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Further Analysis of the Number of Spanning Trees in Circulant Graphs

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ABSTRACT

Let $1 \leq s_1 < s_2 < \cdots < s_k \leq [n/2]$ be given integers. An undirected circulant graph, $C_n^{s_1,s_2,\cdots,s_k}$, has n vertices $0,\ 1,\ 2,\ \cdots,\ n-1$, and for each s_i $(1 \leq i \leq k)$ and j $(0 \leq j \leq n-1)$ there is an edge between j and $j + s_i \mod n$. Let $T(C_n^{s_1,s_2,\cdots,s_k})$ stand for the number of spanning trees of $C_n^{s_1,s_2,\cdots,s_k}$. For this special class of graphs, a general and most recent result is obtained in [Y. P. Zhang, X. Yong, M. J. Golin, The number of spanning trees in circulant graphs, Discrete Mathematics 223 (2000) 337-350.] where it is shown that $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where a_n satisfies a linear recurrence relation of order 2^{s_k-1}.

In this paper we obtain further properties of the numbers a_n by considering their combinatorial structures. Using these properties we answer the open problem posed in the Conclusion of [Y. P. Zhang, X. Yong, M. J. Golin.]. As examples, we describe our technique and asymptotic properties of the numbers.

1 Introduction

All graphs here are undirected. Multiple edges and self-loops are permitted in any graph. A *spanning tree* in a graph G is a tree having the same vertex set as G. The study of the number of spanning trees in a graph has a long history and has been very active because the number is useful in estimating the reliability of a network; analyzing energy of masers in investigating the possible particle transitions; and designing electrical circuits etc. ([3], [5], [7]). A classic result for this problem is the *Matrix Tree Theorem* [9] which expresses the number of spanning trees $T(G)$ of graph G in terms of the determinant of a matrix that can be easily constructed from G's incidence matrix. However, counting spanning trees by calculating this determinant is not acceptable for large graphs. For this reason people have developed methods to get around it (see, for example, [4] and its references) and have paid much attention to deriving formulas for some special classes of graphs. These are the ones devoted to determining explicit, simple formulas for the number of spanning trees in such graphs, e.g. [2, 5, 7, 12, 14].

Let $s_1, s_2, \cdots, s_k$ be fixed positive integers. In our considerations, without loss of generality we may assume that $1 \leq s_1 < s_2 < \cdots < s_k \leq [n/2]$. An undirect circulant graph $C_n^{s_1,s_2,\cdots,s_k}$ has n vertices lebelled $0, 1, 2, \cdots, n-1$, with each vertex i $(0 \leq i \leq n-1)$ adjacent to k vertices $i+s_1, i+s_2, \cdots, i+s_k$ *modn*. Note that $C_n^{s_1,s_2,\cdots,s_k}$ is a $2k$-regular graph. We use $T(C_n^{s_1,s_2,\cdots,s_k})$ to signify the number of spanning trees of $C_n^{s_1,s_2,\cdots,s_k}$.

During the past decades, for some special s_js the recurence formulae for $T(C_n^{s_1,s_2,\cdots,s_k})$ have been studied extensively. Starting from proving $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, which was originally conjectured by Bedrosian [3] and subsequently proven by Kleitman and Golden [9] and was also conjectured by Boesch and Wang [6] without the knowledge of [9]), different proofs of this formula, the formulas for $T(C_n^{1,3})$ and $T(C_n^{1,4})$ and formulas for some graphs related to these graphs etc. have been obtained in [10, 2, 5, 1, 12, 13, 15]. A general and most recent result is derived in [13], that is, if $\gcd(s_1, s_2, \cdots, s_k) = 1$ then $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$ where a_n satisfies a recurrence relation of order 2^{s_k-1}.

This paper continues the above work and focuses on exploring further properties of the numbers a_n by considering their combinatorial structures. With these properties we will obtain the asymptotic properties of $T(C_n^{s_1,s_2,\cdots,s_k})$. This will answer the open problem posed in the Conclusion of [13]. The problem asked is to characterize if the asymptotics of (when n tends to ∞)

$$T(C_{n+1}^{s_1,s_2,\cdots,s_k})/T(C_n^{s_1,s_2,\cdots,s_k})$$

depends only on some of these s_js like the results obtained in [14] for direct circulant graphs. As examples, we describe our technique and asymptotic properties of the numbers.

2 Basic results

In this section, we will consider a polynomial that plays an essential role in counting the numbers of spanning trees in the circulant graphs, which will help show that the number of spanning trees satisfies a (or anti-) reciprocal recurrince relation in n. Following lemma is known.

Lemma 1 *(Biggs[4]) For any integer* $1 \leq s_1 < s_2 < \cdots < s_k \leq [\frac{n}{2}]$.

$$\begin{aligned} T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(2k - \varepsilon^{-s_1 j} - \cdots - \varepsilon^{-s_k j} - \varepsilon^{s_1 j} - \cdots - \varepsilon^{s_k j}), \\ T(\overrightarrow{C}_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1}(k - \varepsilon^{s_1 j} - \varepsilon^{s_2 j} - \cdots - \varepsilon^{s_k j}) \end{aligned}$$

where ε^{-j} *is the conjugate of* ε^j, $\varepsilon = e^{2\pi i/n}$.

We manipulate the formula into a more convenient form and let

$$g_{s_1,s_2,\cdots,s_k}(x) = 2k - x^{-s_1} - x^{-s_2} - \cdots - x^{-s_k} - x^{s_1} - x^{s_2} - \cdots - x^{s_k}, \tag{1}$$

$$f_{s_1,s_2,\cdots,s_k}(x) = c_0 x^{2s_k-2} + c_1 x^{2s_k-3} + \cdots + c_{s_k-1}x^{s_k-1} + \cdots + c_1 x + c_0, \tag{2}$$

where

$$\begin{aligned} &c_{s_k-s_j-i} = s_k + s_{k-1} + \cdots + s_{j+1} - (s_j + i - 1)(k - j), \\ &j = 0, 1, \cdots, k-1, \quad i = 1, 2, \cdots, s_{j+1} - s_j, \quad s_0 = 0. \end{aligned}$$

Then we have the relation between (1) and (2) :

$$f_{s_1,s_2,\cdots,s_k}(x) = -\frac{x^{s_k}}{(x-1)^2} g_{s_1,s_2,\cdots,s_k}(x). \tag{3}$$

In fact, noting that

$$x^{2n} + 2x^{2n-1} + \cdots + (n+1)x^n + \cdots + 2x + 1 = (1 + x + x^2 + \cdots + x^n)^2. \tag{4}$$

Eq(3) can be seen directly by (4).

Note that the coefficients of $f_{s_1,s_2,\cdots,s_k}(x)$ are actually defined by the numbers $k, s_1, s_2, \cdots, s_k$. Therefore the roots of $f_{s_1,s_2,\cdots,s_k}(x)$ depend continuously on these s_js. Combining the above with Lemma 2 in [13], we have the following Lemma 2 which will be used for calculating the initial numbers of spanning trees.

Lemma 2

$$T(C_n^{s_1,s_2,\cdots,s_k}) = n\frac{(-1)^{(s_k-1)(n-1)}}{f_{s_1,s_2,\cdots,s_k}(1)}|I - M^n_{s_1,s_2,\cdots,s_k}|, \tag{5}$$

where

$$M_{s_1,s_2,\cdots,s_k} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{s_k-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_2 \\ 0 & 0 & \cdots & 1 & -c_1 \end{pmatrix},$$

is the companion matrix of $f_{s_1,s_2,\cdots,s_k}(x)$ *and* $|X|$ *represents the determinant of matrix* X.

Lemma 3 *For any integer* $1 \le s_1 < s_2 < \cdots < s_k \le [\frac{n}{2}]$, *let*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2,$$

then

$$a_n = \frac{1}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}} \sum_{i=1}^{2^{s_k-2}} (-1)^i \begin{cases} r_i^n + r_i^{-n} & s_k \ \ is \ \ odd \\ r_i^n - (-r_i)^{-n} & s_k \ \ is \ \ even \end{cases}$$

where r_i, $1 \le i \le 2^{s_k-2}$, *is a complex number.*

Proof. From (1), (2) and the above discussions,

$$\begin{aligned} T(C_n^{s_1,s_2,\cdots,s_k}) &= \frac{1}{n}\prod_{j=1}^{n-1} \varepsilon^{-s_kj}(\varepsilon^j-1)^2 f_{s_1,s_2,\cdots,s_k}(\varepsilon^j) \\ &= n(-1)^{(s_k-1)(n-1)}\prod_{j=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j). \end{aligned}$$

Thus

$$a_n^2 = (-1)^{(s_k-1)(n-1)}\prod_{j=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j). \tag{6}$$

Note that if α is a real root of $f_{s_1,s_2,\cdots,s_k}(x)$ then $1/\alpha$ is also a root of $f_{s_1,s_2,\cdots,s_k}(x)$; If β is a complex root of $f_{s_1,s_2,\cdots,s_k}(x)$ then $1/\beta, \bar{\beta}$ and $1/\bar{\beta}$, where $\bar{\beta}$ is the conjugate of β, are also roots of $f_{s_1,s_2,\cdots,s_k}(x)$ because $f_{s_1,s_2,\cdots,s_k}(x)$ is a real reciprocal polynomial of degree $2s_k-2$. Now we may assume, without lose of generality, that $f_{s_1,s_2,\cdots,s_k}(x)$ has $2v$ real roots and $4u$ complex roots. So $2v+4u = 2s_k-2$ and

$$\begin{aligned} \prod_{j=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j) &= \prod_{j=1}^{n-1}\prod_{i=1}^{v}(\varepsilon^j-\alpha_i)(\varepsilon^j-\alpha_i^{-1}) \\ &\quad \prod_{i=1}^{u}(\varepsilon^j-\beta_i)(\varepsilon^j-\beta_i^{-1})(\varepsilon^j-\bar{\beta}_i)(\varepsilon^j-\bar{\beta}_i^{-1}) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{v} \frac{(1-\alpha_i^n)(1-\alpha_i^{-n})}{(1-\alpha_i)(1-\alpha_i^{-1})} \\
&\quad \prod_{i=1}^{u} \frac{(1-\beta_i^n)(1-\beta_i^{-n})(1-\bar{\beta}_i^{\,n})(1-\bar{\beta}_i^{\,-n})}{(1-\beta_i)(1-\beta_i^{-1})(1-\bar{\beta}_i)(1-\bar{\beta}_i^{\,-1})} \\
&= \frac{(-1)^{v+2u}}{f_{s_1,s_2,\cdots,s_k}(1)} \prod_{i=1}^{v} \frac{(1-\alpha_i^n)^2}{\alpha_i^n} \prod_{i=1}^{u} \frac{[(1-\beta_i^n)(1-\bar{\beta}_i^{\,n})]^2}{\|\beta_i\|^{2n}}.
\end{aligned}
$$

Since $v+2u = s_k - 1$, v is even (or odd) if and only if s_k is odd (or even) and since $\alpha_i < 0$ for $1 \le i \le v$, we have

$$
a_n^2 = \frac{(-1)^{n(s_k-1+v)}}{f_{s_1,s_2,\cdots,s_k}(1)} \left\{ \frac{\prod_{i=1}^{v}(1-\alpha_i^n)\prod_{i=1}^{u}(1-\beta_i^n)(1-\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n} \right\}^2. \tag{7}
$$

Where $|a|$ stands for the absolule value of a real number a; $\|c\|$ stands for the modulus of a complex number c. Let

$$
\begin{aligned}
&\frac{\prod_{i=1}^{v}(x-\alpha_i^n)\prod_{i=1}^{u}(x-\beta_i^n)(x-\bar{\beta}_i^{\,n})}{\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n} \\
&= w_0 x^{s_k-1} - w_1 x^{s_k-2} + \cdots + (-1)^{s_k-1} w_{s_k-1}.
\end{aligned}
$$

Suppose

$$
r_i^n := \left[\frac{\alpha_{i_1}\cdots\alpha_{i_\gamma}\beta_{j_1}\cdots\beta_{j_\delta}\bar{\beta}_{k_1}\cdots\bar{\beta}_{k_\rho}}{\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|} \right]^n, \qquad i_\gamma + j_\delta + k_\rho = l, \tag{8}
$$

is a term of w_l for $0 \le l \le s_k - 1$. Then from $Vieta\ formula$, there is a another term

$$
r_j^n := \left[\frac{\alpha_{q_1}\cdots\alpha_{q_\mu}\beta_{p_1}\cdots\beta_{p_\nu}\bar{\beta}_{d_1}\cdots\bar{\beta}_{d_\tau}}{\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|} \right]^n, q_\mu + p_\nu + d_\tau = s_k - 1 - l, \tag{9}
$$

of w_{s_k-1-l} such that

$$
\begin{aligned}
&(\alpha_{i_1},\cdots,\alpha_{i_\gamma},\beta_{j_1},\cdots,\beta_{j_\delta},\bar{\beta}_{k_1},\cdots,\bar{\beta}_{k_\rho}) \cap \\
&(\alpha_{q_1},\cdots,\alpha_{q_\mu},\beta_{p_1},\cdots,\beta_{p_\nu},\bar{\beta}_{d_1},\cdots,\bar{\beta}_{d_\tau}) = \phi,
\end{aligned}
$$

$$
\begin{aligned}
&(\alpha_{i_1},\cdots,\alpha_{i_\gamma},\beta_{j_1},\cdots,\beta_{j_\delta},\bar{\beta}_{k_1},\cdots,\bar{\beta}_{k_\rho}) \cup \\
&(\alpha_{q_1},\cdots,\alpha_{q_\mu},\beta_{p_1},\cdots,\beta_{p_\nu},\bar{\beta}_{d_1},\cdots,\bar{\beta}_{d_\tau}) \\
&= (\alpha_1,\cdots,\alpha_v,\beta_1,\cdots,\beta_u,\bar{\beta}_1,\cdots,\bar{\beta}_u),
\end{aligned}
$$

where ϕ represents an empty set, so

$$\begin{aligned} r_i^n r_j^n &= \left[\frac{\alpha_1 \cdots \alpha_v \|\beta_1 \cdots \beta_u\|^2}{|\alpha_1 \cdots \alpha_v| \|\beta_1 \cdots \beta_u\|^2}\right]^n \\ &= (-1)^{vn}. \end{aligned}$$

Thus considering the signs of w_l and w_{s_k-1-l}, we have

$$\begin{aligned} (-1)^l r_i^n + (-1)^{s_k-1-l} r_j^n &= (-1)^l r_i^n + (-1)^{s_k-1-l}(-1)^{vn} r_i^{-n} \\ &= (-1)^l \{r_i^n + (-1)^{s_k-1}[(-1)^v r_i]^{-n}\}. \end{aligned}$$

This proves the lemma. □

From the table 2 of [13], we see that the recurrence formulas for a_n have coefficients like $\sqrt{c}$, where c is a positive integer. Following Corollary 1 implies that such numbers can also happen in the general formulas.

Corollary 1 *Let*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2.$$

Then $|f_{s_1,s_2,\cdots,s_k}(-1)|$ *is a factor of* a_n^2 *if* n *is even.*

Proof. From the proof of Lemme 3

$$\begin{aligned} a_{2n}^2 &= (-1)^{(s_k-1)(2n-1)} \prod_{j=1}^{2n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^j) \\ &= (-1)^{(s_k-1)(2n-1)} \prod_{i=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^i) f_{s_1,s_2,\cdots,s_k}(\varepsilon^{2n-i}) f_{s_1,s_2,\cdots,s_k}(\varepsilon^n) \\ &= (-1)^{(s_k-1)(2n-1)} \prod_{i=1}^{n-1} f_{s_1,s_2,\cdots,s_k}(\varepsilon^i) \overline{f_{s_1,s_2,\cdots,s_k}(\varepsilon^i)} f_{s_1,s_2,\cdots,s_k}(-1) \\ &= |f(-1)| \prod_{i=1}^{n-1} \|f_{s_1,s_2,\cdots,s_k}(\varepsilon^i)\|^2. \end{aligned}$$

□

3 Main results

In this section, we will give further properties of a_n, with which we will reduce, by half, the complexity of finding their recurrence relations. Two examples will be given to illustrate the technique introduced.

Theorem 4 *For integers* $1 \le s_1 < s_2 < \cdots < s_k \le [\frac{n}{2}]$, *let*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2.$$

Then a_n *satisfies a recurrence relation of the form*

$$\begin{cases} a_n + a_{n-2^{s_k-1}} = \sum_{i=1}^{2^{s_k-2}} b_i(a_{n-i} + a_{n-2^{s_k-1}+i}) & s_k \ \textit{is odd} \\ a_n + a_{n-2^{s_k-1}} = \sum_{i=1}^{2^{s_k-2}} b_i(a_{n-i} + (-1)^i a_{n-2^{s_k-1}+i}) & s_k \ \textit{is even} \end{cases}$$

Proof. If s_k is odd, then from Lemma 3 we know that the characteristic polynomial, $p(x)$, of a_n can be written as

$$\begin{aligned} p(x) &= \prod_{i=1}^{2^{s_k-1}} (x - r_i)(x - r_i^{-1}) \\ &= x^{2^{s_k-2}} p(x^{-1}). \end{aligned}$$

This indicates that $p(x)$ is a reciprocal polynomial.
Similarly, If s_k is even, then from Lemma 3

$$\begin{aligned} p(x) &= \prod_{i=1}^{2^{s_k-1}} (x - r_i)(x - (-r_i^{-1})) \\ &= x^{2^{s_k-2}} p(-x^{-1}). \end{aligned}$$

i.e, $p(x)$ is anti-reciprocal. Using these properties and the theory of difference equations we infer that a_n satisfy the relations in the Theorem 4. $\square$

Following Examples 1, 2 are the results obtained in [13]. We obtain them here in complexity by half of their work.

Example 1 *(Case 1)* $s_1 = 1, s_2 = 2, s_3 = 3, s_4 = 5$.

$$T(C_n^{1,2,3,5}) = na_n^2,$$

where a_n *satisfies the recurrence relation:*

$$\begin{aligned} (a_n + a_{n-16}) &= \sqrt{3}(a_{n-1} + a_{n-15}) + (a_{n-2} + a_{n-14}) + \sqrt{3}(a_{n-3} + a_{n-13}) \\ &- (a_{n-4} + a_{n-12}) + 9\sqrt{3}(a_{n-5} + a_{n-11}) - 17(a_{n-6} + a_{n-10}) \\ &- \sqrt{3}(a_{n-7} + a_{n-9}) - a_{n-8} \end{aligned}$$

with initial conditions

$$\begin{aligned} &a_1 = 1, a_2 = \sqrt{3}, a_3 = 3, a_4 = 5\sqrt{3}, a_5 = 11, a_6 = 27\sqrt{3}, a_7 = 113, \\ &a_8 = 155\sqrt{3}, a_9 = 729, a_{10} = 979\sqrt{3}, a_{11} = 4531, a_{12} = 6615\sqrt{3}, \\ &a_{13} = 28717, a_{14} = 42601\sqrt{3}, a_{15} = 185163, a_{16} = 272645\sqrt{3}. \end{aligned}$$

Proof. Since $s_4 = 5$ is odd, from Theorem 4, there exist b_i, $1 \le i \le 8$ such that

$$(a_n + a_{n-16}) = \sum_{i=1}^{8} b_i(a_{n-i} + a_{n+i-16}).$$

This gives the following linear equations:

$$\sum_{i=1}^{8} b_i(a_{n-i} + a_{n+i-16}) = (a_n + a_{n-16}), \qquad n = 17, 18, \cdots, 24.$$

The values of a_i for $i \le 24$ can easily be obtained by Lemma 2. With these 24 values we solve the 8 b_is (in [13] we need to find 16 b_is): $b_1 = \sqrt{3}, b_2 = 1, b_3 = \sqrt{3}, b_4 = -1$, $b_5 = 9\sqrt{3}, b_6 = -17, b_7 = -\sqrt{3}, b_8 = -1/2$.
$\square$

Example 2 *(Case 2)* $s_1 = 2, s_2 = 3, s_3 = 4$.

$$T(C_n^{2,3,4}) = na_n^2,$$

where a_n satisfies the recurrence relation:

$$(a_n + a_{n-8}) = (a_{n-1} - a_{n-7}) + (a_{n-2} + a_{n-6}) + 2(a_{n-3} - a_{n-5}) + 4a_{n-4},$$

with initial conditions

$$a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 11, a_6 = 20, a_7 = 43, a_8 = 93.$$

Proof. Since $s_3 = 4$ is even, from Theorem 4 there exist 4 b_is, $1 \le i \le 4$, such that

$$(a_n + a_{n-8}) = \sum_{i=1}^{4} b_i(a_{n-i} + (-1)^i a_{n+i-8}).$$

This gives the following linear equations:

$$\sum_{i=1}^{4} b_i(a_{n-i} + (-1)^i a_{n+i-8}) = (a_n + a_{n-8}), \qquad n = 9, 10, 11, 12.$$

Similarly, the values of a_i for $i \le 12$ can easily be obtained by Lemma 2. Using these 12 values we solve for the 4 b_is to derive that $b_1 = 1, b_2 = 1, b_3 = 2, b_4 = 2$. (in [13] we need to find 8 b_is) $\square$

4 Asymptotic Properties

In this section we will consider the asymptotic properties of $T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2$. Without lose of generality, in the proof of Lemma 3 we may assume that α_i, $|\alpha_i| \geq 1$, and β_j, $\|\beta_j\| \geq 1$, ,$1 \leq i \leq v, 1 \leq j \leq u$, are the real, and the complex roots of $f_{s_1,s_2,\cdots,s_k}(x)$. From (8) or (9)

$$\begin{aligned}\max_{1\leq i\leq 2^{s_k-2}} \|r_i\| &= \left\| \frac{\alpha_1\cdots\alpha_v\beta_1\cdots\beta_u\bar{\beta}_1\cdots\bar{\beta}_u}{\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|} \right\| \\ &= \sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|.\end{aligned}$$

We note from Section 2 that the coefficients of $f_{s_1,s_2,\cdots,s_k}(x)$ is defined actually by $k, s_1, s_2, \cdots, s_k$. Therefore, combining all the above observations we have the following theorem.

Theorem 5 *For any integer* $1 \leq s_1 < s_2 < \cdots < s_k \leq [\frac{n}{2}]$, *let*

$$T(C_n^{s_1,s_2,\cdots,s_k}) = na_n^2.$$

Then

$$\begin{aligned}a_n &\sim \frac{1}{\sqrt{f_{s_1,s_2,\cdots,s_k}(1)}}\left(\sqrt{|\alpha_1\cdots\alpha_v|}\|\beta_1\cdots\beta_u\|\right)^n \\ &= c(k, s_1, s_2, \cdots s_k)\phi(k, s_1, s_2, \cdots s_k)^n.\end{aligned}$$

Therefore,

$$T(C_{n+1}^{s_1,s_2,\cdots,s_k})/T(C_n^{s_1,s_2,\cdots,s_k}) \sim \phi(k, s_1, s_2, \cdots, s_k).$$

Theorem 5 answer the problem posed in the Conclusion of [13]. That is, the asymptotics depends continuously on all these parameters $k, s_1, s_2, \cdots, s_k$. As an example, we consider the asymptotic property for the case $s_1 = 1, s_2 = 2$ (i.e., the conjecture stated in the Introduction). The other cases with $s_k \leq 7$ are listed in the table below. Now

$$T(C_n^{1,2}) = na_n^2.$$

Then from (2), we have $f_{1,2}(x) = x^2 + 3x + 1$ and its two roots are $(-3 \pm \sqrt{5})/2$. Thus

$$a_n \sim \frac{1}{\sqrt{5}}\left(\sqrt{\left|\frac{-3-\sqrt{5}}{2}\right|}\right)^n.$$

We should point out that in considering each of these asymptotic values, it is not necessary to calculate the recurrence relation. All we need to do is to find the all roots with modulus greater than 1 of $f_{s_1,s_2,\cdots,s_k}(x)$.

Table of $c = c(k, s_1, s_2, \cdots s_k)$ and $\phi = \phi(k, s_1, s_2, \cdots s_k)$ for all $s_k \leq 7$

$s_1 \cdots s_7$	c	ϕ	$s_1 \cdots s_7$	c	ϕ	$s_1 \cdots s_7$	c	ϕ
1200000	0.447213	1.618033	1204560	0.110431	2.904809	0004507	0.105409	2.254335
1230000	0.267261	2.102256	1234560	0.104828	3.187066	1200067	0.105409	2.602895
1030000	0.316227	1.700015	1034560	0.107211	2.911946	1230067	0.100503	2.922559
0230000	0.277350	1.722083	0234560	0.105409	2.916682	1030067	0.102597	2.607813
1204000	0.218217	2.147396	0034560	0.107832	2.610672	0230067	0.101015	2.617883
1234000	0.182574	2.509600	1004560	0.113227	2.590648	1204067	0.097128	2.927791
1034000	0.196116	2.165786	0204560	0.111111	2.598096	1234067	0.093250	3.209231
0234000	0.185695	2.181934	1000560	0.127003	2.210094	1034067	0.094915	2.928098
0034000	0.200000	1.754877	0200560	0.124034	2.239729	0234067	0.093658	2.932972
1004000	0.242535	1.736814	0030560	0.119522	2.231087	0034067	0.095345	2.612937
1200500	0.182574	2.183136	0004560	0.113960	2.243994	1004067	0.099014	2.616071
1230500	0.160128	2.537090	1000060	0.164423	1.766268	0204067	0.097590	2.612262
1030500	0.169030	2.200509	0030060	0.447213	1.618033	1200567	0.093250	2.930018
0230500	0.162221	2.189798	0000560	0.128527	1.776331	1230567	0.089802	3.216789
1204500	0.147441	2.555258	1200007	0.136082	2.219366	1030567	0.091287	2.938370
1234500	0.134839	2.866403	1230007	0.125988	2.581326	0230567	0.090166	2.940467
1034500	0.140028	2.563611	1030007	0.130189	2.231281	1204567	0.087370	3.221048
0234500	0.136082	2.572032	0230007	0.127000	2.234550	1234567	0.084515	3.480374
0034500	0.141421	2.224978	1204007	0.119522	2.592650	1034567	0.085749	3.225522
1004500	0.154303	2.194750	1234007	0.112508	2.900819	0234567	0.084818	3.228568
0204500	0.149071	2.211484	1034007	0.115469	2.587496	0034567	0.086066	2.950340
1000500	0.196116	1.755601	0234007	0.113227	2.596523	1004567	0.088735	2.939200
0200500	0.185695	1.759575	0034007	0.116227	2.215354	0204567	0.087705	2.943677
0030500	0.171500	1.764393	1004007	0.123091	2.237258	1000567	0.094916	2.605997
0004500	0.156158	1.769046	0204007	0.120385	2.230466	0200567	0.093658	2.622414
1200060	0.156173	2.204211	1200507	0.112508	2.597047	0030567	0.091669	2.629261
1230060	0.141421	2.560838	1230507	0.106600	2.911441	0004567	0.089087	2.634776
1030060	0.147441	2.210395	1030507	0.109108	2.610707	1000067	0.107893	2.219127
0230060	0.142857	2.213885	0230507	0.107211	2.595657	0200067	0.106001	2.252594
1204060	0.132453	2.570020	1204507	0.102597	2.919860	0030067	0.103143	2.240414
1234060	0.123091	2.884395	1234507	0.098058	3.199528	0004067	0.099503	2.258590
1034060	0.127000	2.581329	1034507	0.100000	2.923710	0000567	0.095360	2.255153
0234060	0.124034	2.580030	0234507	0.098532	2.923309	1000007	0.141143	1.772881
0034060	0.128036	2.225460	0034507	0.100503	2.614814	0200007	0.137443	1.773957
1004060	0.137360	2.227485	1004507	0.104828	2.610447	0030007	0.131477	1.775511
1200560	0.123091	2.584959	0204507	0.103142	2.605826	0004007	0.123923	1.777297
1230560	0.115470	2.897523	1000507	0.115469	2.243072	0000507	0.115379	1.779138
1030560	0.118678	2.590973	0200507	0.113175	2.216749	0000067	0.107908	1.780732
0230560	0.116247	2.593290	0030507	0.109766	2.246622	$\cdots$	$\cdots$	$\cdots$

The values of c and ϕ for $gcd(s_1, s_2, \cdots, s_k) = d \neq 1$ case is not reported since, as described in [13], $T(C_n^{s_1,s_2,\cdots,s_k}) = 0$ for even n and $T(C_n^{s_1,s_2,\cdots,s_k}) = T(C_n^{s_1/d,s_2/d,\cdots,s_k/d})$ for odd n.

5 Conclusion

In this paper, we gave tight formulas for determining the exact number of spanning trees in circulant graphs $C^{s_1,s_2,\cdots,s_k}$. The asymptotics of the numbers of spanning trees of these graphs are proven to depend upon $k, s_1, s_2, \cdots, s_k$. An interesting question would be to find out their exact asymptotic functions.

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