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Recognizing k-complete bipartite bihypergraphs

by

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ABSTRACT

Let H^0 and H^1 be hypergraphs with the same vertex-set V. The ordered pair $H = (H^0, H^1)$ is called a *bihypergraph.* A set $S \subseteq V(H)$ is *stable* in H^i if S contains no hyperedges of H^i, $i = 0, 1$. A bihypergraph $H = (H^0, H^1)$ is called *bipartite* if there exists an ordered partition $S^0 \cup S^1 = V(H)$ such that the set S^i is stable in H^i for $i = 0, 1$.
In Section 1, we give a survey numerous applications of bipartite bihypergraphs. In Section 2, we show that recognizing bipartite bihypergraphs within classes of k-complete bihypergraphs can be done in polynomial time. A bihypergraph $H = (H^0, H^1)$ is called *k-complete*, $k \geq 0$, if each k-subset of $V(H)$ contains a hyperedge of H, i.e., a hyperedge of H^0 or H^1. Moreover, we can construct all bipartitions of a k-complete bihypergraph, if any, in polynomial time.
A bihypergraph $H = (H^0, H^1)$ is called *strongly bipartite* if each maximal stable set of H^0 is a transversal of H^1. We show that recognizing strongly bipartite bihypergraphs (H^0, H^1) is a co-NP-complete problem even in the case where H^0 is a graph and H^1 has exactly one hyperedge. Some examples of strongly bipartite bihypergraphs are given.
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1 Bipartite bihypergraphs

We use standard graph-theoretic terminology, see Melnikov, Sarvanov, Tyshkevich, Yemelichev, and Zverovich [13]. In our model, a hypergraph may contain multiple hyperedge. Also, the empty set may be a hyperedge. A set $S \subseteq V(H)$ in a hypergraph H is *stable* in H if S contains no hyperedges of H.

Definition 1. *Let H^0 and H^1 be hypergraphs with the same vertex-set V. The ordered pair $H = (H^0, H^1)$ is called a* bihypergraph.

Every hyperedge of either H^0 or H^1 is consider as a *hyperedge* of H.

Definition 2. *A bihypergraph $H = (H^0, H^1)$ is called* bipartite *if there exists an ordered partition $S^0 \cup S^1 = V(H)$, called a* bipartition*, such that the set S^i is stable in H^i for $i = 0, 1$. We denote by $\mathcal{BIPBIH}$ the class of all bipartite bihypergraphs.*

Equivalently, a bihypergraph is bipartite if it has the following Property S.

(Property S) there exists an ordered partition $T^0 \cup T^1 = V(H)$ such that the set T^i is a transversal in H^i for $i = 0, 1$.

Recall that a *transversal* in a hypergraph is a vertex subset that intersects all hyperedges. It is obvious that a set $S \subseteq V(H)$ is stable in a hypergraph H if and only if $V(H) \setminus S$ is a transversal in H.

Decision Problem 1 (Bipartite Bihypergraph).

Instance: *A bihypergraph $H = (H^0, H^1)$.*

Question: *Is H bipartite?*

The problem can be formulated as a system of Boolean equations. Let $H = (H^0, H^1)$ be a bihypergraph on vertex-set $V = \{1, 2, \ldots, n\}$. If there exists a bipartition $S^0 \cup S^1$ of H, then we define Boolean variables $x_1, x_2, \ldots, x_n$:

$$x_i^* = \begin{cases} 0 & \text{if} \quad i \in S^0, \\ 1 & \text{if} \quad i \in S^1. \end{cases}$$

The vector $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is a solution to the system

$$\begin{cases} \bigvee\limits_{i \in e} x_i = 1 & \text{for each} \quad i \in E(H^0) \\ \prod\limits_{i \in e} x_i = 0 & \text{for each} \quad i \in E(H^1). \end{cases} \tag{1}$$

Conversely, each solution to the system (1) determines a bipartition of H.

Proposition 1. *A bihypergraphs is bipartite if and only if the system (1) is consistent.*

Since $\prod_{i\in e} x_i = 0$ if and only if $\bigvee_{i\in e} \overline{x}_i = 1$, the system (1) can be written as

$$\begin{cases} \bigvee\limits_{i\in e} x_i = 1 \quad \text{for each} \quad i \in E(H^0) \\ \bigvee\limits_{i\in e} \overline{x}_i = 1 \quad \text{for each} \quad i \in E(H^1). \end{cases} \tag{2}$$

The system (2) defines an instance to SAT problem. Therefore recognizing bipartite bihypergraphs can be reduced to SAT in polynomial time.

In general, recognizing bipartite bihypergraphs is NP-hard. Since 3-SAT is NP-complete, Bipartite Bihypergraph is NP-complete for bihypergraphs $H = (H^0, H^1)$ such that H^0 is a 1-regular graph, and each hyperedge of H^1 has at most three vertices. However, if both H^0 and H^1 are graphs, it it easy to recognize whether (H^0, H^1) is bipartite. Gavril [9] called this problem 2-Colors Graph Partition, and he constructed a parallel algorithm for it requiring $O(\log n)$ time and $O(n^3/(\log_4 n)^{1.5})$ processors on a CRCW PRAM.

Bipartite Hypergraph Problem is a particular case of Decision Problem 1. Indeed, a hypergraph H is bipartite if and only if the corresponding bihypergraph (H, H) is bipartite. A hypergraph H is bipartite if and only if

(Property B) there exists a transversal in H which is a stable set.

Here "B" stands for Felix Bernstein who noted in 1908 that a countable system of infinite sets has Property B. Sufficient conditions for a hypergraph to be bipartite and many other related properties were obtained by Erdös [4, 5, 6], Erdös and Hajnal [7], Miller [14], and Woodall [21]. Woodall [21] and Stein [18] have found interesting connections between bipartite hypergraphs and planar graphs. For example, Four Color Conjecture is equivalent to the following: the family of all odd circuits of a planar graph [considered as sets of edges] has property B.

There are numerous applications of bipartite bihypergraphs to Satisfiability Problem, graph colorings, distinct representatives, and graph vertex bipartitions with prescribed properties.

Zverovich [22] found that bipartite bihypergraphs constitute a natural model for studying hereditary classes of graphs defined in terms of vertex bipartitions. For a set of graphs Z, a graph G is called Z*-free* if no graph of Z is an induced subgraph of G. Let $\mathcal{P}_i$ be a hereditary class of all Z_i-free graphs, $i = 0, 1$. We assume that each $\mathcal{P}_i$ is given by the set Z_i.

Decision Problem 2 (Graph $(\mathcal{P}_0, \mathcal{P}_1)$-Bipartition).

Instance: *A graph G.*

Question: *Is there a bipartition $V_0 \cup V_1 = V(G)$ such that the induced subgraph $G(V_i)$ belongs to $\mathcal{P}_i$ for $i = 0, 1$?*

The sets Z_0 and Z_1 defines two hypergraphs H^0 and H^1 on $V(G)$, namely $E(H^i) = \{X \subseteq V(G) : G(X) \in Z_i\}$, $i = 0, 1$.

Proposition 2. *A graph G $(\mathcal{P}_0, \mathcal{P}_1)$-bipartition if and only if the bihypergraph (H^0, H^1) is bipartite.*

Zverovich [22] applied Proposition 2 to (p, q)-split graphs of Gyárfás [10] and (α, β)-polar graphs of Tyshkevich and Chernyak [19]. For further development see Zverovich and Zverovich [23] and Zverovich and Zverovich [24]. This approach can easily be extended to other hereditary systems, such as subgraphs, homeomorphic subgraphs, minors, etc.

We propose a new application of Bipartite Bihypergraph to Boolean dualization.

2 Applications to Boolean dualization

Let $\mathcal{B}^n$ denote the n-dimensional Boolean cube. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{B}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathcal{B}^n$, we write $\mathbf{x} \leq \mathbf{y}$ (respectively, $\mathbf{x} \geq \mathbf{y}$) if $x_i \leq y_i$ (respectively, $x_i \geq y_i$) for $i = 1, 2, \ldots, n$. Also, $\mathbf{x} < \mathbf{y}$ (respectively, $\mathbf{x} > \mathbf{y}$) means that $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ (respectively, $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$).

Definition 3. *A Boolean function $f : \mathcal{B}^n \to \mathcal{B}$ is called* monotone *if $\mathbf{x} \leq \mathbf{y}$ and $f(\mathbf{x}) = 1$ imply $f(\mathbf{y}) = 1$.*

For a Boolean function $f : \mathcal{B}^n \to \mathcal{B}$, $T(f) = \{\mathbf{x} : f(\mathbf{x}) = 1\}$ is the set of *true points*. Similarly, $F(f) = \{\mathbf{x} : f(\mathbf{x}) = 0\}$ is the set of *false points*. If f is monotone, then $MT(f) = \{\mathbf{x} \in T(f) : f(\mathbf{y}) = 0 \text{ for each } \mathbf{y} < \mathbf{x}\}$ is the set of *minimal true points*. Similarly, $MF(f) = \{\mathbf{x} \in F(f) : f(\mathbf{y}) = 1 \text{ for each } \mathbf{y} > \mathbf{x}\}$ is the set of *maximal false points*. Note that a monotone Boolean function f is uniquely determined by the set $MT(f)$.

As usual, $\overline{x} = 1 - x$ is the *negation* of $x \in \{0, 1\}$. Accordingly, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{B}^n$. The operation $\mathbf{x} \to \overline{\mathbf{x}}$ is also known as *complementation*. We define the *complement* $\overline{f}$ of a Boolean function $f : \mathcal{B}^n \to \mathcal{B}$ by $\overline{f}(\mathbf{x}) = \overline{f(\mathbf{x})}$ for all $\mathbf{x} \in \mathcal{B}^n$.

Definition 4. *The* dual *of a Boolean function $f : \mathcal{B}^n \to \mathcal{B}$ is the Boolean function $f^d : \mathcal{B}^n \to \mathcal{B}$ defined by $f^d(\mathbf{x}) = \overline{f}(\overline{\mathbf{x}})$.*

It is well-known and easy to see that the dual of a monotone Boolean function is also a monotone Boolean function. Suppose that a monotone Boolean function f is given by the set $MT(f)$ of its minimal true points. We want to construct the set $MT(f^d)$ to determine the dual of f. A related decision problem can be formulated as follows.

Decision Problem 3 (Boolean Dualization).

Instance: *A monotone Boolean function f given by the set $MT(f)$, and a subset S of $MT(f^d)$.*

Question: *Is S equal to $MT(f^d)$?*

Complexity of this problem is unknown. Boolean Dualization Problem has a simple hypergraph interpretation. A *clutter* is a hypergraph $H = (V, E)$ such that $e \subseteq e'$ for $e, e' \in E$ implies $e = e'$. There is a natural bijection between monotone Boolean functions and clutters. Let $V = \{1, 2, \dots, n\}$. The *characteristic vector* of a subset $X \subseteq V$ is $\mathrm{char}(X) = (x_1, x_2, \dots, x_n) \in \mathcal{B}^n$, where $x_i = 1$ if and only if $i \in X$. Conversely, the *set corresponding to* $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathcal{B}^n$ is $\mathrm{set}(\mathbf{x}) = \{i \in V : x_i = 1\}$. A clutter $C = (V, E)$ defines a monotone Boolean function f_C on $\mathcal{B}^n$: $MT(f) = \{\mathrm{char}(e) : e \in E\}$. Conversely, minimal true points of a monotone Boolean function f defines hyperedge-set of a clutter C_f on V. The following results are folklore and straightforward.

Proposition 3. $MT(f^d) = \{\overline{\mathbf{x}} : \mathbf{x} \in MF(f)\}$ *for each monotone Boolean function f.*

Let $\mathrm{Trans}(C)$ be the set of all minimal transversals of a clutter C.

Proposition 4. *For each monotone Boolean function f, $\{\mathrm{set}(\overline{\mathbf{x}}) : \mathbf{x} \in MF(f)\}$ is* $\mathrm{Trans}(C_f)$.

Thus, Boolean Dualization is the same problem as to find all minimal transversals of a given clutter.

Decision Problem 4 (Clutter Transversal).

Instance: *A clutter C and a set $T \subseteq \mathrm{Trans}(C)$.*

Question: *Is T equal to* $\mathrm{Trans}(C)$*?*

Proposition 5. Boolean Dualization *and* Clutter Transversal *are polynomial-time equivalent decision problems.*

Noe we establish a connection with bipartite bihypergraphs.

Proposition 6. Boolean Dualization *is a particular case of* Bipartite Bihypergraph Problem.

Proof. According to Proposition 4 we may deal with Clutter Transversal Problem. Let a clutter C and a set $T \subseteq \text{Trans}(C)$ be an instance to Decision Problem 4. We define a bihypergraph $H = (C, C')$ on $V(C)$, where $E(C') = T$.

Suppose (S, S') is a bipartition of H. Since S is a stable set in C, $S' = V(C) \setminus S$ is a transversal of C. The set S' is stable in C', therefore S' does not contain any member of T. Thus, S' contains minimal transversal which not in T, i.e., the answer to Clutter Transversal is "no" $[T \neq \text{Trans}(C)]$.

Now suppose that $T \neq \text{Trans}(C)$. We consider a minimal transversal $S' \in \text{Trans}(C) \setminus T$. Clearly, S' is a stable set in C'. Since S' is a transversal of C, the set $S = V(C) \setminus S'$ is set in C. Thus, (S, S') is a bipartition of H. $\square$

3 Recognizing k-complete bipartite bihypergraphs

Since Bipartite Bihypergraph is a hard problem, it is natural to impose additional conditions to obtain polynomial-time recognizible classes of bipartite bihypergraphs. We define a one-parametric family of such classes. If $X \subseteq Y$ and $|X| = k$, then we say that X is a *k-subset* of Y.

Definition 5 (Zverovich [22]). *A bihypergraph $H = (H^0, H^1)$ is called* k-complete*, $k \geq 0$, if each k-subset of $V(H)$ contains a hyperedge of H.*

We denote by $\mathcal{COMPBIH}(k)$ the class of all k-complete bihypergraphs.

The condition of being k-complete is very natural in view of Ramsey's theorem (Ramsey [16]). Indeed if $k = r(m, n)$ is the Ramsey number, then each vertex k-subset in a graph contains either an m-clique or a stable n-set. Thus, bounded cliques and stable sets may be considered as hyperedges of a k-complete bihypergraph. Clearly,

$$\mathcal{COMPBIH}(0) \subseteq \mathcal{COMPBIH}(1) \subseteq \cdots \subseteq \mathcal{COMPBIH}(k) \subseteq \cdots$$

and $\bigcup\limits_{i=k}^{\infty} \mathcal{COMPBIH}(k)$ contains all bihypergraphs having at least one hyperedge. We show that recognizing bipartite bihypergraphs within each class $\mathcal{COMPBIH}(k)$ can be done in polynomial time. Moreover, we can construct all bipartitions, if any, also in polynomial time. We start with two auxiliary results.

Claim 1. *Let $H = (H^0, H^1) \in \mathcal{COMPBIH}(k)$. If S^i is a stable set in H^i, $i = 0, 1$, then $|S^0 \cap S^1| < k$.*

Proof. The set $S^0 \cap S^1$ does not contain a hyperedges of H^0 as being a subset of a stable set S^0 of H^0. Similarly, it cannot contain any hyperedge of H^1. Since H is a k-complete bihypergraph, the result follows. $\square$

Claim 2. *Let $H = (H^0, H^1)$ be a bipartite k-complete bihypergraph. Suppose that S^0 and S^1 are disjoint stable sets of H^0 and H^1, respectively. If $S^0 \cup S^1 \neq V(H)$, then there exist sets $X^i \subseteq S^i$, $i = 0, 1$, and disjoint sets $Y^0, Y^1 \subseteq (V(H) \setminus (S^0 \cup S^1)) \cup (X^0 \cup X^1)$ such that*

(B1) $|X^i| < k$, $i = 0, 1$,

(B2) $(S^i \setminus X^i) \cup Y^i$ *is a stable set in* H^i, $i = 0, 1$, *and*

(B3) $|Y^0 \cup Y^1| = |X^0 \cup X^1| + 1$.

Proof. Since H is a bipartite bihypergraph, there exists a bipartition $A^0 \cup A^1$ of $V(H)$. We denote $X^0 = S^0 \cap A^1$ and $X^1 = S^1 \cap A^0$. By Claim 1, $|X^i| < k$ for $i = 0, 1$, i.e., (B1) holds.

Since $S^0 \cup S^1 \neq V(H)$, the set $R = (V(H) \setminus (S^0 \cup S^1)) \cup (X^0 \cup X^1)$ contains at least $N = |X^0 \cup X^1| + 1$ vertices. We choose an N-subset Y in R. It remains to define $Y^i = Y \cap A^i$, $i = 0, 1$, and note that $(S^i \setminus X^i) \cup Y^i$ is a stable set in H^i as being a subset of A^i. Both (B2) and (B3) follow. $\square$

Here is our main result.

Theorem 1. *Recognizing bipartite bihypergraphs within each class $\mathcal{COMPBIH}(k)$ can be done in polynomial time. Moreover, it is possible to construct at least one bipartition of a k-complete bipartite bihypergraph in polynomial time.*

Proof. Let $H = (H^0, H^1)$ be a k-complete bihypergraph. If $\emptyset$ is a hyperedge of H, then H is not bipartite, since either H^0 or H^1 has no stable sets. Suppose it is not the case. To recognize whether H is a bipartite, we choose initial stable sets $S_0^0 = \emptyset$ in H^0 and $S_0^1 = \emptyset$ in H^1. Suppose that we have already constructed disjoint stable sets $S_i^0 = \emptyset$ in H^0 and $S_i^1 = \emptyset$ in H^1 for some $i \geq 0$. If $S_i^0 \cup S_i^1 = V(H)$, then we have a bipartition of H.

If $S_i^0 \cup S_i^1 \neq V(H)$, then we apply Claim 2, that is we consider the set $\mathcal{T}$ of all 4-tuples $(X_i^0, X_i^1, Y_i^0, Y_i^1)$ such that

- $X_i^0 \subseteq S_i^0$ and $X_i^1 \subseteq S_i^1$,
- $|X_i^0| < k$ and $|X_i^1| < k$,
- $Y_i^0, Y_i^1 \subseteq (V(H) \setminus (S_i^0 \cup S_i^1)) \cup (X_i^0 \cup X_i^1)$ are disjoint sets, and
- $|Y_i^0 \cup Y_i^1| = |X_i^0 \cup X_i^1| + 1$.

Since $|X_i^0| < k$, $|X_i^1| < k$, and $|Y_i^0|+|Y_i^1| = |X_i^0|+|X_i^1|+1 < 2k$, we can generate the set $\mathcal{T}$ in polynomial time. For each $(X_i^0, X_i^1, Y_i^0, Y_i^1)$ in $\mathcal{T}$, we check in polynomial time whether $(S_i^0 \setminus X_i^0) \cup Y_i^0$ is a stable set in H^0 and $(S_i^1 \setminus X_i^1) \cup Y_i^1$ is a stable set in H^1. If yes, we have found new disjoint stable sets in H^0 and H^1, namely $S_{i+1}^0 = (S_i^0 \setminus X_i^0) \cup Y_i^0$ and $S_{i+1}^1 = (S_i^1 \setminus X_i^1) \cup Y_i^1$. In this case we continue the construction with S_{i+1}^0 and S_{i+1}^1. If not, H is not a bipartite bihypergraph by Claim 2.

Let $n = |V(H)|$. Since $|S_0^0 \cup S_0^1| = 0$ and $|S_{i+1}^0 \cup S_{i+1}^1| = |S_i^0 \cup S_i^1| + 1$ for all $i \geq 0$, either we can construct $S_n^0 \cup S_n^1$, thus obtaining a bipartition of H, or we reject H as being non-bipartite on some step $i < n$. Thus, our algorithm runs in polynomial time. □

Corollary 1. *It is possible to construct all bipartitions of a k-complete bihypergraph in polynomial time.*

Proof. Let $H = (H^0, H^1)$ be a k-complete bihypergraph. According to Theorem 1, we can recognize bipartiteness of H in polynomial time. Moreover, if H is bipartite, we can construct at least one bipartition $A^0 \cup A^1$ of H also in polynomial time.

Let $B^0 \cup B^1$ be an arbitrary bipartition of H. By Claim 1, $|A^0 \cap B^1| < k$ and $|A^1 \cap B^0| < k$. Hence to obtain an arbitrary bipartition from the initial bipartition $A^0 \cup A^1$, it is sufficient to consider all pairs (X^0, X^1) such that $X^i \subseteq A^i$, and $|X^i| < k$, $i = 0, 1$. For each pair (X^0, X^1), we check, whether the sets $B^0 = (A^0 \setminus X^0) \cup X^1$ and $B^1 = (A^1 \setminus X^1) \cup X^0$ are stable in H^0 and H^1, respectively. The result follows from the obvious fact that there are polynomially many pairs (X^0, X^1). □

4 Strongly bipartite bihypergraphs

Here we consider a subclass of bipartite bihypergraphs.

Definition 6. *A bihypergraph $H = (H^0, H^1)$ is called* strongly bipartite *if each maximal stable set of H^0 is a transversal of H^1.*

Given a bihypergraph $H = (H^0, H^1)$, it is easy to find a maximal stable set S^0 in H^0. If S^0 is a transversal of H^1, then $S^1 = V(H) \setminus S^0$ is a stable set of H^1. Thus, it is easy to recognize bipartiteness of strongly bipartite bihypergraphs.

Decision Problem 5 (Strongly Bipartite Bihypergraph).

Instance: *A bihypergraph $H = (H^0, H^1)$.*

Question: *Is H strongly bipartite?*

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of $0-1$ variables. We define the set of literals over X, $L_X = \{x_i, \overline{x}_i : i = 1, 2, \ldots, n\}$, where $\overline{x}_i = 1 - x_i$ is the *negation* of x_i. A *truth assignment to* X a mapping $t : X \to \{0, 1\}$ that assigns a value $t(x_i) \in \{0, 1\}$ to each variable $x_i \in X$. We extend t to L_X putting $t(\overline{x}) = \overline{t(x)}$. A literal $l \in L_X$ is *true under* t if $t(l) = 1$. A *clause* over X is a conjunction of some literals of L_X. Let $C = \{c_1, c_2, \ldots, c_m\}$ be a set of clauses over X. A truth assignment t to X *satisfies* a clause $c_j \in C$ if c_j involves at least one true literals under t. The following Satisfiability Problem, or SAT, is well-known.

Decision Problem 6 (SAT).

Instance: *A set of clauses C over X.*

Question: *Is there a truth assignment to X that satisfies all clauses in C?*

Theorem 2. *Recognizing strongly bipartite bihypergraphs (H^0, H^1) is a co-NP-complete problem even in the case where H^0 is a graph and H^1 has exactly one hyperedge.*

Proof. We use a polynomial-time reduction from SAT. Let (C, X) be an instance to SAT with $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$. Recall that $L_X = \{x_i, \overline{x}_i : i = 1, 2, \ldots, n\}$. We define a bihypergraph (G, H) on $C \cup L_X$:

- in G, L_X induces a matching $x_i\overline{x}_i$, $i = 1, 2, \ldots, n$; vertices $l \in L_X$ and $c_j \in C$ are adjacent if and only if the clause C_j involves the literal l, and
- C is the only hyperedge of H.

Suppose that there exists a truth assignment ϕ that satisfies C. The set of true literals, considered as a vertex subset of L_X, is a maximal stable set in G. By definition, if the bihypergraph (G, H) is not strongly bipartite.

Conversely, suppose that C is not satisfiable. It means that there is no a stable subset $S \subseteq L_X$ that dominates C. In other words, each maximal stable set in G intersects C, i.e., (G, H) is a strongly bipartite bihypergraph. □

Below we propose two examples of strongly bipartite bihypergraphs.

4.1 Triangle graphs

An interesting class of intersection graphs was introduced by McAvaney, Robertson, and DeTemple [12], see also Anbeek, DeTemple, McAvaney, and Robertson [1] and DeTemple, Dineen, Robertson, and McAvaney [3]. A *general partition graph* is the intersection graph G of a family S of sets with the property that every maximal independent set in G corresponds to a partition of S. All general partition graphs satisfy the triangle condition below.

Definition 7. *A graph G is called a* triangle graph *if it satisfies the following condition*

(Triangle Condition) *for every maximal stable set I and every edge uv in $G-I$, there exists a vertex $w \in I$ such that $\{u, v, w\}$ induces a triangle in G.*

Let $\mathcal{TRIANGLE}$ denote the class of all triangle graphs.

Many interesting properties of triangle graphs were found. Recently Orlovich and Zverovich [15] proved that INDEPENDENT DOMINATION is an NP-complete problem within $K_{1,4}$-free triangle graphs. Kloks, Lee, Liu, and Muller [11] showed that the triangle condition can be checked in polynomial time for AT-free graphs, planar graphs and for circle graphs.

Conjecture 1 (Kloks, Lee, Liu, and Muller [11]). *Recognizing triangle graphs is a co-NP-complete problem.*

We define the *proper neighborhood* of an edge $e = uv$ in a graph G as $PN_G(e) = N_G[u] \cap N_G[v]$, i.e., $PN_G(e)$ consists of u, v and all vertices adjacent to both u and v.

Definition 8. *Given a graph $G = (V, E)$, we define the* proper hypergraph *of G, denoted by $PH(G) = (V, E')$, where $E' = \{PN_G(e) : e \in E\}$. The* proper bihypergraph *of G is $(G, PH(G))$.*

Proposition 7. *A graph G satisfies the triangle condition if and only if the proper bihypergraph $(G, PH(G))$ is strongly bipartite.*

Proof. Necessity. For every maximal stable set I and every edge uv in $G-I$, there exists a vertex $w \in I$ such that $\{u, v, w\}$ induces a triangle in G. In other words, I intersects all sets $PN_G(e)$, i.e., I is a transversal of $PH(G)$.

Sufficiency is similar. □

Definition 9. *An edge e is* tristable *if each maximal stable set intersects $PN(e)$, otherwise it is* non-tristable*.*

The triangle condition means that each edge is tristable. Conjecture 1 states that it is NP-complete to decide whether a graph has a non-tristable edge.

Decision Problem 7 (Tristable Edge).

Instance: *A graph G and an edge e of G.*

Question: *Is e a tristable edge?*

Tristable Edge is a particular case of Strongly Bipartite Bihypergraph Problem.

Corollary 2. Tristable Edge *is a co-NP-complete problem.*

Proof. Given an instance (C, X) to SAT, we construct (C', X') by adding two new variables x_{n+1} and x_{n+2} to X, and two new clauses $c_{m+1} = x_{n+1}$ and $c_{m+2} = x_{n+2}$ to C. Clearly, (C, X) is satisfiable if and only if (C', X') is satisfiable. We use the bihypergraph (G, H) for C', X' as in the proof Theorem 2. For $e = c_{m+1}c_{m+2}$, we have $PM_G(e) = C'$. Thus, the edge e is tristable if and only if the bihypergraph (G, H) is strongly bipartite. As in the proof of Theorem 2, the latter condition is equivalent to non-satisfiability of C'. □

4.2 Stable graphs

Ravindra [17] observed that in a P_4-free graph every maximal stable set meets every maximal clique. A *clique* in a graph is a vertex set that induces a complete subgraph.

Definition 10. *A graph G is called a* stable graph *if each maximal stable set in G intersects all maximal cliques of G.*

For a graph G, we define the *clique hypergraph* $Cl(G)$ on $V(G)$ by $E(Cl(G)) = \{X \subseteq V(G) : X \text{ induces a maximal clique in } G\}$. A graph G is stable if and only if the corresponding bihypergraph $(G, Cl(G))$ is bipartite.

Conjecture 2. *Recognizing stable graphs is a co-NP-complete problem.*

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