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Generalizations of interval-filament graphs

by

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ABSTRACT

Let I be a family of intervals on a line L in a plane PL such that every two intersecting intervals have a common segment. Let $V=\{v|\ i(v)\in I\}$ be a vertex set. In PL, above L, construct to each interval $i(v)\in I$ a filament (curve) $a(v)$ connecting its two endpoints and bounded in PL by the endpoints of $i(v)$; $FI=\{a(v)|\ i(v)\in I\}$ is a *family of 2D-interval-filaments* and its intersection graph is a *2D-interval-filament graph* [GA4]. Let PP be a plane perpendicular to PL whose intersection with PL is exactly the family of *2D*-interval-filaments FI. In PP, above PL, for every $a(v)\in FI$, connect its endpoints by a filament v such that if $a(u),a(v)$ overlap the two filaments u,v intersect and if $a(u),a(v)$ are disjoint, the two filaments u,v do not intersect; $FF=\{v|\ a(v)\in FI\}$ is a family of *3D-interval-filaments* and its intersection graph is a *3D-interval-filament graph*. A graph $G(V,E)$ is $\boldsymbol{G}$ *mixed* if its edge set can be partitioned into two disjoint subsets $E1$, $E2$ such that $G(V,E1)\in\boldsymbol{G}$, $G(V,E2)$ is transitive and for every three vertices u,v,w if $u\rightarrow v\in E2$ and $(v,w)\in E1$ then $(u,w)\in E1$.

We prove that *the family of complements of 3D-interval-filament graphs is exactly the family of co(2D-interval-filament) mixed graphs* and define various subfamilies of *3D*-interval-filament graphs characterizing them as complements of families of $\boldsymbol{G}$ mixed graphs. We present another generalization of the *2D*-interval-filament graphs, namely the *k-filament graphs*. We describe polynomial time algorithms for holes in **co**($\boldsymbol{G}$ mixed) graphs.

1. INTRODUCTION

We consider only finite graphs $G(V,E)$ with no parallel edges and no loops, where V is the set of vertices and E the set of edges. For $U \subseteq V$, $G(U)$ is the vertex subgraph defined by U. For $F \subseteq E$, $G(F)$ is the edge subgraph defined by F on V. Two vertices connected by an edge $u—v$ are called *adjacent* and we denote this by (u,v), without regard for the orientation of the edge; $coG(V,coE)$ is *the complement of G* where $coE=\{(u,v) \mid u \neq v, (u,v) \notin E\}$. For $E1 \subseteq coE$ we denote $co[coG(E1)]$ by $G(coE1)$. A directed edge from u to v is denoted $u \rightarrow v$. We also denote $N_G(v)=\{u \mid (u,v) \in E\}$ and $N_G[v]=N_G(v) \cup \{v\}$. By a path $p=v_1—...—v_k$ we always mean a simple path, denoted $p(v_1,v_k)$; p is an *induced path* if it has no chords. A *hole* $h=v_1—...—v_k—v_1$ is a chordless cycle with four or more vertices, denoted $h(v_1,v_k)$. A *subpath* of h, clockwise from v_i to v_j, is denoted $h(v_i,v_j)$. A subset of V is a *clique* (an *independent set*) if every two of its vertices are adjacent (not adjacent, respectively). Two subsets of a set *intersect* if they have a non-empty intersection; two subsets *overlap* if they intersect but none is contained in the other.

A graph G is an *intersection graph* of a family S of distinct subsets of a set if there is a one-to-one correspondence between the vertices of G and the subsets in S such that two subsets intersect iff their corresponding vertices are adjacent; S is a *representation* of G. Intersection graphs are of interest in various domains such as computer science, genetics and ecology [RO]. Intersection graphs of intervals on a line, subtrees on a tree and arcs on a circle are called *interval, chordal* and *circular-arc graphs* [GA3,GA2], respectively. An oriented graph $G(V,E)$ is called *transitive* if it is acyclic and for every three vertices $u,v,w \in V$, $u \rightarrow v, v \rightarrow w \in E$ implies $u \rightarrow w \in E$ [GRU,EPL]; its underlying undirected graph is called a *comparability graph*. A transitive graph can also be defined as follows: an oriented graph $G(V,E)$ is transitive if it is acyclic and for every directed path with three vertices $p=u_1 \rightarrow u_2 \rightarrow u_3$, p induces a clique if each of the two subpaths $u_1 \rightarrow u_2$, $u_2 \rightarrow u_3$ induces a clique. We define a generalization of the transitive graphs: For a fixed k, an oriented graph is *k-transitive* if it is acyclic and for every directed path $p=u_1 \rightarrow u_2 \rightarrow ... \rightarrow u_{k+2}$ with $k+2$ vertices, p induces a clique if each of the two subpaths $u_1 \rightarrow u_2 \rightarrow ... \rightarrow u_{k+1}$ and $u_2 \rightarrow ... \rightarrow u_{k+2}$ induces a clique. Thus, a graph is transitive iff it is *1*-transitive. Transitive graphs are perfect, but k-transitive graphs with $k \geq 2$ may not be perfect since odd holes are k-transitive.

A family $\boldsymbol{H}$ of graphs is *hereditary* if $H(V,E) \in \boldsymbol{H}$ implies $H(U) \in \boldsymbol{H}$ for every $U \subseteq V$; the families of intersection graphs are hereditary. We denote $\boldsymbol{coH}=\{coH \mid H \in \boldsymbol{H}\}$. Consider a

hereditary family ***H*** of graphs. A graph *H(V,E)* is called ***H*** *mixed* [GA4] if its edge set can be partitioned into two disjoint subsets *E1*, *E2* such that $H(V,E1)\in\boldsymbol{H}$, *H(V,E2)* is transitive and for every three distinct vertices *u,v,w* if $u\rightarrow v\in E2$ and $(v,w)\in E1$ then $(u,w)\in E1$. The letter ***H*** is generic and can be replaced by names of specific families.

Gavril [GA4] defined a new family of intersection graphs: Let *I* be a family of intervals on a line *L*, every two intersecting intervals having a common segment; we denote by *l(i),r(i)* the endpoints of an interval *i*. Let $V=\{v|\ i(v)\in I\}$ be a vertex set. In the plane *PL* containing *L*, above *L*, we construct to each interval $i(v)\in I$ a filament (curve) *a(v)* connecting *l(i(v)),r(i(v))*, bounded by the perpendiculars to *L* at *l(i(v)),r(i(v))*, and not containing its endpoints (Figures 1,2). A filament *a(v)* can have more than two endpoints subject to the conditions that it does not contain its endpoints, is continuous and all endpoints are in *L*; a filament can be a piece of (contained in) another filament. $FI=\{a(v)|\ i(v)\in I\}$ is *a family of 2D-interval-filaments* and its intersection graph is a *2D-interval-filament graph*. Clearly, if two intervals are disjoint, their filaments do not intersect. The family of complements of *2D*-interval-filament graphs is exactly the family of cointerval mixed graphs [GA4]. The family of *2D*-interval-filament graphs contains the family of polygon-circle graphs and the latter includes the circular-arc, the circle trapezoid, the circle and the chordal graphs [GA2,FMW,GA1,GA3].

For a given family ***G***, the families of ***coG*** mixed graphs and of ***co***(***G*** mixed) graphs are distinct. For example, when ***G*** is the family of cointerval graphs, the cocointerval mixed graphs are the interval mixed graphs, while the co(cointerval mixed) graphs are the *2D*-interval-filament graphs. Every cointerval mixed graph is a transitive mixed graph and every transitive mixed graph is a co(*2D*-interval-filament) mixed graph.

In a family of *2D*-interval-filaments let *x* be an intersection point of filaments (Figure 1). If $x\notin L$, then, by definition, every filament containing *x* has endpoints on opposite sides of *x*. We replace *x* by a very short segment *x,y* adding this segment to all *2D*-interval-filaments containing *x*. If $x\in L$, we replace *x* by a small curve *x,y*, with $x,y\in L$, and add it to all *2D*-interval-filaments containing *x*. No additional intersections are generated and the intersection of every two *2D*-interval-filaments is a curve segment. Therefore, we will assume that the intersections of two *2D*-interval-filaments *u,v* are segments.

Let *GA(V)* be the intersection graph of a family of *2D*-interval-filaments $FI=\{a(v)|\ v\in V\}$ constructed on a family of intervals *I* on a line *L* in a plane *PL*. Let *PP* be a plane perpendicular to *PL* whose intersection with *PL* is exactly the family of *2D*-interval-

filaments *FI*. For $a(v) \in FI$, let *PP(a(v))* be the part of *PP* whose intersection with *PL* is exactly *a(v)*. In *PP(a(v))*, above *PL* we connect the endpoints of *a(v)* by a filament *v* such that if *a(u),a(v)* overlap the two filaments intersect and if *a(u),a(v)* are disjoint, the two filaments do not intersect; *v* does not contain its endpoints and is continuous. We call such a filament *v* a *3D-interval-filament*. The intersection graph *G(V,E)* of a family *FF* of *3D*-interval-filaments on a line *L* is called a *3D-interval-filament graph*. To every *3D*-interval-filament *v* corresponds the region *s(v)* in *PP(a(v))* bounded by $v \cup a(v)$. Therefore, two vertices *u*,*v* in *G(V,E)* are adjacent iff the regions *s(u),s(v)* overlap, i.e., *PP(a(u))*, *PP(a(v))* overlap or say $PP(a(u)) \subset PP(a(v))$ and the filaments *u*,*v* intersect in *PP(a(u))*. The edge set $coE=\{(u,v) |\ u \cap v = \phi\}$ of *coG(V,coE)* can be partitioned into two disjoint subsets $E1=\{(u,v) |\ a(u) \cap a(v) = \phi\}$ and $E2=\{u \rightarrow v |\ a(u) \subset a(v) \text{ and } u \cap v = \phi\}$. Clearly, $(u,v) \in E1$ iff $s(u) \cap s(v) = \phi$, while $u \rightarrow v \in E2$ iff $s(u) \subset s(v)$. The graph *coG(V,E2)* is transitive since $u \rightarrow v, v \rightarrow w \in E2$ implies $a(u) \subset a(v) \subset a(w)$ and $u \cap v = \phi$, $v \cap w = \phi$, i.e., $s(u) \subset s(v) \subset s(w)$, hence $u \cap w = \phi$ and $u \rightarrow w \in E2$. The complement *G(coE1)* of *coG(V,E1)* is exactly *GA* since $(u,v) \notin E1$ iff $a(u) \cap a(v) \neq \phi$ iff $s(u) \cap s(v) \neq \phi$. Let *u,v,w* be three distinct vertices such that $w \rightarrow v \in E2$ and $(u,v) \in E1$; then, $a(w) \subset a(v)$ and $a(u) \cap a(v) = \phi$, implying that $a(w) \cap a(u) = \phi$ and $(w,u) \in E1$. Thus, *coG(V,coE)* is a ***coG*** mixed graph where ***G*** is the family of *2D*-interval-filament graphs. Therefore, the complements of *3D*-interval-filament graphs are co(*2D*-interval-filament) mixed graphs. Similarly (families to be defined), the complements of polygon-circle-filament graphs are co(polygon-circle) mixed graphs, the complements of cocomparability-filament graphs are comparability mixed graphs and the complements of permutation-filament graphs are permutation mixed graphs. For an intersection graph *G(V,E)* of a family *FI* of *3D*-interval-filaments, we denote by *i(v)*, *a(v)* and *v* the interval, the *2D*-interval-filament and the *3D*-interval-filament corresponding to a vertex *v* of *G*.

Consider a tree *T* in a plane *PL* and a family *ST* of subtrees of *T*. To each vertex *x* of *T* we add a branch *(x,y)*, adding it also to every subtree in *ST* containing *x*; the intersection of every two subtrees in *ST* contains an edge of *T*. Let *PP* be a surface perpendicular to *PL* such that $PP \cap PL = T$. On *PP*, above *T*, consider a family $FT=\{f_t |\ t \in ST\}$ of filaments, called *subtree filaments*, each f_t contained in *PP(t)* and connecting all the endpoints of *t*, such that if two subtrees overlap, their curves intersect, and if two subtrees are contained one in another, their curves may or may not intersect. Clearly, if two subtrees are disjoint, their curves do not intersect. The intersection graph *G(V,E)* of *FT* is *a subtree filament graph*.

The family of complements of subtree-filament graphs is exactly the family of cochordal mixed graphs [GA4].

In Section 2 we define various subfamilies of *3D*-interval-filament graphs and characterize them as complements of corresponding families of ***coG*** mixed graphs. The main result is that *the family of complements of 3D-interval filament graphs is exactly the family of co(2D-interval-filament) mixed graphs.*

Gavril [GA4] described a polynomial time algorithm to find maximum weight cliques in ***H*** mixed graphs when ***H*** has such an algorithm. Thus, we can find maximum weight independent sets in *3D*-interval-filament graphs since *2D*-interval-filament graphs have such an algorithm [GA4]. Other polynomial time algorithms in *2D*-interval-filament and subtree-filament graphs were given for finding maximum weight cliques [GA4], maximum weight induced paths [GA5], maximum weight induced matchings [CA], and holes and antiholes of given parity [GA6]. Polynomial time algorithms for maximum independent sets and cliques in subtree overlap graphs were given in [CS].

In Section 3 we present an algorithm to find maximum weight cliques in *k*-transitive graphs and prove that the complements of the *k*-filament graphs (to be defined) are *k*-transitive mixed graphs. In Section 4 we describe polynomial time algorithms to find holes of a given parity in ***co***(***H***-mixed) graphs and *3D*-interval-filament graphs. When we discuss an ***H*** mixed graph *H(V,E1,E2)*, we assume that the partition *E1*,*E2* is given, and when *H(V,E1)* is the complement of an intersection graph, we assume that the intersection representation is also given. Sections 5,6 contain additional observations.

Figure 1

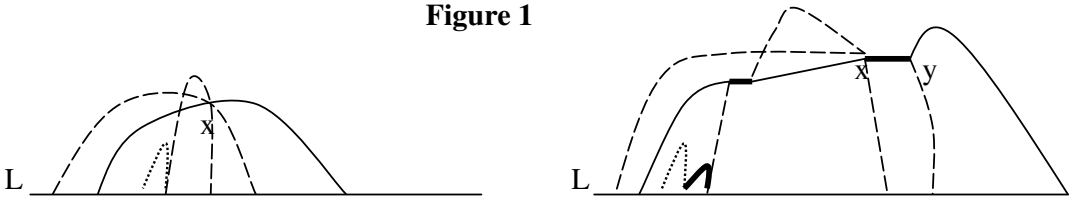
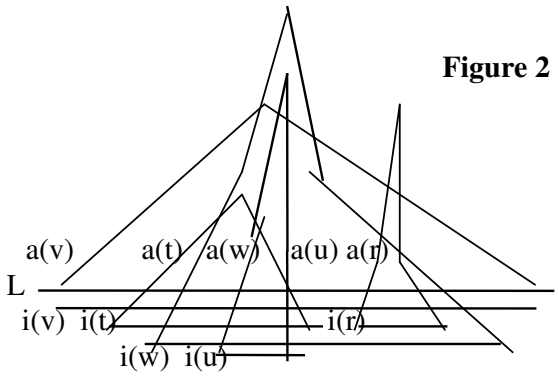
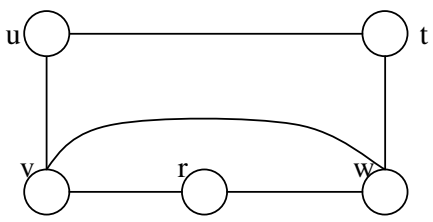


Figure 2





2. *3D*-INTERVAL-FILAMENT GRAPHS

In this Section we define families of intersection graphs of *3D*-interval-filaments as complements of ***coG*** mixed graphs for various families ***G*** of *2D*-interval-filament graphs. Consider a family ***G*** of *2D*-interval-filament graphs.

Lemma 1. Let *coG(V,E1,E2)* be a ***coG*** mixed graph such that *G(coE1)* is an intersection graph of a family *FI* of *2D*-interval-filaments on a line *L*. Then, $u \rightarrow v \in E2$ implies $N_{G(coE1)}(u) \subseteq N_{G(coE1)}(v)$. If the union of two intersecting elements in *FI* is of the same geometrical type as the elements in *FI* then there exists an intersection representation of *G(coE1)* by a family of *2D*-interval-filaments such that $u \rightarrow v \in E2$ implies $a(u) \subset a(v)$.

Proof. Let *FI* be a representation of *G(coE1)*. Assume that $u \rightarrow v \in E2$ in *coG(V,E1,E2)* and there is a vertex $w \in N_{G(coE1)}(u) - N_{G(coE1)}(v)$, that is, $(w,u) \in coE1$ and $(w,v) \notin coE1$. But $(w,v) \in E1$ implies, by the definition of ***coG*** mixed graphs, that $(w,u) \in E1$ contradicting the assumption that $(w,u) \in coE1$. Thus, $N_{G(coE1)}(u) \subseteq N_{G(coE1)}(v)$, i.e., if $a(w) \cap a(u) \neq \phi$ then $a(w) \cap a(v) \neq \phi$. Assume that $a(u) \not\subset a(v)$. Since $u \rightarrow v \in E2$, it follows that $(u,v) \in coE1$, $a(u) \cap a(v) \neq \phi$ and $i(u) \cap i(v) \neq \phi$, therefore $a(u) \cup a(v)$ is continuous and is within the perpendiculars to *L* at the endpoints of $i(u) \cup i(v)$. If $a(u) \cup a(v)$ is of the same geometrical type as the elements of *FI*, we can replace $a(v)$ by $a(u) \cup a(v)$ without changing the intersection relation; the new *2D*-interval-filament $a(v)$ is continuous and does not contain its endpoints. The graph *coG(E2)* is transitive, and by going from its sources to its sinks, we obtain an intersection representation of *G(coE1)* in which $u \rightarrow v \in E2$ implies $a(u) \subset a(v)$.□

Some geometrical types are preserved under union of intersecting elements: the union of two intervals on a line is an interval, the union of two subtrees of a tree is a subtree, the union of two *2D*-interval-filaments is a *2D*-interval-filament and the union of two *3D*-interval-filaments is a *3D*-interval-filament. But, the union of two intersecting polygons inscribed in a circle may not be a polygon inscribed in a circle; yet, as proved by Cameron [CA], their union can be replaced by the convex hull of the two polygons, without changing the intersection relationship. A graph is a polygon-circle graph iff it is an intersection graph of a family of polygonal filaments on a line [GA4], implying that the polygon-circle graphs are *2D*-interval-filament graphs; using Cameron's result, a representation by polygonal filaments fulfilling Lemma 2 can be obtained.

Theorem 2 [GA4]. A graph is a *2D*-interval-filament graph iff its complement is a cointerval mixed graph. Every *2D*-interval-filament graph *G* has a representation by a family of piecewise linear filaments each filament with at most $O(|V|)$ linear pieces and intersection points.

Proof. A co(*2D*-interval-filament) graph is a cointerval mixed graph [GA4].

Conversely, let the complement *coG(V,E1,E2)* of a graph *G* be a cointerval mixed graph such that *coG(V,E2)* is a transitive graph and *coG(V,E1)* is the complement of an interval graph with an interval representation *I* on a line *L*. By Lemma 1, we can assume that $u \rightarrow v \in E2$ implies $i(u) \subset i(v)$.

Now (Figure 2), to each interval $i(u) \in I$ we add, above *L*, a triangular hook *a(u)* connecting its two endpoints, such that if $i(u) \subset i(v)$ the two triangular hooks *a(u),a(v)* do not intersect. For every pair (one pair at a time) of adjacent vertices *u*,*v* of *G* such that the filaments *a(u),a(v)* do not intersect yet (this happens when $i(u) \subset i(v)$), we pick a point on *a(u)* and stretch it in a spike (perpendicular to *L*) to reach above and intersect *a(v)*. Also, for every triangular hook *a(w)* we meet on the way such that $u \rightarrow w \in E2$ (implying $i(u) \subset i(w)$) we stretch it also in a spike, such that no intersections between the spikes of *a(u),a(w)* (and the spikes of two distinct *a(w)*'s) occur; thus, we obtain that $a(v) \cap a(w) \neq \phi$. Let us prove that *w*,*v* are adjacent in *G*. Assume that the vertices *w*,*v* are adjacent in *coG(V,E1,E2)*. Since $i(w) \cap i(v) \neq \phi$, it follows that $(w,v) \notin E1$; thus, $(w,v) \in E2$ and $i(v) \subset i(w)$ or $i(w) \subset i(v)$. By the above construction, the spike of *a(u)* meets *a(w)* before *a(v)* implying that $i(w) \subset i(v)$. Thus $w \rightarrow v \in E2$ and by the transitivity of *E2*, $u \rightarrow w, w \rightarrow v \in E2$ implies $u \rightarrow v \in E2$, in contradiction to the assumption that *u*,*v* are not adjacent in *coG(V,E1,E2)*. By the construction, every filament *a(v)* is piecewise linear containing at most $O(|V|)$ pieces and intersection points.

Let us prove that *G* is the intersection graph of the family of *2D*-interval-filaments *{a(u)}*, with the spikes. Consider two adjacent vertices *u*,*v* of *G*; hence $(u,v) \notin E1 \cup E2$, implying $(u,v) \in coE1$ and $i(u) \cap i(v) \neq \phi$. If *i(u)*, *i(v)* overlap, then *a(u),a(v)* intersect. If $i(u) \subset i(v)$, then, by the above construction *a(u),a(v)* intersect. Consider now two non-adjacent vertices *u*,*v* of *G*; thus $(u,v) \in E1 \cup E2$. If $(u,v) \in E1$, i.e., $i(u) \cap i(v) = \phi$, then by the above construction *a(u),a(v)* do not intersect. Otherwise, $u \rightarrow v \in E2$ and $i(u) \subset i(v)$. Assume that $a(u) \cap a(v) \neq \phi$. This can appear during the construction of the spike of a curve *a(w)* to a curve *a(t)*, when $w \rightarrow u \in E2$ and $i(w) \subset i(u)$. By the transitivity of *E2*, we have $w \rightarrow v \in E2$, thus *a(v)* was also stretched together with *a(w),a(u)* to have a spike above the spikes of *a(u),a(w)*

to intersect *a(t)*, implying that *a(u)* and *a(v)* do not intersect, which is a contradiction. Therefore, *G* is the intersection graph of the family *{a(u)}* of *2D*-interval-filaments.□

Theorem 3. A graph is a *3D*-interval-filament graph iff its complement is a co(*2D*-interval-filament) mixed graph. Every *3D*-interval-filament graph has a representation by a family of piecewise linear filaments, each filament with $O(|V|^2)$ linear pieces.

Proof. As proved in the Introduction, the complement of a *3D*-interval-filament graph is a co(*2D*-interval-filament) mixed graph.

Conversely, let the complement *coG(V,E1,E2)* of a graph *G* be a co(*2D*-interval-filament) mixed graph such that *coG(V,E2)* is a transitive graph and *coG(V,E1)* is the complement of a *2D*-interval-filament graph with a representation as an intersection graph of a family *{a(v)}* of *2D*-interval-filaments on a line *L*. By Lemma 1 we can assume that $u \rightarrow v \in E2$ implies $a(u) \subset a(v)$ but every *a(v)* has $O(|V|^2)$ linear pieces. By Theorem 2, we can assume that each *2D*-interval-filament has $O(|V|)$ linear pieces and intersection points. Also (see Figure 1), we can assume that the intersections of *2D*-interval-filaments are curve segments. The relation $a(u) \subset a(v)$ defines a transitive relation on *V*, which defines a topological ordering on *V* obtained by the elimination of sources in this transitive relation. Assume that the line *L* is in a plane *PL* and let *PP* be a surface perpendicular to *PL* whose intersection with *PL* is exactly the union of all filaments. For every vertex *v*, we consider above *PL* a plane PQ_v parallel to *PL*, the planes PQ_v being ordered by the above topological ordering of *V*; hence, if two vertices *u*,*v* have $a(u) \subset a(v)$, then the plane PQ_u is between the planes PQ_v and *PL*. For every *v*, we consider the intersection between PQ_v and *PP(a(v))* which is in fact a projection *b(v)* of *a(v)* on PQ_v. We connect the endpoints of *b(v)* with the endpoints of *a(v)* at a slight angle, denoting this new *3D*-interval-filament by *v*, such that *v* is contained in *PP(a(v))*. By the construction, for two *3D*-interval-filaments *u*,*v*, if $a(u) \subset a(v)$, then *u* is contained in the region *s(v)* bounded by $v \cup a(v)$. We assume that the *3D*-interval-filaments do not contain their endpoints on *PL*. Clearly, every *v* has $O(|V|^2)$ linear pieces.

Now, consider a pair of adjacent vertices *u*,*v* of *G* such that *u*,*v* do not intersect yet; this can happen when $s(u) \subset s(v)$ or when *s(u)*,*s(v)* overlap, intersecting in a segment *x(u,v)*, while on the common surface *PP(x(u,v))* *3D*-filament *v* passes above *3D*-filament *u*. For every such pair *u*,*v*, one pair at a time, we pick a point in *u*, on *PP(x(u,v))* and stretch it in a spike to reach above and intersect *v*. Also, for every filament *w* met on the way such that $u \rightarrow w \in E2$ (implying $s(u) \subset s(w)$) we stretch it also in a spike, such that no intersections

between the spikes of u,w (and the spikes of two distinct w's) occur; thus, we obtain that $v \cap w \neq \phi$. Let us prove that w,v are adjacent in G. Assume that the vertices w,v are adjacent in $coG(V,E1,E2)$. Since $a(w) \cap a(v) \neq \phi$, it follows that $(w,v) \notin E1$; thus, $(w,v) \in E2$ and $a(v) \subset a(w)$ or $a(w) \subset a(v)$. By the construction, the spike of u meets w before v implying that $a(w) \subset a(v)$. Hence $w \rightarrow v \in E2$ and by the transitivity of $E2$, $u \rightarrow w, w \rightarrow v \in E2$ implies $u \rightarrow v \in E2$, in contradiction to the assumption that u,v are not adjacent in $coG(V,E1,E2)$. By the construction, every filament v is piecewise linear containing at most $O(|V|^2)$ pieces.

Let us prove that $G(V,E)$ is the intersection graph of the family of $3D$-interval-filaments $\{u\}$, with the spikes. Consider two adjacent vertices u,v of G; hence $(u,v) \notin E1 \cup E2$, implying $(u,v) \in coE1$. Thus, both when $a(u)$, $a(v)$ overlap or $a(u) \subset a(v)$, u,v intersect, by the construction of the spikes. Consider now two non-adjacent vertices u,v of G; thus $(u,v) \in E1 \cup E2$. If $(u,v) \in E1$, i.e., $a(u) \cap a(v) = \phi$, then by the above construction u,v do not intersect. Otherwise, $u \rightarrow v \in E2$ and $a(u) \subset a(v)$. Assume that $u \cap v \neq \phi$. This could appear only during the construction of the spike of a filament w to a filament t, when $w \rightarrow u \in E2$ and $a(w) \subset a(u)$. But, by the transitivity of $E2$, we have $w \rightarrow v \in E2$, thus v was also stretched together with w and u to have a spike above the spikes of u and w to intersect t, implying that u and v do not intersect, which is a contradiction. Therefore, G is the intersection graph of the family $\{u\}$ of $3D$-interval-filaments.□

Reference [GRU] proved without details, that a cocomparability graph is an intersection graph of piecewise linear filaments between two parallel lines. In fact this can be seen as follows: Given a transitive graph coG, we label a source vertex by v_1, eliminate it and continue labeling the remaining vertices with v_2 to v_n by eliminating sources. We arrange a family of parallel linear segments between two vertical parallel lines, and label the linear segments from the lowest to the highest with v_1 to v_n. Consider segment v_1. We pick a point on v_1 and stretch it into a spike, above the highest segment labeled v_i fulfilling $(v_1,v_i) \in G$. Also, for every segment v_j met on the way, fulfilling $(v_1,v_j) \notin G$, i.e. $v_1 \rightarrow v_j \in coG$, we stretch also v_j in a spike, such that no intersections between the spikes of v_1 and v_j occurs. The intersections added are between v_k and v_j, where v_k is not stretched with a spike since $(v_1,v_k) \in G$. Assume $(v_k,v_j) \notin G$. Therefore $v_j \rightarrow v_k \in coG$ and $v_1 \rightarrow v_j \rightarrow v_k \in coG$, implying $v_1 \rightarrow v_k \in coG$ by transitivity, in contradiction to the assumption that $(v_1,v_k) \in G$.

Consider a cocomparability graph G and its representation as a family of filaments between two parallel vertical lines $L1$, $L2$ in a plane PL. By the proof in [GRU], a filament must have at least two endpoints, one on $L1$, and another on $L2$, but it can have additional

endpoints on *L1*, *L2*. Therefore the union of two filaments is also a filament. For two intersecting filaments *a(u)*, *a(v)*, if every filament intersecting *a(u)* intersects *a(v)*, we replace *a(v)* by *a(u)∪a(v)* without changing the intersection relationship and still we have a representation by filaments. In fact, every filament delimitates a continuous region between *L1*, *L2* such that *G* is also the intersection graph of these regions. We can assume that the intersections of the filaments are curve segments. Perpendicular to *PL*, we consider a plane *PP* whose intersection with *PL* is the set of above filaments. For every filament *a(v)*, we connect its endpoints with a filament *v* in the plane *PP(a(v))* such that *v* does not contain its endpoints, if *a(u)*,*a(v)* overlap the two filaments intersect and if *a(u)*,*a(v)* are disjoint, the two filaments do not intersect (the proof of Theorem 3 shows how to obtain this). These filaments are called *cocomparability-filaments* and their intersection graphs are called *cocomparability-filament graphs*. Similarly for permutation graphs: A graph [EPL] *G(V,E)* is a permutation graph iff both *G* and *coG* are comparability graphs iff *G* is an intersection graph of linear segments between two vertical parallel lines *L1*,*L2*. The union of two such linear segments is not a linear segment. For two intersecting linear segments *a(u)*,*a(v)*, if every linear segment intersecting *a(u)* intersects *a(v)*, we replace *a(v)* by *a(u)∪a(v)* without changing the intersection relationship. We still have a representation by filaments, but each filament is a collection of linear segments, each collection corresponding to a connected subgraph of *G*. In fact, every such filament delimitates a continuous region between *L1*, *L2* such that *G* is also the intersection graph of these regions. We can define *3D*-filament graphs for permutation graphs, called *permutation-filament graphs*, in the same way we defined for cocomparability graphs. Thus, we obtain:

Corollary 4. A graph is a cocomparability-filament (permutation-filament) graph iff its complement is a transitive (permutation, respectively) mixed graph. Every such graph has a representation by a family of piecewise linear filaments, each filament with at most $O(|V|)$ linear pieces.

We remark that the family of permutation mixed graphs is the same as the family of copermutation mixed graphs since the complement of a permutation graph is also a permutation graph. Now, we give now a proof of a result in [GA4].

Theorem 5. A graph is a subtree-filament graph iff its complement is a cochordal mixed graph. Every subtree-filament graph *G(V,E)* has a representation by a family of piecewise linear filaments, each filament with at most $O(|V|)$ linear pieces.

Proof. As proved in the Introduction, the complement of a subtree-filament graph is a cochordal mixed graph.

Conversely, let the complement *coG(V,E1,E2)* of a graph *G(V,E)* be a cochordal mixed graph such that *coG(V,E2)* is a transitive graph and *coG(V,E1)* is the complement of a chordal graph *G(V,coE1)*. A chordal graph *G(V,coE1)* has at most $|V|$ maximal cliques. Also, there exists a tree *T*, whose vertices are in a one-to one correspondence with the maximal cliques of *G(V,coE1)*, such that for every $v \in V$, the family of maximal cliques containing *v* defines a subtree *a(v)* of *T*; *G(V,coE1)* is the intersection graph of $\{a(v) | v \in V\}$. By the definition of T, if $u \rightarrow v \in E2$ then $N_{coG(E1)}(u) \subseteq N_{coG(E1)}(v)$ and $a(u) \subset a(v)$. We add to every vertex *x* of *T* a branch *(x,y)* adding *(x,y)* to every subtree containing *x*. Therefore, we can assume that the intersection of the subtrees corresponding to a maximal clique of *G(V,coE1)* contains an edge of *T*. The relation $a(u) \subset a(v)$ defines a transitive relation on *V*, which defines a topological ordering on *V* obtained by the elimination of sources in this transitive relation. Assume that *T* is in a plane *PL* and let *PP* be a surface perpendicular to *PL* whose intersection with *PL* is exactly *T*. For every vertex *v*, we consider above *PL* a plane PQ_v parallel to *PL*, such that the planes are ordered by the above topological ordering of *V*: if two vertices *u*,*v* have $a(u) \subset a(v)$, then the plane PQ_u is between the planes PQ_v and *PL*. For every *v*, we consider the intersection between PQ_v and *PP(a(v))* which is in fact a projection *b(v)* of *a(v)* on PQ_v. We connect the endpoints of *b(v)* with the endpoints of *a(v)* at a slight angle denoting this subtree-filament by *v*, such that *v* is contained in *PP(a(v))*. By the construction, for two subtree-filaments *u*,*v*, if $a(u) \subset a(v)$, then *u* is contained in the region *s(v)* bounded by $v \cup a(v)$. We assume that the subtree-filaments do not contain their endpoints on plane *PL*. Clearly, every *v* contains $O(|V|)$ linear pieces.

Now, consider a pair of adjacent vertices *u*,*v* of *G* such that *u*,*v* do not intersect yet; this can happen when $s(u) \subset s(v)$ or when *s(u)*,*s(v)* overlap and on the common surface *PP(x(u,v))* subtree-filament *v* passes above subtree-filament *u*. For every such pair *u*,*v*, one pair at a time, we pick a point in *u*, on *PP(x(u,v))* and stretch it in a spike to reach above and intersect *v*. Also, for every filament *w* met on the way such that $u \rightarrow w \in E2$ (implying $s(u) \subset s(w)$) we stretch it also in a spike, such that no intersections between the spikes of *u*,*w* (and the spikes of two distinct *w*'s) occur; thus, we obtain that $v \cap w \neq \phi$. By the construction, every *v* is piecewise linear containing at most $O(|V|)$ pieces.

The proof that $G(V,E)$ is the intersection graph of the family of subtree-filaments $\{u\}$, with the spikes, is similar to the one of Theorem 3.□

Consider a subtree-filament graph $G(V,E)$ with a representation on a tree T. For a point x in T, let $V_x=\{v|\ v\in V,\ x\in a(v)\}$. The subgraph $G(V_x)$ is a cocomparability graph: for every two non-intersecting filaments $u,v\in V_x$, we have $x\in a(u)\cap a(v)\neq\phi$, thus $s(u)\subset s(v)$ or $s(v)\subset s(u)$ implying $u\rightarrow v\in E2$ or $v\rightarrow w\in E2$ in $coG(V_x)$, where $E2$ is a transitive relation; [GA4] uses this in an algorithm to find maximum cliques in subtree-filament graphs.

A similar family of *3D*-filaments can be defined using families of polygonal filaments on a line obtained from polygons in a circle. By a proof similar to the proofs of Theorems 3−5, we obtain:

Corollary 6. A graph is an intersection graph of a family of polygon-circle filaments iff its complement is a co(polygon-circle) mixed graph.

3. ALGORITHMS FOR MAXIMUM WEIGHT CLIQUES IN *k*-TRANSITIVE AND *k*-FILAMENT GRAPHS

Consider a transitive, i.e., a *1*-transitive, graph $G(V,E)$ and let $v_1,...,v_n$ be a topological ordering of its vertices; we denote $N^-(v)=\{u|\ u\rightarrow v\in E\}$. The algorithm to find a maximum weight clique in G works as follows: For every i we construct a maximum weight clique c_i in $G(V_i)$ such that $v_i\in c_i$. Assume that we have already $c_1,...,c_i$. Let us construct c_{i+1}. If $N^-(v_{i+1})=\phi$ then $c_{i+1}=\{v_{i+1}\}$. Assume that $N^-(v_{i+1})\neq\phi$. By the transitivity of G, for every $v_j\in N^-(v_{i+1})$, $c_j\cup\{v_{i+1}\}$ is a clique of $G(V_i)$. Thus, $c_{i+1}=c_j\cup\{v_{i+1}\}$ where c_j is a clique with maximum weight in $\{c_j|\ v_j\in N^-(v_{i+1})\}$. In this algorithm we remember with every c_i, its highest indexed vertex v_i, from which to continue to v_{i+1}. A similar algorithm works also for k-transitive graphs G. For every v_i and for every clique $\{v_{i,1},...,v_{i,k-1},v_i\}$ with k vertices in $G(V_i)$, we remember a maximum weight clique c of $G(V_i)$ whose k highest indexed vertices are $\{v_{i,1},...,v_{i,k-1},v_i\}$. Then, by the definition of the k-transitive graphs, $c\cup\{v_{i+1}\}$ is a clique of $G(V_{i+1})$ iff $\{v_{i,1},...,v_{i,k-1},v_i,v_{i+1}\}$ is a clique. Below is a formal description of the algorithm.

Consider a weighted k-transitive graph $G(V,E)$ and let $v_1,...,v_n$ be a topological ordering of its vertices; for $U\subseteq V$ we denote $\omega(U)=\sum_{v\in U}\omega(v)$. For a clique c of G with $|c|\geq k$, we denote the set of its k highest indexed vertices $\{u_{|c|-k+1},...,u_{|c|}\}$ by $tail(c)$. The algorithm for finding a maximum weight clique in G works by constructing for each v_i, $i=1,2,...,n$, the set T_i of all triples $[b,pr(b),\Omega(b)]$ where: (i) b is a clique with k vertices in $G(V_i)$ such that

$v_i \in b$; (ii) $\Omega(b)=max\{\omega(c) \mid c$ *is a clique in* $G(V_i)$ *having* $|c| \geq k$ *and* $b=tail(c)\}$; (iii) $pr(b)$ is the highest indexed vertex in c-b, c being a clique giving the maximum to $\Omega(b)$;

In addition, with each v_i we keep a pair b_i,Ω_i defined as follows: $\Omega_i=max\{\omega(c) \mid c$ *is a clique in* $G(V_i,E)$ *and* $v_i \in c\}$; let c be a clique for which $\Omega_i=\omega(c)$. If $|c| \geq k$ then $b_i=tail(c)$ and if $|c|<k$ then $b_i=c$. The algorithm for evaluating T_i,b_i,Ω_i for every i is given below.

1. ***for*** $i=1$ ***to*** n set $T_i=\phi$;
2. ***for*** $i=1$ ***to*** n ***do***
   $\Omega_i=0$;
3. let $C_i=\{b \mid b$ *is a clique of* $G(V_i)$, $|b|=k$, $v_i \in b\}$;
4. ***for*** every $b \in C_i$
   let $\Omega(b)=\omega(b)$, $pr(b)=\Lambda$;
   let v_r be the lowest indexed vertex in b;
5. ***for*** every $u \in N(v_i) \cap N(v_r)$ such that $b \cup \{u\}$ is a clique
   let $c=b \cup \{u\}-\{v_i\}$;
   let v_j be the highest indexed vertex in c;
6. find $[c,pr(c),\Omega(c)]$ in T_j;
7. ***if*** $\Omega(b)<\Omega(c)+\omega(v_i)$ ***then*** $\Omega(b)=\Omega(c)+\omega(v_i)$ and $pr(b)=u$;
8. add $[b,pr(b),\Omega(b)]$ to T_i;
9. ***if*** $\Omega_i<\Omega(b)$ ***then*** $\Omega_i=\Omega(b)$, $b_i=b$;
10. ***for*** every maximal clique c of $G(V_i)$ fulfilling $|c|<k$, $v_i \in c$
    if $\Omega_i<\omega(c)$ ***then*** $\Omega_i=\omega(c)$ and $b_i=c$;
11. ***end***.

To find a maximum weight clique in $G(V,E)$, we find a vertex v_i having $\Omega_i=max_{1 \leq j \leq n}\Omega_j$ and backtrack on the b_i's and the $pr(b_i)$'s.

Lemma 7. The algorithm finds a maximum weight clique in a k-transitive graph.

Proof. Let us first prove by induction that the sets T_i,b_i,Ω_i constructed by the algorithm are correct. Assume that this is true for $j<i$. In Steps 3-9 we consider every clique b of $G(V_i)$ such that $|b|=k$, $v_i \in b$, and every clique c in $G(N(v_i))$ such that $|c|=k$, $b=tail(c \cup \{v_i\})$; let v_j be the highest indexed vertex in c. In Step 6 we find $[c,pr(c),\Omega(c)]$ in T_j. Thus, by induction, $\Omega(c)$ is the weight of a maximum weight clique d in $G(V_j)$ having c

as tail. Let us prove that $d \cup \{v_i\}$ is a clique of $G(V_i,E)$. Consider a vertex $u \in d-c$. The set $\{u\} \cup c$ is a clique and $\{u\} \cup c \cup \{v_i\}$ is a directed path starting in u with $k+2$ vertices, its first $k+1$ vertices being the clique $\{u\} \cup c$ and its last $k+1$ vertices being the clique $c \cup \{v_i\}$. Thus, by the definition of the k-transitivity $\{u\} \cup c \cup \{v_i\}$ is a clique, implying that u is adjacent to v_i. Hence, $d \cup \{v_i\}$ is a clique having maximum weight among all cliques in $G(V_i,E)$ which have $c \cup \{v_i\}$ as their highest indexed vertices. Thus, in Step 7, $\Omega(b)$ is the maximum weight of a maximum weight clique c' in $G(V_i,E)$ such that $b=tail(c')$ and $pr(b)$ is the highest indexed vertex in $c'-b$. Therefore the construction of T_i is correct. In Step 9 we evaluate the weight of a maximum weight clique c' in $G(V_i,E)$ such that $|c'| \geq k$ and $v_i \in c'$. In Step 10 we compare this weight with the weights of the maximal cliques c of $G(V_i,E)$ such that $|c| < k$ and $v_i \in c$, and take Ω_i as the maximum weight and b_i as the highest indexed k vertices of a maximum weight clique. Thus, the evaluation of Ω_i and b_i is correct.

Consider a maximum weight clique c of G; let v_i be its highest indexed vertex and let $b=tail(c)$. If $|c| < k$, then c is found in Step 10 and $\omega(c)=\Omega_i$. If $|c| \geq k$ then by Steps 3-9, the triple $[b,pr(b),\Omega(b)]$ appears in T_i, and Ω_i is set equal to $\omega(c)$. Thus, the algorithm finds c or a clique of equal weight.□

By keeping every T_i in lexicographic order and inserting appropriate pointers, we can perform the searching in Step 6 in $k|V|$ steps. Thus, the algorithm works in time $O(k|V|^{k+2})$. When $k=1$, every T_i has exactly one triple $[\{v_i\},pr(\{v_i\}),\Omega(\{v_i\})]$ and the algorithm is identical to the one in [EPL] working in time $O(|E|)$.

Consider a family I of intervals on a line L and a vertex set $V=\{v|\ i(v) \in I\}$. In the plane containing L, above L, we construct to each interval $i(v) \in I$ a filament v connecting its two endpoints; two filaments may intersect even when their intervals are disjoint. Let $F=\{v|\ i(v) \in I\}$ and let $G(V,E)$ be the intersection graph of F. For $u,v \in F$, we denote $i(u) < i(v)$ whenever $i(u) \cap i(v) = \phi$ and $i(u)$ appears at the left of $i(v)$. Consider two filaments u,v such that $u \cap v \neq \phi$ and $i(u) < i(v)$. Let $K(u,v)$ be the maximum size of a subfamily D of F fulfilling: (i) every $w \in D$ intersects neither u nor v and has $i(u) < i(w) < i(v)$; (ii) every pair $w,z \in D$ has $w \cap z = \phi$ and $i(w) \cap i(z) = \phi$.

Let $k=1+max\{K(u,v)|\ u,v \in F,\ u \cap v \neq \phi$ *and* $i(u) < i(v)\}$. We say that F is a *k-filament family* and its intersection graph G is a *k-filament graph*. In other words, F is a k-filament family if for every two filaments u,v having $u \cap v \neq \phi$ and $i(u) < i(v)$, there are no k filaments w with $i(u) < i(w) < i(v)$ which intersect neither u nor v, are mutually disjoint and have mutually

disjoint intervals. Clearly, the *2D*-interval-filament graphs are *1*-filament graphs. Let $s(v)$ be the region bounded by $v \cup i(v)$. Two filaments u,v intersect iff $s(u),s(v)$ overlap. The intersection graph *GS* of $\{s(v) \mid v \in V\}$ is called a *k*-region graph.

Theorem 8. The complement of a *k*-filament graph *G* is a co-*k*-region mixed graph and a co-*k*-region mixed graph is a *k*-transitive mixed graph $coG(V,E1,E2)$ where $coG(V,E1)$ is *k*-transitive, $coG(V,E2)$ is transitive, and both $coG(V,E1)$, $coG(V,E2)$ have a common topological ordering.

Proof. Let *G* be a *k*-filament graph represented by a family *F* of *k*-filaments. The edge set *coE* of $coG(coE)$ can be partitioned into two disjoint subsets $E1=\{u \rightarrow v \mid u,v \in V, u \cap v = \phi \text{ and } i(u) < i(v)\}$, $E2=\{u \rightarrow v \mid u,v \in V, u \cap v = \phi \text{ and } i(u) \subset i(v)\}$. For two vertices u,v we have $(u,v) \in coE1$ iff $u \cap v \neq \phi$ or $i(u) \subset i(v)$, iff $s(u) \cap s(v) \neq \phi$. Hence, $G(coE1)=GS$. Consider three vertices $u,v,w \in V$ such that $u \rightarrow v \in E2$ and $(v,w) \in E1$. Then $i(u) \subset i(v)$, $u \cap v = \phi$ and $i(v) \cap i(w) = \phi$, $v \cap w = \phi$ implying $u \cap w = \phi$, $i(u) \cap i(w) = \phi$, i.e., $(u,w) \in E1$. Therefore, the complement of *G* is a co-*k*-region mixed graph.

Let $coG(V,E1,E2)$ be a co-*k*-region mixed graph where $G(coE1)$ is a *k*-region graph *GS* and $E1=\{u \rightarrow v \mid s(u) \cap s(v) = \phi \text{ and } i(u) < i(v)\}$. Clearly $coG(E2)$ is transitive and $coG(V,E1)$, $coG(V,E2)$, $coG(V,coE)$ are acyclically oriented with a common topological ordering. Consider a directed path $p(u_1,u_{k+2})$ with $k+2$ vertices in $coG(V,E1)=coGS$ such that each of the two subpaths $p(u_1,u_{k+1})$ and $p(u_2,u_{k+2})$ induces a clique. By the definition of *E1*, $i(u_1) < i(u_2) < \ldots < i(u_{k+2})$. Assume that $u_1 \rightarrow u_{k+2} \notin E1$, i.e., $i(u_1) \cap i(u_{k+2}) \neq \phi$. Since the subpaths $p(u_1,u_{k+1})$ and $p(u_2,u_{k+2})$ of *p* are cliques, it follows that $u_2,\ldots,u_{k+1}$ intersect neither u_1 nor u_{k+2}, are mutually non-intersecting and $i(u_2) < \ldots < i(u_{k+1})$, contradicting the fact that *F* is a *k*-filament family. Thus $u_1 \rightarrow u_{k+2} \in E1$ and $\{u_1,u_2,\ldots,u_{k+2}\}$ is a clique of $coG(V,E1)$. Therefore, $coG(V,E1)=GS$ is *k*-transitive and $coG(V,E1,E2)$ is a *k*-transitive mixed graph.□

By Theorem 8, the complement $coG(V,E1,E2)$ of a *k*-filament graph *G* is a *k*-transitive mixed graph with $coG(V,E1)$,$coG(V,E2)$ having a common topological ordering. Thus, we can find a maximum weight independent set in *G* in time $O(k|V|^{k+3})$, using the algorithms in [GA4] and above.

4. HOLES OF GIVEN PARITY IN *co(H* MIXED) GRAPHS

Consider a family of ***H*** mixed graphs, such that the family ***G=coH*** (***H=coG***) has a polynomial time algorithm to find an induced path of a given parity between two vertices;

this implies that ***G*** also has a polynomial time algorithm to find an induced hole of a given parity. Consider a graph $G(V,E)$ such that $coG(V,E1,E2)$ is a ***coG*** mixed graph.

Lemma 9. Let $p(v_1,v_k)$ be an induced path in G and consider some i, $1 \le i \le k-2$.

a) If $v_i \rightarrow v_k \in E2$, then for every v_j, $1 \le j \le k-2$, we have $v_j \rightarrow v_k \in E2$.

b) If $(v_i,v_k) \in E1$ and for every v_j, $1 \le j \le k-2$, $v_k \rightarrow v_j \notin E2$, then for every vertex $v_s \in p(v_1,v_{k-2})$ there exists $(v_1,v_s) \in E1$.

c) If some $v_s \in p$ fulfils that $v_s \rightarrow v_j \notin E2$ for every $v_j \in p$, then between $p(v_1,v_{s-2})$ and $p(v_{s+2},v_k)$ there are only $E1$ edges in coG.

Proof. a) Assume that $v_i \rightarrow v_k \in E2$. By the definition of the ***coG*** mixed graphs, we cannot have $(v_{i+1},v_k) \in E1$, since this would imply $(v_i,v_{i+1}) \in E1$. Thus, $v_{i+1} \rightarrow v_k \in E2$; and so on, for every vertex v_j, $1 \le j \le k-2$, to the right and left of v_i in p.

b) Assume that $(v_i,v_k) \in E1$ and for every v_j, $1 \le j \le k-2$, $v_k \rightarrow v_j \notin E2$. Assume that $(v_r,v_k) \notin E1$ for some r, $1 \le r \le k-2$. Thus, $v_r \rightarrow v_k \in E2$ implying that $v_i \rightarrow v_k \in E2$ by (a), contradicting the assumptions.

c) Assume that some $v_s \in p$ fulfils that $v_s \rightarrow v_j \notin E2$ for every $v_j \in p$, and $(v_r,v_t) \notin E1$ for some r,t, $1 \le r \le s-2$, $s+2 \le t \le k$. Thus, $v_r \rightarrow v_t \in E2$ or $v_t \rightarrow v_r \in E2$. If $v_r \rightarrow v_t \in E2$, then $v_s \rightarrow v_t \in E2$ by (a), contradicting the assumptions. Similarly when $v_t \rightarrow v_r \in E2$.□

Lemma 10. Any hole $h(v_1,v_k)$ of G, which is not a hole of $G(coE1)$, has two non-adjacent vertices v_i,v_j such that $v_i \rightarrow v_j \in E2$ and has no three vertices v_i,v_j,v_k such that $v_i \rightarrow v_j, v_j \rightarrow v_k \in E2$.

Proof. If h is not a hole of $G(coE1)$, then h has two non-adjacent vertices v_i,v_j such that $v_i \rightarrow v_j \in E2$. Assume that h has three vertices v_i,v_j,v_k such that $v_i \rightarrow v_j, v_j \rightarrow v_k \in E2$; w.l.o.g. assume that $i < k < j$. But, Lemma 9a applied to $v_i \rightarrow v_j \in E2$ and the path $p(v_i,v_j)$ implies that for every $v_r \in p(v_i,v_{j-2})$, we have $v_r \rightarrow v_j \in E2$, hence $v_k \rightarrow v_j \in E2$.□

Lemma 11. Consider a hole $h(v_1,v_k)$ of G which is not a hole of $G(coE1)$. There is a vertex v_i in h such that for every $v_j \in h(v_{i+2},v_{i-2})$, we have $v_j \rightarrow v_i \in E2$ and $h(v_{i+2},v_{i-2})$ is an induced path of $G(coE1)$.

Proof. Since h is not a hole of $G(coE1)$, h has two non-adjacent vertices v_i,v_j such that $v_j \rightarrow v_i \in E2$. Thus, by Lemma 9a applied to the path $h(v_{i+2},v_{i-2})$, for every $v_j \in h(v_{i+2},v_{i-2})$, we have $v_j \rightarrow v_i \in E2$. By Lemma 10, there are no $E2$ edges between vertices in $h(v_{i+2},v_{i-2})$, thus $h(v_{i+2},v_{i-2})$ is an induced path of $G(coE1)$.□

The algorithm to find a hole $h(v_1,v_k)$ of a given parity in the complement G of a ***coG*** mixed graph $coG(V,E1,E2)$, when the family of $G(coE1)$ graphs has a polynomial time algorithm to find an induced path of a given parity between two vertices, works as follows: By Lemma 11, h is a hole of $G(coE1)$, which we can find directly, or h has a vertex v_i such that for every $v_j \in h(v_{i+2},v_{i-2})$, $v_j \rightarrow v_i \in E2$. We take every vertex of G as candidate for v_i of h, every two non-adjacent vertices in $N_G(v_i)$ as candidates for v_{i-1},v_{i+1} and every $v \in N_G(v_{i-1})-N_G[v_i]$, $w \in N_G(v_{i+1})-N_G[v_i]$ as candidates for v_{i-2},v_{i+2}. In $G((V-N_G[v_i,v_{i-1},v_{i+1}]) \cup \{v_{i-2},v_{i+2}\}, coE1)$ we find an induced path of the needed parity from v_{i+2} to v_{i-2}.

The above algorithm can be adjusted to find maximum weighted holes of a given parity, when the family $G(coE1)$ has a polynomial time algorithm to find a maximum weight induced path of a given parity between two vertices. The algorithms work in time $O(|V|^5|V|^c)$ where $O(|V|^c)$ is the time needed to find a (maximum weight) induced path of a given parity in $G(coE1)$. Since the family of *2D*-interval-filament graphs has an $O(|V|^{12})$ polynomial time algorithm to find a weighted induced path of a given parity between two vertices [GA5], the above algorithm can find maximum weighted holes of a given parity in *3D*-interval-filament graphs in $O(|V|^{17})$ time.

Consider a hole $h(v_1,v_r)$ in a k-filament graph G (Section 4); by Theorem 8, $coG(V,E1,E2)$ is a co-k-region mixed graph. A hole $h(v_1,v_r)$ of the k-region graph $G(coE1)=GS$ must contain two adjacent vertices, say v_1,v_r such that $i(v_1) \cap i(v_r) = \phi$ and for every $w \in h(v_3,v_{k-1})$ it exists $i(v_1) \prec i(w) \prec i(v_r)$. Thus, $r \prec 2k+3$, otherwise h would contain k mutually non-adjacent vertices, not adjacent to v_1,v_r, corresponding to k filaments v_j with $i(v_1) \prec i(v_j) \prec i(v_r)$ which intersect neither v_1 nor v_r, are mutually disjoint and have mutually disjoint intervals. Thus, by Lemma 11, if $h(v_1,v_r)$ has $r \geq 2k+3$ vertices, then h has a vertex v_1 such that for every $v_j \in h(v_3,v_{k-1})$, we have $v_j \rightarrow v_1 \in E2$ and $h(v_3,v_{k-1})$ is an induced path of GS. We look for an induced path $h(v_3,v_{k-1})$ of such a hole h of GS by going on L from left to right, remembering for each such path the last $2k$ filaments, and adding the next filament v only to those paths whose last $2k$ filaments, except the last one, do not intersect v. Therefore, for each parity and every induced path X with $2k$ filaments, we find an induced path from v_3 whose last $2k$ filaments form X and continue to the filaments intersecting only the last one in X. Thus, we can find in G a hole of a given parity $h(v_1,v_r)$ with $r \geq 2k+3$ using this algorithm, and with $r \prec 2k+3$ directly, k being constant.

5. EXAMPLES

Consider the bipartite graph G composed of a hole $h(v_1,v_8)$ and two vertices w_1 adjacent to v_1,v_5, and w_2 adjacent to v_3,v_7. Assume that G is a $2D$-interval-filament graph. By Lemma 11, h contains a vertex v such that for every $u \in h\text{-}N_G[v]$, we have $u \rightarrow v \in E2$ and for every two non-adjacent vertices $u,x \in h\text{-}N_G[v]$ we have $(u,x) \in E1$. Assume that $v=v_1$. In the hole $w_2 — v_7 — v_8 — v_1 — v_2 — v_3 — w_2$, we also have $w_2 \rightarrow v_1 \in E2$. In the hole $h1=w_2 — v_3 — v_4 — v_5 — v_6 — v_7 — w_2$, w_2 must be the vertex required by Lemma 11 to have incoming $u \rightarrow w_2 \in E2$ edges in coG from vertices $u \in h1\text{-}N_G[w_2]$. But then, by Lemma 9 applied to the induced path $w_2 — v_7 — v_6 — v_5 — w_1 — v_1 — v_2$ we obtain that $v_1 \rightarrow w_2 \in E2$, contradicting $w_2 \rightarrow v_1 \in E2$. Assume that $v=v_2$. In the hole $w_2 — v_7 — v_8 — v_1 — v_2 — v_3 — w_2$, we also have $w_2 \rightarrow v_2 \in E2$. In the hole $h1=w_2 — v_3 — v_4 — v_5 — v_6 — v_7 — w_2$, the vertex required by Lemma 11 to have $E2$ incoming edges in coG from its other non-adjacent vertices of $h1$ is w_2 or v_3; if it is w_2, we obtain the contradiction $v_2 \rightarrow w_2 \in E2$, as above, and if it is v_3, by symmetry, we are in the same situation as $v=v_1$. Therefore G is not a $2D$-interval-filament graph. The graph $G\text{-}\{v_1\}$ is a $2D$-interval-filament graph since we represent $h1$ as a hole with w_2 as the vertex required by Lemma 11 to have incoming $E2$ edges in coG from its other non-adjacent vertices of $h1$; then, we attach filaments to endpoints of v_3,v_5,v_7 denoted v_2,w_1,v_8.

Consider the bipartite graph $H(V)$ composed of two copies of the above graph $G_i(h_i(v_{1,i},v_{8,i}) \cup \{w_{1,i},w_{2,i}\})$, $i=1,2$, and four additional vertices z_1,z_3,z_5,z_7, every z_j adjacent to $v_{j,1}$ and $v_{j,2}$. Assume that H is a $3D$-interval-filament graph. The graph G_1 is not a $2D$-interval-filament graph, hence coG_1 must contain an $E2$ edge $v_{j,1} \rightarrow v_{s,1} \in E2$. By deleting $N_H[v_{s,1}]$ from H we obtain a connected graph, hence every $u \in V\text{-}N_H[v_{s,1}]$ has $u \rightarrow v_{s,1} \in E2$. Also, through every two non-adjacent vertices $v_{i,2}$, $v_{j,2}$ of G_2 and $v_{s,1}$ there is a hole of H implying that $(v_{i,2},v_{j,2}) \in E1$. Therefore coG_2 has only $E1$ edges and must be a $2D$-interval-filament graph, contradicting the previous example. The graph $H\text{-}\{v_{1,1},v_{3,2}\}$ is a $2D$-interval-filament graph since we represent $h1$ of G_1 as before, and we draw G_2 inside $v_{6,1}$ to connect $v_{5,1}$ with $v_{7,1}$, through z_5,z_7. Now, we represent $v_{1,1}$ by a $3D$-interval-filament where $a(v_{1,1})$ is the union of all these $2D$-interval-filaments, $v_{1,1}$ intersecting outside the plane PL with $v_{2,1},v_{8,1},w_{1,1},z_1$. Hence, $H\text{-}\{v_{3,2}\}$ is a $3D$-interval-filament graph.

We define recursively a sequence of families of graphs: {co[$(i+1)D$-interval-filament] graphs}={co[iD-interval-filament] mixed graphs}. In a similar way to the

previous example, by taking i copies of G_1 and connecting every two with z's, we obtain bipartite graphs which are not $(i+1)D$-interval-filament graphs.

6. ADDITIONAL OBSERVATIONS

Let $G(V,E)$ be the intersection graph of a family of $3D$-interval-filaments $FF=\{v| v\in V\}$. Using the observation that the union of two intersecting $2D$-interval-filaments is a $2D$-interval-filament, Cameron[CA] proved that a maximum induced matching in a $2D$-interval-filament graph can be found in polynomial time. Since the union of two intersecting $3D$-interval-filaments is also a $3D$-interval-filament, we obtain a similar result: $FF1=\{u\cup v| \ u\cap v\neq\phi, \ u,v\in FF\}$ is also a family of $3D$-interval-filaments and we denote its intersection graph by H. Two filaments in $FF1$ do not intersect iff they correspond to two edges $e1,e2$ of G such that $e1,e2$ have no vertex in common in G and no vertex of $e1$ is adjacent in G to a vertex of $e2$. Therefore, a maximum weight independent set in H is a maximum weight induced matching in G and vice-versa. A maximum weight independent set of the $3D$-interval-filament graph H can be obtained by the algorithm in [GA4].

Consider a maximum clique C of G. For every two vertices $u,v\in C$ the $3D$-interval-filaments u,v intersect, thus $s(u)\cap s(v)\neq\phi$ and $i(u)\cap i(v)\neq\phi$. By the Helly property, there is a point x on L contained in all the intervals $i(v)$, $v\in C$, every v having endpoints on both sides of x. Let G_x be the subgraph of G corresponding to the $3D$-interval-filaments having endpoints on both sides of x, and open L at x into two lines $L1,L2$. We obtain a family of cocomparability-filaments, hence G_x is a cocomparability-filament graph. Therefore, a maximum clique of G is contained in a cocomparability-filament graph obtained in this way, and could be found if a polynomial time algorithm to find a maximum clique in a cocomparability-filament graph were known. Similarly, a clique intersecting all maximum independent sets of a perfect $3D$-interval-filament graph is contained in such a cocomparability-filament graph. Let $h(v_1,v_k)$ be a hole in $coG(V,E1,E2)$. The co($2D$-interval-filament) graph $coG(E1)$ has a partition of its edge set $E1$ into $E11$ corresponding to pairs of $2D$-interval-filaments having disjoint intervals and $E12$ corresponding to pairs of non-intersecting $2D$-interval filaments having intervals contained one in another. If h has an $E11$ edge (v_1,v_k), then $i(v_1)\cap i(v_k)=\phi$ and for a point x in L between the $i(v_1),i(v_k)$, every $3D$-interval-filament v_i, $2<i<k-1$, intersects both v_1,v_k having endpoints on both sides of x. Hence $h(v_3,v_{k-2})$ is an induced path in the complement of the cocomparability-filament

graph G_x. If $h(v_1,v_k)$ has no *E11* edges, then every two intervals $i(v_i),i(v_j)$, $1\leq i,j\leq k$, intersect and by the Helly property there is a point x in L contained in all of them. Hence h is an induced hole in coG_x. Therefore, a hole $h(v_1,v_k)$ of coG is either a hole of some coG_x or for some i, $h(v_{i+2},v_{i-2})$ is an induced path in some coG_x.

A permutation graph can be represented as the intersection graph of a family of segments connecting two parallel lines *L1,L2*. When we connect *L1,L2* at one endpoint, we obtain a family of *2D*-interval-filaments on a line L fulfilling that no two intersecting *2D*-interval-filaments u,v have $i(u)\subset i(v)$; such *2D*-interval-filament graphs may have additional interesting properties.

Consider a *2D*-interval filament graph G based on a family of *proper intervals* on a line L whose complement is $coG(V,E1,E2)$; as in Lemma 1, when $u\rightarrow v\in E2$ we replace $i(v)$ by $i(u)\cup i(v)$ to obtain that $i(u)\subset i(v)$. The graph $G(coE1)$ is a proper interval graph, thus every $N_{G(coE1)}(v)$ is a cobipartite graph, since every interval $i(u)$, $u\in N_{G(coE1)}(v)$, must contain one of $i(v)$ endpoints. Since $u\rightarrow v\in E2$ implies $i(u)\subset i(v)$, it follows that every such u must end at one of v's endpoints. This enables to construct intersection representations as *2D*-interval-filaments for complements of co-proper-interval mixed graphs and for ***coG*** mixed graphs where every $N_{G(coE1)}(v)$ is a cobipartite graph. Similar definitions exist for *3D*-interval-filament graphs.

Consider a *2D*-circular-arc-filament graph $G(V,E)$ [GA4] of a family $\{a(v)\}$ of *2D*-arc-filaments based on a family of arcs in a circle such that (i) no two arcs cover the circle, (ii) $N_G(u)\subseteq N_G(v)$ implies $a(u)\subset a(v)$ and (iii) the intersection of filaments are segments. We can define *3D*-circular-arc-filaments and intersection graphs in a similar way to the *3D*-interval-filaments and intersection graphs; it can be proved that a graph is a *3D*-circular-arc-filament graph iff its complements is a co(*2D*-circular-arc-filament) mixed graph.

There is a simple generalization of the polygon-circle graphs: a *polygon-cactus* is a cactus in which every block is a subtree or a polygon. *A polygon-cactus graph* is the intersection graph of a family F of polygon-cacti on a cactus CA such that every polygon of a polygon-cactus in F, is inscribed in a circle of CA. As proved in [GA4], a graph is a polygon-circle graph iff it is an intersection graph of a family of polygonal filaments on a line L, implying that the polygon-circle graphs are *2D*-interval-filament graphs. Similarly, the polygon-cactus graphs on a cactus CA are contained in the family of *2D*-interval-filament graphs, obtained first by inscribing on L the basic tree of the cactus CA and then the polygonal lines in the circles.

Consider a graph $H(V,F)$. The relation $N_H(v) \subseteq N_H(u)$ is transitive and we can add to H a transitive subset of edges $E2$ fulfilling that $u \rightarrow v \in E2$ implies $N_H(v) \subseteq N_H(u)$. The new graph, with $E1=F$, is a ***G*** mixed graph, since $u \rightarrow v \in E2$ and $w \in N_H(v)$ implies $w \in N_H(u)$. In this way, we obtain larger families of graphs having polynomial time algorithms for maximum cliques.

Let $G(V,E1,E2)$ be a ***G*** mixed graph. For a vertex v let $U_v=\{u|\ (u,v) \in E1\}$. Assume that we wont to transform some of the $(u,v) \in E1$ edges into $u \rightarrow v \in E2$ edges. We can do this only in entire connected components of $H(U_v, coE1\}$, otherwise we obtain a vertex $w \in U_v$ having $(v,w) \in E1$, $u \rightarrow v \in E2$ and $(u,w) \notin E1$ contradicting the definition of ***G*** mixed graphs. Therefore, for transforming a ***G*** mixed graph G into another ***G*** mixed graph $G1$ by changing some $E1$ edges into $E2$ edges, we do this in entire connected components of $H(U_v, coE1)$, replacing every $(u,v) \in E1$ by $u \rightarrow v \in E2$ and keeping $E2$ transitive.

Let $H(V,E1,E2)$ be a ***G*** mixed graph. In coH, every $N_{coH}[v]$ is a star cut-set separating between the two vertex sets $X_v=\{u|\ u \rightarrow v \in E2\}$ and $Y_v=\{u|\ (u,v) \in E1$ *or* $v \rightarrow u \in E2\}$. Let a star cut-set decomposition be $N_{coH}[v] \cup X_v$, $N_{coH}[v] \cup Y_v$. By decomposing coH by star cut-sets until no more possible, we obtain a family of ***G*** mixed subgraphs $H(Z)$ such that $coH(Z)$ has no star cut-sets. Assume that such a subgraph $H(Z)$ has $E2$ edges and let $Z1$ be the subset of its $E2$ sinks. $H(Z1)$ has no edges of $E1 \cup E2$ and $H(Z)$ has no three vertices x,y,z having $x \rightarrow y$, $y \rightarrow z, x \rightarrow z \in E2$, otherwise a star cut set exists in $coH(Z)$. Therefore $H(Z)$ is a subgraph of $H(E1)$ or Z can be decomposed into two subsets $Z1$ and $Z-Z1$ such that $H(Z1)$ is an independent set, $H(Z-Z1)$ is a subgraph of $H(E1)$ and for every two vertices $v \in Z1$, $u \in Z-Z1$, u,v are not adjacent in H or $u \rightarrow v \in E2$. For example, if coH is a $2D$-interval-filament graph, then $coH(Z1)$ is a clique of coH and $coH(Z-Z1)$ is an interval graph; if coH is a $3D$-interval-filament graph, then $coH(Z1)$ is a clique of coH and $coH(Z-Z1)$ is a $2D$-interval-filament graph.

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