}c_{n,p,q}^2 T(C_n^{s_1,s_2,\cdots,s_k}),$$

where $\forall q$, we have $a_{n,p} \leq c_{n,p,q} \leq b_{n,p}$, and both $a_{n,p}$, $b_{n,p}$ satisfy linear recurrence relations of order p^{s_k}.

Proof. Let $\varepsilon = e^{\frac{2\pi i}{n}}$ and $\varepsilon_p = e^{\frac{2\pi i}{pn}}$. Notice that $\cos(\frac{2\pi s_i j}{pn}) = \cos(\frac{2\pi s_i l}{n})$, and $\cos(\frac{2\pi q n l}{pn}) = 1$ when $j = pl$, $0 \leq l \leq n-1$. From Lemma 1, we have the following product expression of $T(C_{pn}^{s_1,s_2,\cdots,s_k,qn})$:

$$\begin{aligned}
T(C_{pn}^{s_1,s_2,\cdots,s_k,qn}) &= \frac{1}{pn}\prod_{j=1}^{pn-1} g_{s_1,s_2,\cdots,s_k,qn;2k+2}(\varepsilon_p^j) \\
&= \frac{1}{pn}\prod_{\substack{j=1 \\ p\nmid j}}^{pn-1} g_{s_1,s_2,\cdots,s_k,qn;2k+2}(\varepsilon_p^j)\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k;2k}(\varepsilon^j) \\
&= \frac{1}{p}c_{n,p,q}^2 T(C_n^{s_1,s_2,\cdots,s_k}),
\end{aligned}$$

where $c_{n,p,q}^2 = \prod_{\substack{j=1 \\ p\nmid j}}^{pn-1} g_{s_1,s_2,\cdots,s_k,qn;2k+2}(\varepsilon_p^j)$. Note that we have

$$b_{n,p}^2 = \frac{\prod_{j=1}^{pn-1} g_{s_1,s_2,\cdots,s_k;2k}(\varepsilon_p^j)}{\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k;2k}(\varepsilon^j)} \leq c_{n,p,q}^2 \leq \frac{\prod_{j=1}^{pn-1} g_{s_1,s_2,\cdots,s_k;2k+4}(\varepsilon_p^j)}{\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k;2k+4}(\varepsilon^j)} = a_{n.p}^2.$$

Now $f_{s_1,s_2,\cdots,s_k;2k+4}(x)$ have roots α_1, α_2, $\cdots$, α_{s_k}, $1/\alpha_1$, $1/\alpha_2$, $\cdots$, and $1/\alpha_{s_k}$, where $|\alpha_i| \geq 1$ for each i, $1 \leq i \leq s_k$. Therefore making use of the relations:

$$\begin{aligned}\prod_{j=1}^{n-1}(-\varepsilon^{-s_k j}) &= (-1)^{(s_k-1)(n-1)},\\ \prod_{j=1}^{n}(\varepsilon^j - x) &= (-1)^{n-1}(1-x^n) \ for\ any\ x,\end{aligned}$$

we have

$$\begin{aligned}\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k;2k+4}(\varepsilon^j) &= \prod_{j=1}^{n-1}(-\varepsilon^{-s_k j}) f_{s_1,s_2,\cdots,s_k;2k+4}(\varepsilon^j)\\ &= (-1)^{(s_k-1)(n-1)}\frac{\prod_{j=1}^{n}\prod_{i=1}^{s_k}(\varepsilon^j-\alpha_i)(\varepsilon^j-1/\alpha_i)}{\prod_{i=1}^{s_k}(1-\alpha_i)(1-1/\alpha_i)}\\ &= (-1)^{(s_k-1)(n-1)}\frac{\prod_{i=1}^{s_k}\prod_{j=1}^{n}(\varepsilon^j-\alpha_i)(\varepsilon^j-1/\alpha_i)}{\prod_{i=1}^{s_k}(1-\alpha_i)(1-1/\alpha_i)}\\ &= (-1)^{(s_k-1)(n-1)}\frac{\prod_{i=1}^{s_k}(1-\alpha_i^n)(1-1/\alpha_i^n)}{\prod_{i=1}^{s_k}(1-\alpha_i)(1-1/\alpha_i)},\end{aligned}$$

where if there is a $\alpha = 1$, we view $\frac{1-\alpha^n}{1-\alpha} = \lim_{r\to+0}\frac{1-(\alpha+r)^n}{1-(\alpha+r)} = n$. Similarly,

$$\prod_{j=1}^{pn-1} g_{s_1,s_2,\cdots,s_k;2k+4}(\varepsilon_p^j) = (-1)^{(s_k-1)(pn-1)}\frac{\prod_{i=1}^{s_k}(1-\alpha_i^{pn})(1-1/\alpha_i^{pn})}{\prod_{i=1}^{s_k}(1-\alpha_i)(1-1/\alpha_i)}.$$

Thus,

$$\begin{aligned}a_{n,p}^2 &= (-1)^{(s_k-1)n}\left(\prod_{i=1}^{s_k}(1+\alpha_i^n+\cdots+\alpha_i^{p(n-1)})(1+1/\alpha_i^n+\cdots+1/\alpha_i^{p(n-1)})\right)\\ &= (-1)^{(s_k-1)n}\frac{\prod_{i=1}^{s_k}(1+\alpha_i^n+\cdots+\alpha_i^{p(n-1)})^2}{\prod_{i=1}^{s_k}\alpha_i^{p(n-1)}}.\end{aligned}$$

Consequently, we may let $a_{n,p} = \sum_{t=1}^{p^{s_k}} r_t^n$ for some numbers r_t, $1 \leq t \leq 2^{s_k}$. Then from the generatingfunctionology [10], $a_{n,p}$ satisfies a linear recurrence relation of order p^{s_k}. Analogically, we can prove the similar result for $b_{n,p}$. □

From the proof we note that in general the formula for $c_{n,p,q}$ is not easy to derive. Nevertheless, this lemma produces the following corollary which states that $c_{n,p,q}$ satisfies a linear recurrence relation for some p, q.

Corollary 1 *For given integers s_1, s_2, $\cdots$, s_k, $1 \le s_1 < s_2 < \cdots < s_k$, we have*

$$(1) \qquad T(C_{2n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{2}c_n^2 T(C_n^{s_1,s_2,\cdots,s_k}),$$

where $c_n = c_{n,2,1}$ satisfies a linear recurrence relation of order 2^{s_k}. and

$$(2) \qquad T(C_{3n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{3}c_n^2 T(C_n^{s_1,s_2,\cdots,s_k}),$$

where $c_n = c_{n,3,1}$ satisfies a linear recurrence relation of order 3^{s_k}.

Proof. (1) This is the case that $p = 2, q = 1$. Noticing that $\cos(j\pi) = -1$ when j is odd and $\cos(j\pi) = 1$ when j is even, therefore from Lemma 3, we have $c_{n,2,1} = a_{n,2}$. (2) This is the case that $p = 3, q = 1$. Noticing that for any integer j, $\cos(2j\pi/3) = \cos(4j\pi/3) = -1/2$ if j is not divisible by 3 and $\cos(2j\pi/3) = 1$ otherwise, therefore, it is seen that

$$c_{n,3,1}^2 = \frac{\prod_{j=1}^{3n-1} g_{s_1,s_2,\cdots,s_k;2k+3}(\varepsilon_3^j)}{\prod_{j=1}^{n-1} g_{s_1,s_2,\cdots,s_k;2k+3}(\varepsilon^j)}.$$

The rest of the proof follows similarly from the proof of Lemma 3. $\square$

Note that Lemma 3 gives an estimation of $T(C_{pn}^{s_1,s_2,\cdots,s_k,qn;n})$ using $T(C_n^{s_1,s_2,\cdots,s_k})$. Corollary 1 provides a method for calculating $T(C_{2n}^{s_1,s_2,\cdots,s_k,n})$ and $T(C_{3n}^{s_1,s_2,\cdots,s_k,n})$ for fixed integers s_1, s_2, $\cdots$, s_k where $1 \le s_1 < s_2 < \cdots < s_k$ are given. The $T(C_n^{s_1,s_2,\cdots,s_k})$ are known already [16]. The initial conditions of c_n can be obtained exactly from the *Matrix Tree Theorem*. Hence, by solving a system of linear equations, we can establish the exact recurrence relation for c_n. The method is applied to several examples in the following section.

3 Recurrence relations and asymptotic properties

Following theorems provide recurrence formulas for some special cases of Corollary 1.

Theorem 4

$$T(C_{2n}^{1,n}) = \frac{1}{2}c_n^2 T(C_n^1) = \frac{n}{2}c_n^2,$$

where c_n satisfies the recurrence relation $c_n = 2\sqrt{2}c_{n-1} - c_{n-2}$ with initial conditions $c_1 = 2\sqrt{2}$ and $c_2 = 6$.

Proof. Notice that $T(C_n^1) = n$.

From Corollary 1, $T(C_{2n}^{1,n})$ has the form in the theorem, where c_n satisfies a linear recurrence relation of order 2. The initial conditions of c_n can be derived by using the *Matrix Tree Theorem.* We have $c_1 = 2\sqrt{2}$, $c_2 = 6$, and $c_3 = 10\sqrt{2}$. By letting $c_n = d_1c_{n-1} + d_2c_{n-2}$, d_1 and d_2 can be obtained easily by plugging the initial c_i into the formula and solving a system of linear equations. $\square$

Theorem 5
$$T(C_{2n}^{1,2,n}) = \frac{1}{2}c_n^2 T(C_n^{1,2}),$$
where c_n satisfies the recurrence relation $c_n = 2\sqrt{2}c_{n-1} + c_{n-2} - 2\sqrt{2}c_{n-3} - c_{n-4}$ with initial conditions $c_1 = 2\sqrt{2}$, $c_2 = 10$, $c_3 = 16\sqrt{2}$ and $c_4 = 62$.

Proof. From Corollary 1,
$$T(C_{2n}^{1,2,n}) = \frac{1}{2}c_n^2 T(C_n^{1,2}),$$
where c_n satisfies a linear recurrence relation of order 4. Suppose that $c_n = d_1 c_{n-1} + d_2 c_{n-2} + d_3 c_{n-3} + d_4 c_{n-4}$ for some d_1, d_2, d_3 and d_4. The initial conditions can be derived by using the *Matrix Tree Theorem*, they are $c_1 = 2\sqrt{2}$, $c_2 = 10$, $c_3 = 16\sqrt{2}$, $c_4 = 62$, $c_5 = 118\sqrt{2}$, $c_6 = 460$, $c_7 = 898\sqrt{2}$ and $c_8 = 3518$. After plugging these c_i into the relation and solving the linear equations, we have $d_1 = 2\sqrt{2}$, $d_2 = 1$, $d_3 = -2\sqrt{2}$ and $d_4 = -1$. □

It is already known that $T(C_n^{1,2}) = na_n^2$, where a_n are *Fibonacci* numbers. So Theorem 5 provides a easy method to calculate $T(C_{2n}^{1,2,n})$, which is in linear time of n in the sense of the analysis of algorithms.

Similarly we have the following recurrence formulas. We would like to omit their proofs here.

Theorem 6
$$T(C_{2n}^{1,2,3,n}) = \frac{1}{2}c_n^2 T(C_n^{1,2,3}),$$
where c_n satisfies the recurrence relation $c_n = 2\sqrt{3}c_{n-1} - 11c_{n-4} + 2\sqrt{3}c_{n-7} - c_{n-8}$ with initial conditions $c_1 = 2\sqrt{3}$, $c_2 = 24$, $c_3 = 24\sqrt{3}$, $c4 = 100$, $c_5 = 178\sqrt{3}$, $c_6 = 936$, $c_7 = 1622\sqrt{3}$ and $c_8 = 8636$.

Theorem 7
$$T(C_{3n}^{1,n}) = \frac{1}{3}c_n^2 T(C_n^{1}) = \frac{n}{3}c_n^2,$$
where c_n satisfies the recurrence relation $c_n = 6c_{n-1} - 6c_{n-2} + c_{n-3}$ with initial conditions $c_1 = 6$, $c_2 = 24$ and $c_3 = 111$.

Theorem 8
$$T(C_{3n}^{1,2,n}) = \frac{1}{3}c_n^2 T(C_n^{1,2}),$$
where c_n satisfies the recurrence relation $c_n = 9c_{n-1} - 9c_{n-2} - 55c_{n-3} + 63c_{n-4} + 63c_{n-5} - 55c_{n-6} - 9c_{n-7} + 9c_{n-8} - c_{n-9}$ with initial conditions $c_1 = 9$, $c_2 = 63$, $c_3 = 321$, $c4 = 2079$, $c_5 = 13239$, $c_6 = 86991$, $c_7 = 573057$, $c_8 = 3794175$ and $c_9 = 25152441$.

Lemma 3 and Corollary 1 have given us more information than the recurrence relations. From their proofs it is easy to see the asymptotic properties of $T(C_{2n}^{s_1,s_2,\cdots,s_k,n})$ and $T(C_{3n}^{s_1,s_2,\cdots,s_k,n})$. We have the following theorems.

Theorem 9 *Let $s_1,\ s_2,\ \cdots,\ s_k$ be fixed positive integers such that $1 \le s_1 < s_2 < \cdots < s_k$. Suppose $f_{s_1,s_2,\cdots,s_k;2k+4}(x)$ have roots $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_{s_k},\ 1/\alpha_1,\ 1/\alpha_2,\ \cdots,\ 1/\alpha_{s_k}$ with $|\alpha_i| > 1$ for each i. Then for the c_n in the formula $T(C_{2n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{2}c_n^2 T(C_n^{s_1,s_2,\cdots,s_k})$, $c_n \sim |\sqrt{\prod_{i=1}^{s_k}\alpha_i}|^n$.*

Theorem 10 *Let $s_1,\ s_2,\ \cdots,\ s_k$ be fixed positive integers such that $1 \le s_1 < s_2 < \cdots < s_k$. Suppose $f_{s_1,s_2,\cdots,s_k;2k+3}(x)$ have roots $\beta_1,\ \beta_2,\ \cdots,\ \beta_{s_k},\ 1/\beta_1,\ 1/\beta_2,\ \cdots,\ 1/\beta_{s_k}$ with $|\beta_i| > 1$ for each i. Then for the c_n in the formula $T(C_{3n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{3}c_n^2 T(C_n^{s_1,s_2,\cdots,s_k})$, $c_n \sim |\prod_{i=1}^{s_k}\beta_i|^n$.*

We would like to repeat that for given integers $s_1,\ s_2,\ \cdots,\ s_k$ with $1 \le s_1 < s_2 < \cdots < s_k$, $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$, where a_n has asymptotic property that $a_n \sim c\phi^n$ for some c and ϕ and that a technique is provided in [14] to derive these c and ϕ values. In the technique what we need to do is the calculation of all roots of a degree $s_k - 1$ polynomial. Combining this result and technique with the Theorems 9 and 10, we have

$$T(C_{2n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{2}nu_n^2,$$

where $u_n \sim c\varphi^n$ for some c and φ; and

$$T(C_{3n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{3}nv_n^2.$$

where $u_n \sim c\vartheta^n$ for some c and ϑ.

For all cases of $1 \le s_1 < s_2 < \cdots < s_k \le 5$, the c values are given in [16], the φ and ϑ values are calculated here and listed in the following two tables. Notice that in deriving each of these asymptotic properties, it is not necessary to calculate any recurrence relation using the *Matrix Tree Theorem*. All we need to do is to calculate the all roots of a degree $s_k - 1$ polynomial and a degree $2s_k$ polynomial.

4 Conclusion

A method that uses *Chebyshev* polynomial technique for deriving closed formulas for the circulant graphs $C_n^{s_1,s_2\cdots,s_k,\frac{n}{a_1},\frac{n}{a_2},\cdots,\frac{n}{a_l}}$, where $a_1,\ a_2,\ \cdots,\ a_l \in \{2,3,4,6\}$ and for all i, $1 \le i \le l$, $a_i|n$ is given in [15]. This method is also applied to derive closed formulas $T(C_{2n}^{1,n})$, $T(C_{3n}^{1,n})$, $T(C_{4n}^{1,n})$ $T(C_{6n}^{1,n})$ and $T(C_{6n}^{1,2n,3n})$. In this paper our analysis of recurrence relations and asymptotics focused mainly on two classes circulant graphs of form $C_{2n}^{s_1,s_2,\cdots,s_k,n}$ and $C_{3n}^{s_1,s_2,\cdots,s_k,n}$. One can apply our idea to consider more general related graphs.

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$\{s_k\}$	φ	$\{s_k\}$	φ
$\{1\}$	2.4142137	$\{1,3,4\}$	6.7194498
$\{1,2\}$	4.4894342	$\{1,3,5\}$	6.8392650
$\{1,3\}$	4.7264265	$\{1,4,5\}$	6.8108968
$\{1,4\}$	4.8300525	$\{2,3,4\}$	6.7758386
$\{1,5\}$	4.8824824	$\{2,3,5\}$	6.7953888
$\{2,3\}$	4.7889002	$\{2,4,5\}$	6.8704249
$\{2,4\}$	4.4894342	$\{3,4,5\}$	6.9182287
$\{2,5\}$	4.8935533	$\{1,2,3,4\}$	8.5137376
$\{3,4\}$	4.8804810	$\{1,2,3,5\}$	8.6150783
$\{3,5\}$	4.9069606	$\{1,2,4,5\}$	8.6834930
$\{4,5\}$	4.9199007	$\{1,3,4,5\}$	8.7133034
$\{1,2,3\}$	6.5118172	$\{2,3,4,5\}$	8.7468767
$\{1,2,4\}$	6.6605541	$\{1,2,3,4,5\}$	10.5068124
$\{1,2,5\}$	6.7795708		

Figure 1: φ values for the formula $T(C_{2n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{2}nu_n^2$, where $u_n \sim c\varphi^n$ for some c and φ. Notice that in the 2, 4 rows, the values only apply for the odd n.

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$\{s_k\}$	ϑ	$\{s_k\}$	ϑ
$\{1\}$	4.7912878	$\{1,3,4\}$	18.5512644
$\{1,2\}$	10.7448805	$\{1,3,5\}$	18.941730
$\{1,3\}$	11.3609561	$\{1,4,5\}$	18.8147955
$\{1,4\}$	11.6173813	$\{2,3,4\}$	18.7378857
$\{1,5\}$	11.7449405	$\{2,3,5\}$	18.7708247
$\{2,3\}$	11.5169554	$\{2,4,5\}$	19.0174567
$\{2,4\}$	10.7448805	$\{3,4,5\}$	19.1783525
$\{2,5\}$	11.7717440	$\{1,2,3,4\}$	26.1748375
$\{3,4\}$	11.7402214	$\{1,2,3,5\}$	26.5319947
$\{3,5\}$	11.8040842	$\{1,2,4,5\}$	26.7793049
$\{4,5\}$	11.8352470	$\{1,3,4,5\}$	26.8805573
$\{1,2,3\}$	17.9199343	$\{2,3,4,5\}$	27.0094160
$\{1,2,4\}$	18.3769926	$\{1,2,3,4,5\}$	35.4037868
$\{1,2,5\}$	18.7482215		

Figure 2: ϑ values for the formula $T(C_{3n}^{s_1,s_2,\cdots,s_k,n}) = \frac{1}{3}nv_n^2$, where $v_n \sim c\vartheta^n$ for some c and ϑ. Notice that in the 2, 4 rows, the values only apply for the odd n's.

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