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War and Peace in Veto Voting

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ABSTRACT

Let $I = \{i_1, \ldots, i_n\}$ be a set of voters (players) and $A = \{a_1, \ldots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference P_i over the candidates. We assume that P_i is a complete order on A. The preference profile $P = \{P_i, i \in I\}$ is called a *situation*. A situation is called *war* if the set of all voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite. For a simple class of veto voting schemes we prove that the results of elections in all war situations uniquely define the results for all other (*peace*) situations.
Key words: veto, voting scheme, voting by veto, veto power, veto resistance, voter, candidate, player, outcome, coalition, block, effectivity function, veto function, social choice function, social choice correspondence

1 Main Theorem

We follow standard concepts and notation of veto voting theory; see e.g. [1, 2]. Let $I = \{i_1, \ldots, i_n\}$ be a set of voters (players) and $A = \{a_1, \ldots, a_p\}$ be a set of candidates (outcomes). Each voter $i \in I$ has a preference (a complete order) P_i over all candidates. The set of all preferences $P = \{P_i, i \in I\}$ is called a *preference profile* or a *situation.* A situation is called *war* if the set of voters I is partitioned in two coalitions K_1 and K_2 such that all voters of K_i have the same preference, $i = 1, 2$, and these two preferences are opposite.

Further, each voter $i \in I$ has μ_i veto cards and each candidate $a \in A$ has λ_a counter-veto cards. Positive integers μ_i and λ_a are called the *veto power* of $i \in I$ and *veto resistance* of $a \in A$, respectively. The corresponding integral-valued functions. $\mu : I \to \mathbb{Z}_+$ and $\lambda : I \to \mathbb{Z}_+$ are called veto power and veto resistance distributions.

Let us define the *veto order* σ_μ as a word in the alphabet $I = \{i_1, \ldots, i_n\}$ in which every letter $i \in I$ appears exactly μ_i times and hence each word σ_μ has the same length $\sum_{i \in I} \mu_i$. The triplet $(\lambda, \mu, \sigma_\mu)$ is called *veto voting scheme (VVS).* It is realized as follows. In the given order σ_μ the voters put their veto cards against the candidates until all veto cards are finished. The voters have perfect information. It is forbidden to over-veto, that is as soon as a candidate a has got λ_a veto cards he is eliminated and no more veto cards can be used against him. All non-eliminated candidates are elected. Obviously, this set will be empty unless total veto power is strictly less than total veto resistance, that is

$$\sum_{i \in I} \mu_i < \sum_{a \in A} \lambda_a \tag{1.1}$$

If we assume further that

$$\sum_{a \in A} \lambda_a - \sum_{i \in I} \mu_i = 1. \tag{1.2}$$

then exactly one candidate is elected in each situation. However, unlike (1.1), this assumption is not mandatory.

The voters may behave in many different, sometimes rather sophisticated, ways, see [1, 2]. In this paper we consider only the simplest concept of their so-called *sincere* behavior. This means that each voter $i \in I$ always put his veto card against the worst (with respect to his preference P_i) not yet eliminated candidate. Hence, given a VVS $(\lambda, \mu, \sigma_\mu)$, a set of elected candidates $B = B(P) \subseteq A$ is uniquely defined for every situation $P = \{P_i, i \in I\}$.

In general, a mapping $S : P \to 2^A$ which assigns a set of candidates to every preference profile is called a *social choice correspondence (SCC)*, and it is called a *social choice function (SCF)* if only one candidate is elected, that is $|S(P)| = 1$ for each situation P. Thus, every veto voting scheme $(\lambda, \mu, \sigma_\mu)$ defines a SCC $S_{\lambda,\mu,\sigma_\mu}$ which is an SCF whenever (1.2) holds. The SCC or SCF generated by a veto voting scheme are called *veto SCC and SCF*, respectively.

A veto order σ_μ is called *simple* if the voters do not alternate, or more precisely, if there exists a permutation τ of I such that first the voter $\tau^{-1}(i_1)$ put all veto cards, followed by

$\tau^{-1}(i_2)$, etc. Obviously, a simple veto order σ_μ is uniquely determined by μ and τ. The corresponding veto voting scheme and SCC we will call *simple* and denote by (λ, μ, τ) and $S_{\lambda,\mu,\tau}$, respectively.

In this paper we prove that each simple veto SCC is uniquely defined by the values it takes in the war situations. More precisely, the following statement holds.

Theorem 1 *Given two simple veto voting schemes $VVS' = (\lambda', \mu', \tau)$ and $VVS'' = (\lambda'', \mu'', \tau)$ which generate social choice correspondences $S' = S_{\lambda',\mu',\tau}(P)$ and $S'' = S_{\lambda'',\mu'',\tau}(P)$, respectively, if $S'(P) = S''(P)$ for each war situation P then $S'(P) = S''(P)$ for all P.*

Note, however, that Theorem 1 can not be promoted to the rank of a "general law of diplomacy". For example, it is not general enough just because it only holds when the two involved veto orders coincide, moreover, it must be a simple order; see Example 1 below.

Further, let us remark that in a war situation the veto order (simple or not) does not matter at all. In this case all candidates are uniquely ordered and all voters are split in two coalitions which veto candidates "from two opposite ends of this order". Some "moderate centrist" candidates will be elected and the set of these candidates does not depend on the order in which the voters act. More accurately these arguments are summarized as follows.

Lemma 1 *Given distributions λ, μ and two veto orders $\sigma'_\mu, \sigma''_\mu$, the equality $S_{\lambda,\mu,\sigma'_\mu}(P) = S_{\lambda,\mu,\sigma''_\mu}(P)$ holds for each war situation P.*

Yet, for other (peace) situations the result can depend on the veto order.

Example 1 *Let us consider two voters of veto power 3 and 1 and three candidates of veto resistance 1,2, and 2, that is $I = \{i_1, i_2\}, A = \{a_1, a_2, a_3\}$, $\mu_1 = 3, \mu_2 = 1$, $\lambda_1 = 1, \lambda_2 = \lambda_3 = 2$. Note that (1.2) holds and hence this voting scheme generates an SCF. Let the preferences be $a_1 > a_2 > a_3$ and $a_2 > a_1 > a_3$ for i_1 and i_2 respectively. This profile defines a peace situation P.*

First, let us consider two simple veto orders. If i_1 votes first then he eliminates a_3 and puts one remaining veto card against a_2. Still a_2 is not eliminated, yet. Moreover, a_2 will be elected, since i_2 vetoes a_1. If i_2 votes first he puts his only veto card against a_3. This allows i_1 to eliminate both a_3 and a_2. Hence. in this case a_1 is elected.

Now let us consider two veto orders i_1, i_1, i_2, i_1 and i_1, i_2, i_1, i_1. These orders are not simple and they have similar pattern: first i_1, then i_2, and then i_1 again. Yet, these two orders result in electing different candidates. In the first case i_1 eliminates a_3, then i_2 eliminates a_1, and a_2 is elected. In the second case i_1 puts just one veto card against a_3, then i_2 eliminates a_3, and now i_1 can eliminate a_2 by the two remaining veto cards, hence, a_1 is elected.

Finally, let us remark that, according to Lemma 1, all four veto orders considered above would give the same result in each war situation.

2 An equivalent statement

The theorem can be equivalently reformulated in a less emotional way.

The *veto function* is defined as a mapping $V : 2^I \times 2^A \rightarrow \{0,1\}$, that is V has two arguments: a coalition of voters $K \subseteq I$ and a block of candidates $B \subseteq A$. The equalities $V(K,B) = 1$ and $V(K,B) = 0$ mean that K can, and respectively can not, veto B. The complementary function $E(K,B) = V(K, A \setminus B)$ is called *effectivity function*; see [1, 2].

Each pair of distributions $\mu : I \rightarrow \mathbb{Z}_+$ and $\lambda : I \rightarrow \mathbb{Z}_+$, generates a veto function $V = V_{\mu,\lambda}$

$$V(K,B) = 1 \ \text{ iff } \ \sum_{i \in K} \mu_i \geq \sum_{a \in B} \lambda_a. \tag{2.3}$$

In other words, K can veto B if the voters from K has sufficiently many veto-cards to eliminate all candidates from B. In these terms we can reformulate Theorem 1 as follows.

Theorem 2 *Let $VVS' = (\lambda', \mu', \tau)$ and $VVS'' = (\lambda'', \mu'', \tau)$ be two simple veto voting schemes such that they have the same simple veto order τ and their veto functions $V' = V_{\mu',\lambda'}$ and $V'' = V_{\mu'',\lambda''}$ are equal, that is $V'(K,B) = V''(K,B)$ for all $K \subseteq I, B \subseteq A$. Then the SCCs $\mathcal{S}' = \mathcal{S}_{\mu',\lambda',\tau}$ and $S'' = \mathcal{S}_{\mu'',\lambda'',\tau}$ are equal, too, that is $S'(P) = S''(P)$ for every situation P.*

To prove that Theorems 1 and 2 are equivalent we only need to show that Theorem 2 becomes trivial if we restrict ourselves by the war situations only. In other words, given a veto function, the results of elections in all war situations are uniquely defined, and vice versa. Due to Lemma 1, this is true for all (not only simple) veto orders.

Lemma 2 *Given two veto voting schemes $VVS' = (\lambda', \mu', \sigma'_{\mu'})$ and $VVS'' = (\lambda'', \mu'', \sigma''_{\mu''})$ which generate veto functions $V' = V_{\lambda',\mu',\sigma'_{\mu'}}$, $V'' = V_{\lambda'',\mu'',\sigma''_{\mu''}}$ and SCCs $S' = S_{\lambda',\mu',\sigma'_{\mu'}}$, $S'' = S_{\lambda'',\mu'',\sigma''_{\mu''}}$, the following claims are equivalent:*

(i) $V' = V''$, that is $V'(K,B) = V''(K,B)$ for all $K \subseteq I, B \subseteq A$,

(ii) $S'(P) = S''(P)$ for every war situation P.

Proof . Suppose that $V' \neq V''$, say $1 = V'(K,B) \neq V''(K,B) = 0$ for some $K \subseteq I, B \subseteq A$, that is in VVS' coalition K can veto block B but in VVS'' it can not. Consider a complete order P_0 over A such that each candidate from B precedes each candidate from $A \setminus B$. Let a_0 be the last candidate from B in this order. Define a war situation P as follows. All voters from K prefer candidates according to P_0, (that is for them $A \setminus B$ is better than B) and all voters from $I \setminus K$ have the opposite preference. Then obviously, $a_0 \notin S'(P)$, since $V'(K,B) = 1$ and in VVS' coalition K can veto the whole block B including a_0. Yet $a_0 \in S''(P)$ since $V''(K,B) = 0$, that is in VVS'' coalition K has not enough veto power to eliminate B and hence a_0 will remain unvetoed. Thus $S'(P) \neq S''(P)$.

Vice versa, suppose that $S'(P) \neq S''(P)$ for a war situation P defined by a complete order P_0 over A and a partition $K, I \setminus K$. Without loss of generality, we can assume that

$a_0 \in S''(P) \backslash S'(P)$, that is $a_0 \notin S'(P)$ and $a_0 \in S''(P)$. Let B consists of a_0 and all candidates preceding a_0 in order P_0. Then obviously, $V'(K, B) = 1$, otherwise a_0 would be elected in VVS', and $V''(K, B) = 0$, otherwise a_0 would be vetoed in VVS''.

Let us underline again that all above arguments are based on Lemma 1. □

3 Proof of Theorems 2

Now without loss of generality we can assume that permutation τ is identical, that is i_1 distributes all veto cards first, followed by i_2, etc. In this case the argument τ becomes irrelevant and we will omit it in all formulas.

Given a voting scheme (λ, μ), let us fix a voter $i \in I$ and a candidate $a \in A$. We say that i *kills* a if a is eliminated and the last veto card against him is put by i. We say that i *wounds* a if i puts at least one veto card against a but does not eliminate a, that is either a is elected or eliminated (killed) later by some other voter. Finally, we say that i ignores a if i does not put a veto card against a.

Lemma 3 *A voter $i \in I$ can ignore or kill several candidates but wound at most one.*

Proof . Indeed, if i wounds a then i can not switch to another candidate a' before he kills a. This follows from our two basic assumptions: (i) the veto order is simple and (ii) the voting is sincere. Let us remark that both assumptions are important. □

Given a veto voting scheme (λ, μ), we will divide the voting procedure in $|I| = n$ steps. The m-th one begins after $m - 1$ voters are finished. In other words, we will consider voting schemes $VVS_m = (\lambda, \mu, m)$ with a truncated set of voters $I_m = \{i_1, \ldots, i_m\}$, where $m = 1, \ldots, n$.

Given m and a candidate $a \in A$, let $s_m(a)$ be the sequence of those voters from I_m who put at least one veto card against a. The length of this sequence is a non-decreasing function of m. We will also add a special symbol 0 at the end of $s_m(a)$ if after m steps a is already eliminated. This symbol 0 will remain the last since it is not allowed to over-veto.

Now let us compare two veto voting schemes $VVS' = (\lambda', \mu', \tau) = (\lambda', \mu')$ and $VVS'' = (\lambda'', \mu'', \tau) = (\lambda'', \mu'')$. Theorem 2 claims that if two veto functions are equal $V'(K, B) = V''(K, B)$ for all $K \subseteq I$ and $B \subseteq A$ then the result of elections is always the same, that is $S'(P) = S''(P)$ for every situation P.

We would like to prove (by induction on m) a stronger claim: $s'_m(a) = s''_m(a)$ for each m and $a \in A$. Unfortunately, this is not true. Yet, it becomes true if we slightly modify the voting procedure. Let us introduce one additional rule: the last veto card of a voter must wound, not kill. We will refer to this rule as the DNK-rule (though DNK-BYLVC would be more accurately). Suppose that a voter $i \in I$ runs out of his veto cards exactly after killing a candidate $a' \in A$. In this case we give one extra veto card to i and one extra anti-veto card to the candidate a who follows a' in the order P_i. Anyway, these two cards will immediately annihilate each other since i will put his extra card against a by the next (and last) move.

Hence, this modification preserves the SCC $S(P)$ in every situation P. Though the veto function $V(K, B)$ changes, we will see that this does not create a problem.

We will prove by induction on m that $s'_m(a) \neq s''_m(a)$ for all $a \in A$. Let $m = 1$ and suppose that $s'_1(a) \neq s''_1(a)$. There are two possibilities.

Case 1. The candidate a is eliminated in one voting scheme, say VVS', but not in the other, that is $\mu'(i_1) \geq \lambda'(B) = \sum_{b \in B} \lambda'(b), \quad \mu''(i_1) < \lambda''(B) = \sum_{b \in B} \lambda''(b)$, where block B consists of a and all candidates worse than a with respect to P_{i_1}. In other words, i_1 has enough veto power to eliminate B in VVS' but not in VV''. Indeed, in this case $s'_1(a) \neq s''_1(a)$, since $s'_1(a)$, unlike $s''(a)$, has symbol 0 at the end, that is $s'_1(a) = (s''_1(a), 0)$. It is clear that in this case veto functions V' and V'' are not equal either, namely $1 = V'(\{i_1\}, B) \neq V''(\{i_1\}, B) = 0$, and we get a contradiction.

Case 2. In VVS' and VVS'' voter i_1 eliminates the same set of candidates. Yet, in VVS' he runs out of his veto cards exactly after killing a candidate, while in VVS'' he still have some extra veto cards to wound a new candidate a, though not enough for killing a. Though in this case $s'_1(a)$ and $s''_1(a)$ differ, yet we must apply the DNK-rule and they become equal.

Analogous (just a little more sophisticated) arguments work for the general inductive step from $m - 1$ to m. We assume that $s'_{m-1}(a) = s''_{m-1}(a)$ for all $a \in A$ and prove that $s'_m(a) = s''_m(a)$ for all $a \in A$. The proof is indirect. Assume that $s'_m(a) \neq s''_m(a)$ for some $a \in A$ and consider again the same two cases. In case 2 our arguments will not change, except we substitute i_m for i_1.

Case 2. In VVS' and VVS'' voter i_m eliminates the same set of candidates. Yet, in VVS' he runs out of his veto cards exactly after killing a candidate a', while in VVS'' he still have some extra veto cards to wound a new candidate a, though not enough to kill a. In this case $s'_1(a) \neq s''_1(a)$ but we apply the DNK-rule and they become equal.

Case 1. Candidate a is killed by i_m in one voting scheme, say in VVS'', but not in the other. Perhaps, in VVS' voter i_m did not even wound a. Yet obviously, there is a candidate a^0 wounded but not killed by i_m. By induction hypothesis, in VVS'' the voter i_m kills a^0 before a. In this case $s'_1(a^0) \neq s''_1(a^0)$ (more precisely, $(s'_1(a^0), 0) = s''_1(a^0)$) and we will prove that $V' \neq V''$. Let us consider VVS', set $B_1 = \{a^0\}$, and denote by K_1 the coalition of all voters who wound a^0. In particular, $i_m \in K_1$. By Lemma 3, the voters of K_1 wound no other candidate but perhaps they kill some other. By B_2 denote the set of all candidates killed by K_1 (if any) and by K_2 the set of all voters who wound a candidate from B_2. By construction, $K_1 \cap K_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. By Lemma 3, the voters of K_2 wound no other candidates, except the candidates from B_2, but perhaps they kill some other. By B_3 denote the set of all candidates killed by K_2 (if any) and by K_3 the set of all voters who wound a candidate from B_3, etc. Obviously, this process will finish and we get a family of coalitions $K_1, \ldots, K_p$ and blocks $B_1, \ldots, B_q$ such that

(i) $p = q$ or $p = q + 1$;

(ii) the obtained coalitions and blocks are pairwise disjoint;

(iii) the voters from K_j wound only candidates from B_j and kill only candidates from B_{j+1}, and vice versa, the candidates from B_j are wounded only by the voters from K_j and are killed only by the candidates from K_{j-1}.

Let $K = \cup_{j=1}^{p} K_j$ and $B = \cup_{j=1}^{q} B_j$. By (iii), K put all veto cards only against B, none against $A \setminus B$, and vice versa, each veto card put against B belongs to K, none to $I \setminus K$. Yet, K can not eliminate B in VVS', hence $V'(K, B) = 0$. Now let us consider VVS'' and the same pair (K, B). By induction hypothesis, (iii) almost holds again, except the voter i_m after eliminating a^0 can (and must, according to the DNK-BYLVC-rule) spend his remaining veto cards out of B. Hence, $V''(K, B) = 1$, since in VVS'', unlike VVS', a^0 is killed by i_m. Thus $1 = V''(K, B) \neq V'(K, B) = 0$.

Finally, we have to recall that in m steps the original distributions $\lambda', \mu', \lambda'', \mu''$ can change. Yet, $S'(P)$ and $S''(P)$ can not. Let us show that $V'(K, B)$ and $V''(K, B)$ (for the considered pair (K, B)) can not change either. Indeed, in both VVS' and VVS'', the voters from $K \setminus \{i_m\}$ put all veto cards only against B and the candidates from B receive all veto cards only from the voters of K. Hence, the numbers of extra cards given to K and to B are equal. Thus the modified veto functions take in (K, B) the same values as the original ones. In particular, $1 = V''(K, B) \neq V'(K, B) = 0$ and we get a contradiction.

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