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On the Complexities of Some Combinatorial Problems in Reverse Engineering of Protein and Gene Networks

by

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ABSTRACT

In this paper we investigate the computational complexities of a combinatorial problem that arises in the reverse engineering of protein and gene networks.

1 Introduction

In this paper, we study a computational complexity problem that arises in the context of experimental design for reverse engineering of protein and gene networks. We will first pose the problem in linear algebra terms, and then recast it as a combinatorial question. After that, we will briefly discuss its systems biology motivation.

Our problem is described in terms of two matrices

$$A \in \mathbb{R}^{n \times n}, \quad B \in \mathbb{R}^{n \times m}$$

such that:

- A is unknown;
- B is initially unknown, but each of its columns, denoted as B_i, $i = 1, \ldots, m$, can be retrieved with a unit-cost query;
- the columns of B are in general position, meaning that each subset of $k \leq n$ columns of B is linearly independent;
- the *zero structure* of the matrix $C = AB = (c_{ij})$ is known; more precisely: a binary matrix $C^0 = \left(c^0_{ij}\right) \in \{0,1\}^{n \times m}$ is given, and it is known that $c_{ij} = 0$ for each i, j for which $c^0_{ij} = 0$.

The objective, roughly speaking, is to obtain as much information as possible about A (which, in the motivating application, describes regulatory interactions among genes and/or proteins), while performing "few" queries (each of which may represent the measuring of a complete pattern of gene expression, done under a different set of experimental conditions).

Notice that there are intrinsic limits to what can be accomplished: if we multiply each row of A by some nonzero number, then the zero structure of C is unchanged. Thus, the best that we can hope for is to identify the rows of A up to scalings (in abstract mathematical terms, as elements of the projective space $\mathbb{P}^{n-1}$). To better understand these geometric constraints, let us reformulate the problem as follows. Let A_i denote the ith row of A. Then the specification of C^0 amounts to the specification of *orthogonality relations*

$$A_i \cdot B_j = 0$$

for each pair i, j for which $c^0_{ij} = 0$. Suppose that we decide to query the columns of B indexed by

$$J = \{j_1, \ldots, j_\ell\}\,.$$

Then, the information obtained about A may be summarized as:

$$A_i \in \mathcal{H}^{\perp}_{J,i}$$

where "$\perp$" indicates orthogonal complement, and

$$\mathcal{H}_{J,i} = \text{span}\,\{B_j, j \in J_i\}\ ,$$

$$J_i = \{j \mid j \in J \text{ and } c^0_{ij} = 0\}\,. \tag{1}$$

Suppose now that the set of indices of selected queries J has the property:

$$\text{each set } J_i,\ i = 1, \ldots, n, \text{ has cardinality } \geq n - k, \tag{2}$$

for some given integer k. Then, because of the general position assumption, the space $\mathcal{H}_{J,i}$ has dimension $\geq n - k$, and hence the space $\mathcal{H}^{\perp}_{J,i}$ has dimension at most k.

The most desirable special case is that in which $k = 1$. Then

$$\dim \mathcal{H}^{\perp}_{J,i} \leq 1,$$

so that each A_i is uniquely determined up to a scalar multiple, which is the best that could be theoretically achieved. Often, in fact, finding the sign pattern (such as "$(+, +, -, 0, 0, -, \ldots)$") for each row of A is the main experimental goal (this would correspond, in our motivating application, to determining if the regulatory interactions affecting each given gene or protein are inhibitory or catalytic). Assuming that the degenerate case $\mathcal{H}^{\perp}_{J,i} = \{0\}$ does not hold (which would determine $A_i = 0$), once that an arbitrary nonzero element v in the line $\mathcal{H}^{\perp}_{J,i}$ has been picked, there are only two sign patterns possible for A_i (the pattern of v and that of $-v$). If, in addition, one knows at least one nonzero sign in A_i, then the sign structure of the whole row has been *uniquely* determined. (In the motivating biological question, typically one such sign is indeed known; for example, the diagonal elements a_{ii}, i.e. the ith element of each A_i, is known to be negative, as it represents a degradation rate.) Thus, we will be interested in this question:

$$\text{find } J \text{ of minimal cardinality such that } |J_i| \geq n - 1,\ i = 1, \ldots, n. \tag{**Q1**}$$

If queries have variable unit costs (different experiments have a different associated cost), this problem must be modified to that of minimizing a suitable linear combination of costs, instead of the number of queries.

More generally, suppose that the queries that we performed satisfy (2), with $k > 1$ but small. It is not true anymore that there are only two possible sign patterns for any given A_i, but the number of possibilities is still very small. For simplicity, let us assume that we know that no entry of A_i is zero. (If this is not the case, the number of possibilities may increase, but the argument is very similar.) We wish to prove that the possible number of signs is much smaller than 2^n. Indeed, suppose that the queries have been performed, and that we then calculate, based on the obtained B_j's, a basis $\{v_1, \ldots, v_k\}$ of $\mathcal{H}^{\perp}_{J,i}$ (assume $\dim \mathcal{H}^{\perp}_{J,i} = k$; otherwise pick a smaller k). Thus, the vector A_i is known to have the form

$$\sum_{r=1}^{k} \lambda_r v_r$$

for some (unknown) real numbers $\lambda_1, \ldots, \lambda_k$. We may assume that $\lambda_1 \neq 0$ (since, if $A_i = \sum_{r=2}^{k} \lambda_r v_r$, the vector $\varepsilon v_1 + \sum_{r=2}^{k} \lambda_r v_r$, with small enough ε, has the same sign pattern as

A_i, and we are counting the possible sign patterns). If $\lambda_1 > 0$, we may divide by λ_1 and simply count how many sign patterns there are when $\lambda_1 = 1$; we then double this estimate to include the case $\lambda_1 < 0$. Let $v_r = \mathrm{col}\,(v_{1r}, \ldots, v_{nr})$, for each $r = 1, \ldots, k$. Since no coordinate of A_i is zero, we know that A_i belongs to the set

$$\mathcal{C} \;=\; \mathbb{R}^{k-1} \setminus \Big(L_1 \bigcup \ldots \bigcup L_n \Big)$$

where, for each $1 \leq s \leq n$, L_s is the hyperplane in $\mathbb{R}^{k-1}$ consisting of all those vectors $(\lambda_2, \ldots, \lambda_k)$ such that $\sum_{r=2}^{k} \lambda_r v_{sr} = -v_{s1}$. On each connected component of $\mathcal{C}$, signs patterns are constant. Thus the possible number of sign patterns is upper bounded by the maximum possible number of connected regions determined by n hyperplanes in dimension $k-1$. A result of L. Schläfli (see [2, 8], and also [9] for a discussion, proof, and relations to Vapnik-Chervonenkis dimension) states that this number is bounded above by $\Phi(n, k-1)$, provided that $k - 1 \leq n$, where $\Phi(n, d)$ is the number of possible subsets of an n-element set with at most d elements, that is,

$$\Phi(n,d) \;=\; \sum_{i=0}^{d} \binom{n}{i} \;\leq\; 2\frac{n^d}{d!} \;\leq\; \Big(\frac{en}{d}\Big)^d \,.$$

Doubling the estimate to include $\lambda_1 < 0$, we have the upper bound $2\Phi(n, k-1)$. For example, $\Phi(n, 0) = 1$, $\Phi(n, 1) = n + 1$, and $\Phi(n, 2) = \frac{1}{2}(n^2 + n + 2)$. Thus we have an estimate of 2 sign patterns when $k = 1$ (as obtained earlier), $2n + 2$ when $k = 2$, $n^2 + n + 2$ when $k = 3$, and so forth. In general, the number grows only polynomially in n (for fixed k).

These considerations lead us to formulating the generalized problem, for each fixed k: *find J of minimal cardinality such that $|J_i| \geq n - k$* for all $i = 1, \ldots, n$. Recalling the definition (1) of J_i, we see that $J_i = J \bigcap T_i$, where $T_i = \{j \mid c^0_{ij} = 0\}$. Thus, we can reformulate our question purely combinatorially, as a more general version of Question **(Q1)** as follows. Given sets

$$T_i \subseteq \{1, \ldots, m\}\,, \quad i = 1, \ldots, n.$$

and an integer $k < n$, the problem is:

find $J \subseteq \{1, \ldots, m\}$ of minimal cardinality such that $|J \bigcap T_i| \geq n - k$, $1 \leq i \leq n$.

(Q2)

For example, suppose that $k = 1$, and pick the matrix $C^0 \in \{0, 1\}^{n \times n}$ in such a way that the columns of C^0 are the binary vectors representing all the $(n-1)$-element subsets of $\{1, \ldots, n\}$ (so $m = n$); in this case, the set J must equal $\{1, \ldots, m\}$ and hence has cardinality n. On the other hand, also with $k = 1$, if we pick the matrix C^0 in such a way that the columns of C^0 are the binary vectors representing all the 2-element subsets of $\{1, \ldots, n\}$ (so $m = n(n-1)/2$), then J must again be the set of all columns (because, since there are only two zeroes in each column, there can only be a total of 2ℓ zeroes, $\ell = |J|$, in the submatrix indexed by J, but we also have that $2\ell \geq n(n-1)$, since each of the n rows must have $\geq n - 1$ zeroes); thus in this case the minimal cardinality is $n(n-1)/2$.

1.1 Motivations from Systems Biology

This problem was motivated by the setup for reverse-engineering of protein and gene networks described in [6, 7] and reviewed in [10]. We assume that the time evolution of a vector of state variables $x(t) = (x_1(t), \ldots, x_n(t))$ is described by a system of differential equations:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n, p_1, \ldots, p_m) \\ \dot{x}_2 &= f_2(x_1, \ldots, x_n, p_1, \ldots, p_m) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n, p_1, \ldots, p_m) \end{aligned}$$

(in vector form, "$\dot{x} = f(x, p)$"), where $p = (p_1, \ldots, p_m)$ is a vector of parameters, representing for instance the concentrations of certain enzymes which are maintained at a constant value during a particular experiment. There is a reference value $\bar{p}$ of p, which represents "wild type" (that is, normal) conditions, and a corresponding steady state $\bar{x}$. That is, $f(\bar{x}, \bar{p}) = 0$. We are interested in obtaining information about the Jacobian of the vector field f evaluated at $(\bar{x}, \bar{p})$, or at least about the signs of the derivatives $\partial f_i/\partial x_j(\bar{x}, \bar{p})$. For example, if $\partial f_i/\partial x_j > 0$, this means that x_j has a positive (catalytic) effect upon the rate of formation of x_i. The critical assumption, indeed the main point of [6, 7], is that, while we do not know the form of f, we do know that *certain parameters p_j do not directly affect certain variables x_i.* This amounts to *a priori* biological knowledge of specificity of enzymes and similar data. This knowledge will be summarized by the binary matrix $C^0 = \left(c^0_{ij}\right) \in \{0, 1\}^{n \times m}$, where "$c^0_{ij} = 0$" means that p_j does not appear in the equation for $\dot{x}_i$, that is, $\partial f_i/\partial p_j \equiv 0$.

The experimental protocol allows us to make small perturbations in which we change one of the parameters, say the kth one, while leaving the remaining ones constant. (A generalization would allow for the simultaneous perturbation of more than one parameter.) For the perturbed vector $p \approx \bar{p}$, we measure the resulting steady state vector $x = \xi(p)$. (Mathematically, we suppose that for each vector of parameters p in a neighborhood of $\bar{p}$ there is a unique steady state $\xi(p)$ of the system, where ξ is a differentiable function. In practice, each such perturbation experiment involves letting the system relax to steady state, and the use of some biological reporting mechanism, such as microarrays, in order to measure the expression profile of the variables x_i.) For each of the possible m experiments, in which a given p_j is perturbed, we may estimate the n "sensitivities"

$$b_{ij} = \frac{\partial \xi_i}{\partial p_j}(\bar{p}) \approx \frac{1}{\bar{p}_j - p_j} \left(\xi_i(\bar{p} + p_j e_j) - \xi_i(\bar{p})\right) \ , \quad i = 1, \ldots, n$$

(where $e_j \in \mathbb{R}^m$ is the jth canonical basis vector). We let B denote the matrix consisting of the b_{ij}'s. (See [6, 7] for a discussion of the fact that division by $\bar{p}_j - p_j$, which is undesirable numerically, is not in fact necessary.) Finally, we let A be the Jacobian matrix $\partial f/\partial x$ and let C be the negative of the Jacobian matrix $\partial f/\partial p$. From $f(\xi(p), p) \equiv 0$, taking derivatives with respect to p, and using the chain rule, we get that $A = BC$. This brings us to the problem stated in this paper. (The general position assumption is reasonable, since we are dealing with experimental data.)

2 Computational Complexities and Randomized Approximation Algorithms

We will use the following assumptions, notations and terminologies uniformly throughout the rest of this paper:

- $[a, b]$ is the set $\{a, a+1, a+2, \ldots, b\}$ for integers a and b.
- An m-equipartition of $X = [a, b = a + m\beta - 1]$ (for positive integers a, m and β) is a partition of X into m sets $X_1, X_2, \ldots, X_m$ where $X_i = [a+(i-1)\beta, a+i\beta-1]$. An α-shift of this m-equipartition (for $0 \leq \alpha < \beta$) is the partition of X into sets $X'_1, X'_2 \ldots, X'_m$ where $X'_k = \{x' \mid x \in X_k\}$ with $x' = \begin{cases} x + \alpha & \text{if } x + \alpha \leq b \\ a + ((x + \alpha) \bmod (b + 1)) & \text{otherwise} \end{cases}$.
- Unless otherwise stated, we assume that there is at least one valid solution to our problems.

2.1 Problems and Definitions

The k-Covering Problem $\mathbf{CP}_k$, that captures Question **(Q2)** in the previous section, is as follows:

> **Instance** $< m, n, k >$**:** An universe $U = [1, m]$ and sets $T_1, T_2, \ldots, T_n \subseteq U$ with $\cup_{i=1}^{n} T_i = U$.
>
> **Valid Solutions:** A subset $U' \subseteq U$ of the universe such that $|U' \cap T_i| \geq n - k$ for each $i \in [1, n]$.
>
> **Objective:** *Minimize* $|U'|$.

We will give special attention to $\mathbf{CP}_1$ that captures Question **(Q1)** and denote it *simply by* **CP**.

The Set k-cover ($\mathbf{SC}_k$) problem is a known problem in combinatorial algorithms and is defined as follows:

> **Instance** $< n, m, k >$**:** An universe $U = [1, n]$, sets $S_1, S_2, \ldots, S_m \subseteq U$ with $\cup_{j=1}^{m} S_j = U$ and a positive integer k.
>
> **Valid Solutions:** A subset of indices $I \subseteq [1, m]$ such that, for every element $x \in U$, $|j \in I \ : \ x \in S_j| \geq k$.
>
> **Objective:** *Minimize* $|I|$.

$\mathbf{SC}_1$ is simply called the Set Cover problem and denoted by **SC**. We will denote an instance of **SC** simply by $< n, m >$ instead of $< n, m, 1 >$.

An ε-approximate solution (or simply an ε-approximation) of a minimization problem is defined to be a solution with an objective value no larger than ε times the value of the optimum.

2.2 Previously Known Results for $\mathbf{SC}_k$

The following results are known about $\mathbf{SC}_k$.

Fact 1
(a) [3] *Assuming NP$\not\subseteq$DTIME($n^{\log\log n}$), instances $< n, m >$ of the* **SC** *problem cannot be approximated to within a factor of $(1-\varepsilon)\ln n$ for any constant $0 < \varepsilon < 1$ in polynomial time.*
(b) [11] *An instance $< n, m, k >$ of the $\mathbf{SC}_k$ problem can be $(1 + \ln n)$-approximated in $O(nmk)$ time by a simple greedy heuristic that, at every step, selects a new set that covers the maximum number of those elements that has not been covered at least k times yet.*

2.3 Equivalence of $\mathbf{CP}_k$ and $\mathbf{SC}_{n-k}$

We can establish a 1-1 correspondence between an instance $< m, n, k >$ of $\mathbf{CP}_k$ and an instance $< n, m, n-k >$ of $\mathbf{SC}_{n-k}$ by defining $S_i = \{\ j \mid i \in T_j\}$ for each $i \in [1, m]$. It is easy to verify that U' is a solution to the instance of $\mathbf{CP}_k$ if and only if the collection of sets S_u for each $u \in U'$ is a solution to the instance of $\mathbf{SC}_{n-k}$.

2.4 NP-completeness of $\mathbf{CP}_k$

Lemma 2 $\mathbf{CP}_k$ *is NP-complete for all constant k.*

Proof. By the equivalence of Section 2.3, it suffices to show that $\mathbf{SC}_{n-k}$ is NP-complete. Obviously, $\mathbf{SC}_{n-k} in$ NP. To show NP-hardness we provide a reduction from an instance $< n', m' >$ of **SC** (with $n' > k$) to an instance $< n, m, n-k >$ $\mathbf{SC}_{n-k}$. Given the instance $< n', m' >$ of **SC** with sets $S_1, S_2, \ldots, S_{m'}$, we create an instance $< n = n' + 12(n'-k), m = m' + 36(n'-k), n-k >$ of $\mathbf{SC}_{n-k}$ with the following sets:

(old sets) $S_1, S_2, \ldots, S_m$,

(new sets) Let T be the collection of all α-shifts of the 2-equipartition of $[n'+1, n'+12(n'-k)]$ for all $\alpha \in [0, 6(n'-k)-1]$, all β-shifts of the 3-equipartition of $[n'+1, n'+12(n'-k)]$ for all $\beta \in [0, 4(n'-k)-1]$, and all γ-shifts of the 4-equipartition of $[n'+1, n'+12(n'-k)]$ for all $\gamma \in [0, 3(n'-k)-1]$. Let t be an arbitrary set in T. Then, the collection of new sets Q consists of $t' \cup [1, n']$ for every $t' \in T \backslash t$ and the set t.

Note any element $x \in [n'+1, n'+12(n'-k)]$ is covered by only the new sets and the collection of $36(n'-k)$ new sets cover each $x \in [n'+1, n'+12(n'-k)]$ (resp. $x \in [1, n']$) exactly $6(n'-k)+4(n'-k)+3(n'-k) = 13(n'-k) = n-k$ (resp. $13(n'-k)-1 = n-k-1$) times. Thus, $S_{i_1}, S_{i_2}, \ldots, S_{i_t}$ is a solution of **SC** if and only if $S_{i_1}, S_{i_2}, \ldots, S_{i_t}$ plus the collection of $36(n'-k)$ new sets is a solution of $\mathbf{SC}_{n-k}$. ❑

Remark 1 *Similar reductions can be given for other values of k as well when k is a function of n.*

3 Approximation Algorithms for $\mathbf{CP}_k$

Using Fact 1(b) and the equivalence of Section 2.3, it follows that an instance $<m, n, k>$ of $\mathbf{CP}_k$ can be $(1 + \ln n)$-approximated in $O(mnk)$ time by a simple greedy heuristic.

3.1 Analysis of Greedy Heuristic Cannot be Further Improved for $\mathbf{CP}_k$ or $\mathbf{SC}_{n-k}$

Johnson [4] provides an example in which the greedy heuristic for some instance of **SC** over n elements has an approximation ratio of at least $\log_2 n$. Using a similar approach, we can show the following.

Lemma 3 *For any fixed k, the greedy heuristic (as described in Fact 1(b)) has an approximation ratio of at least $\left(\frac{1}{2} - o(1)\right)\left(\frac{n-k}{8n-2}\right)\log_2 n = \Omega(\log n)$ for some instance $<m, n, k>$ of $\mathbf{CP}_k$, or equivalently, for some instance $<n, m, n-k>$ of $\mathbf{SC}_{n-k}$.*

Proof. We will create an instance of $\mathbf{SC}_{n-k}$ with $n = 2\alpha + \gamma \gg k$ where $\alpha \gg k$ be a sufficiently large positive integer that is also a power of 4 and $\gamma = 3 + \log_2 \alpha \gg k$ be the least positive integer such that $2^{\gamma-1} \geq 2\alpha + \gamma - k$. Notice that by our choice of parameters $\frac{2n}{5} < \alpha < \frac{n}{2} \equiv \alpha = \Theta(n)$ and $1 + \log_2 n < \gamma < 2 + \log_2 n \equiv \gamma = \Theta(\log n)$. Let $\mathbf{S} = \{S_1, S_2, \ldots, S_{2^\gamma - 1}\}$ be the collection of all distinct non-empty subsets of $[2\alpha + 1, 2\alpha + \gamma]$. Notice that every $x \in [2\alpha + 1, 2\alpha + \gamma]$ occurs in exactly $2^{\gamma-1} \geq n - k$ sets in the collection $\mathbf{S}$.

A collection of our sets corresponding to an optimal cover of our instance of $\mathbf{SC}_{n-k}$ will consist of the $2^{\gamma+1} - 2$ sets $[1, \alpha] \cup S_i$ and $[\alpha + 1, 2\alpha] \cup S_i$ for each set $S_i \in \mathbf{S}$. Any $x \in [2\alpha + 1, 2\alpha + \gamma]$ occurs in exactly $2^\gamma > 2\alpha + \gamma - 1 = n - k$ of these sets. Also, each $x \in [1, 2\alpha]$ occurs in exactly $2^{\gamma-1} \geq 2\alpha + \gamma - 1 = n - k$ of these sets. Hence, an optimal solution of this instance of $\mathbf{SC}_{n-k}$ uses at most $2^{\gamma+1} - 2 < 8n - 2$ sets. Notice that each set in this optimal cover contains atmost $\alpha + \gamma < \frac{n}{2} + 2 + \log_2 n$ elements.

Now we specify another collection of sets which will force the greedy heuristic to use at least $(n - k)\left(\frac{1}{2} - o(1)\right)\log_2 n$ sets. Partition $[1, \alpha]$ into $p = 1 + \log_4 \alpha$ disjoint sets $P_1, P_2, \ldots, P_p$ such that $|P_i| = \left\lceil \frac{3}{4^i}\alpha \right\rceil$ for $i \in [1, p]$. Observe that $p > \log_4 n$. Similarly, partition $[\alpha + 1, 2\alpha]$ into $p = 1 + \log_4 \alpha$ disjoint sets $Q_1, Q_2, \ldots, Q_p$ such that $|Q_i| = \left\lceil \frac{3}{4^i}\alpha \right\rceil$ for $i \in [1, p]$. Let $\mathbf{S}' = \{S_1, S_2, \ldots, S_{n-k}\} \subseteq \mathbf{S}$. Now, for each $P_i \cup Q_i$ and each distinct $S_j \in \mathbf{S}'$, create a set $T_{i,j} = P_i \cup Q_i \cup S_j$. We claim that greedy will pick the sets $T_{1,1}, \ldots, T_{1,n-k}, T_{2,1}, \ldots, T_{2,n-k}, \ldots, T_{q,1}, \ldots, T_{q,n-k}$ with $q = \left(\frac{1}{2} - o(1)\right)\log_2 n < p$. This can be shown by induction as follows:

- The greedy must start by picking the sets $T_{1,1}, \ldots, T_{1,n-k}$ in some arbitrary order. Until all these sets have been picked, the unpicked ones have at least $\frac{3}{4}2\alpha = \frac{3}{2}\alpha$ elements that have not been covered $n - k$ times, whereas each set in the optimal cover has at most $\alpha + \gamma = \alpha + 3 + \log_2 \alpha$ elements and α is sufficiently large.

- Inductively, suppose that the greedy has picked all sets $T_{i,j}$ with $i < q$ when it considers a $T_{q,r}$ for possible consideration. Obviously $T_{q,r}$ contains at least $\frac{3}{4^q}2\alpha = \frac{6}{4^q}\alpha$ elements

that are not yet covered $n-k$ times. On the other hand, the number of elements that are not yet covered $n-k$ times in any set from our optimal cover is at most

$$\gamma + \left(1 - \sum_{i=1}^{q-1} \frac{3}{4^i}\right)\alpha = \gamma + \frac{1}{4^{q-1}}\alpha = \gamma + \frac{4}{4^q}\alpha$$

and $\frac{6}{4^q}\alpha > \gamma + \frac{4}{4^q}\alpha$ provided $q < \log_4\left(\frac{2\alpha}{\gamma}\right)$. Since $\log_4\left(\frac{2\alpha}{\gamma}\right) > \log_4\left(\frac{4n}{5(2+\log_2 n)}\right) > \log_4\left(\frac{4n}{10\log_2 n}\right) = \left(\frac{1}{2} - o(1)\right)\log_2 n$, the inequality $\frac{6}{4^q}\alpha > \gamma + \frac{4}{4^q}\alpha$ holds for $q \in \left[1, \left(\frac{1}{2} - o(1)\right)\log_2 n\right]$.

❑

3.2 Improved Randomized Approximation Algorithms for $\mathbf{CP}_k$or $\mathbf{SC}_k$

As stated before, an instance $<n, m, k>$ of $\mathbf{SC}_k$ can be $(1+\ln n)$-approximated in $O(mnk)$ time for any k. In this section, we provide a randomized algorithm with an expected performance ratio better than $(1+\ln n)$ for larger k; this in turn will provide a constant expected performance ratio for $\mathbf{CP}_k$ for smaller k. Consider the standard integer programming (IP) formulation of an instance $<n, m, k>$ of $\mathbf{SC}_k$ [11]:

$$\begin{array}{ll} \textit{minimize } \sum_{j=1}^{m} x_j & \\ \text{subject to} & \\ \quad \sum_{i \in S_j} x_j \geq k & \text{for each } i \in [1, n] \\ \quad x_j \in \{0, 1\} & \text{for each } j \in [1, m] \end{array}$$

A linear programming (LP) relaxation of the above formulation is then obtained by replacing each constraint $x_j \in \{0, 1\}$ by $0 \leq x_j \leq 1$. The following randomized approximation algorithm for $\mathbf{SC}_k$ can then be designed:

1. Select an appropriate positive constant $c \geq 1$.
2. Find a solution $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_m^*)$ to the LP relaxation via any polynomial-time algorithm for solving linear programs (e.g. [5]).
3. Form a family of sets $\mathcal{C}_0 = \{S_j : \ cx_j^* \geq 1\}$.
4. Form a family of sets $\mathcal{C}_1 \subseteq \mathcal{S} - \mathcal{C}_0$ by selecting a set $S_j \in \{S_1, S_2, \ldots, S_m\} \backslash \mathcal{C}_0$ with probability cx_j^*.
5. Form a family of sets $\mathcal{C}_2$ by greedy choices: if an $i \in [1, n]$ belongs to fewer than k sets in $\mathcal{C}_0 \cup \mathcal{C}_1$, choose any of the remaining sets that contains i.
6. Return $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$ as the solution.

Theorem 4 *The expected performance ratio of our algorithm is given by*

$$\begin{array}{lll} 2+\left(\mathbf{e}^{-2}+\mathbf{e}^{-9/8}\right)\cdot\frac{n}{k} \;\simeq\; 2+0.46\cdot\frac{n}{k}, & \textit{with } c=2 \textit{ if } \frac{n}{k}\leq 5 \\ 3+\mathbf{e}^{-3/2}\cdot\frac{n}{k} \;\simeq\; 3+0.223\cdot\frac{n}{k}, & \textit{with } c=3 \textit{ if } \frac{n}{k}<9 \\ 2+2\ln\left(\frac{n}{2k}\right), & \textit{with } c=2\ln\left(\frac{n}{2k}\right) \textit{ if } \frac{n}{k}\geq 7 \end{array}$$

Proof. Let OPT denote the minimum number of sets used by an optimal solution. Obviously, OPT$\geq \sum_{i=1}^{m} x_i^* \geq k$.

Each set in $S_j \in \mathcal{C}_0 \cup \mathcal{C}_1$ is selected with probability $\min\{cx_j, 1\}$. Thus $s = E[|\mathcal{C}_0 \cup \mathcal{C}_1|] \leq c\sum_{j=1}^{m} x_j^* \leq c \cdot \text{OPT}$.

Next we estimate an upper bound on $E[|\mathcal{C}_2|]$. For each $i \in [1,n]$ let the random variable ν_i be $\max\{k-d_i, 0\}$ where d_i is the number of sets in $\mathcal{C}_0 \cup \mathcal{C}_1$ that contain i. Clearly, $|\mathcal{C}_2| \leq \sum_{i=1}^{n} \nu_i$. Thus it suffices to estimate $E[\nu_i]$. Because our estimate will not depend on i, we will drop this index i from ν_i for notational simplifications.

Assume that $i \in [1,n]$ is contained in $k-f$ sets from $\mathcal{C}_0$ for some $f < k$. Then, $1 \leq \nu \leq f$ and

$$E[\nu] \;\leq\; \sum_{j=1}^{\infty} \mathbf{Pr}[\nu \geq j] \;=\; \sum_{j=1}^{f} \mathbf{Pr}[\nu \geq j] \;=\; \sum_{j=0}^{f-1} \mathbf{Pr}[\nu \geq f-j] \tag{3}$$

Let the random variable y denote the number of sets in $\mathcal{C}_1$ that contain i. Considering the constraint $\sum_{i \in S_j} x_j^* \geq k$ and the fact that we select a set $S_j \in \{S_1, S_2, \ldots, S_m\} \backslash \mathcal{C}_0$ with probability cx_j^*, it follows that $E[y] \geq cf$. Now,

$$\mathbf{Pr}[\nu \geq f-j] = \mathbf{Pr}[y < (1-\delta_j)cf] \tag{4}$$

where

$$(1-\delta_j)cf = j \;\equiv\; cf\delta_j = cf - j \;\equiv\; \delta_j = \frac{cf-j}{cf} \tag{5}$$

By using Chernoff's bound [1], we have

$$\mathbf{Pr}[y < (1-\delta_j)cf] \leq \mathbf{Pr}[y < (1-\delta_j)E[y]] < \mathbf{e}^{-E[y]\delta_j^2/2} \leq \mathbf{e}^{-cf\delta_j^2/2} \tag{6}$$

where

$$\frac{cf\delta_j^2}{2} \;=\; \frac{(cf-j)^2}{2cf} \;=\; \frac{f}{2}c - j + \frac{j^2}{2cf} \;=\; \zeta(c,f,j) \tag{7}$$

Combining Equations (3), (4), (6) and (7) we get $E[\nu] < \sum_{j=0}^{f-1} \mathbf{e}^{-\zeta(c,f,j)}$.

Lemma 5 *Let* $S(c,f) = \sum_{j=0}^{f-1} \mathbf{e}^{-\zeta(c,f,j)}$.
(a) *If* $c \geq \frac{5}{2}$ *then* $S(c,f)$ *is maximized for* $f=1$ *and this maximum value is* $\mathbf{e}^{-c/2}$.
(b) $S(2,f)$ *is maximized for* $f=2$ *and this maximum value is* $\mathbf{e}^{-2}+\mathbf{e}^{-9/8} < 0.46$.

Proof.
(a) For notational convenience, let $h = c/2$. Note that $S(c,1) = S(2h,1) = \mathbf{e}^{-\zeta(2h,1,0)} = \mathbf{e}^{-h}$. For $f > 1$ we can overestimate $S(2h,f)$ using $\zeta'(2h,f,j) = fh - j$. Thus $S(2h,f) < \mathbf{e}^{-fh}\sum_{\ell=0}^{f-1}\mathbf{e}^{\ell} = \mathbf{e}^{-f(h-1)}\sum_{\ell=1}^{f}\mathbf{e}^{-\ell}$. We have two cases.

Case 1: $\mathbf{f = 2}$. Since $S(2h,2) = \mathbf{e}^{-2h} + \mathbf{e}^{-2h+1-\frac{1}{8h}}$, we have

$$S(2h,2) < S(2h,1) \quad \equiv \quad e^{-2h}(1+e^{1-\frac{1}{8h}}) < \mathbf{e}^{-h} \quad \equiv \quad 1 + \mathbf{e}^{1-\frac{1}{8h}} < \mathbf{e}^{h}$$

It is easy to see that for $1 \le h \le 2$ the left-hand-side of the last inequality has smaller derivative than the right-hand-side. Hence to verify that the last inequality holds for $h \ge 1.25$ it suffices to check it for $h = 1.25$ where it is $1 + 2.46 < 3.49$.

Case 2: $\mathbf{f > 2}$. Since $\sum_{\ell=1}^{f}\mathbf{e}^{-\ell} \le \mathbf{e}^{-1}\sum_{\ell=0}^{\infty}\mathbf{e}^{-\ell} = \frac{1}{\mathbf{e}-1} < \mathbf{e}^{-0.54}$, we have $S(2h,f) < \mathbf{e}^{-f(h-1)-0.54}$. Thus, $S(2h,f) < \mathbf{e}^{-h}$ provided:

$$\mathbf{e}^{-f(h-1)-0.54} < \mathbf{e}^{-h} \quad \equiv \quad f(h-1)+0.54 > h \quad \equiv \quad (f-1)h > f-0.54 \quad \equiv \quad h > \frac{f-0.54}{f-1}$$

The last right-hand-side is decreasing with f, so it suffices to consider $f = 3$; thus the sufficient condition is $h > 1.23$ and hence $c > 2.46$.

(b) $S(2,1) = \mathbf{e}^{-1} < \mathbf{e}^{-2} + \mathbf{e}^{-9/8}$. The following series of arguments shows that $S(2,f) \le S(2,2)$ for all $f > 2$:

- $\zeta(2,f,j) = f - j + \frac{j^2}{4f}$. Also, $\zeta(2,f,f-1) = 1 + \frac{(f-1)^2}{4f} = \frac{f+2+1/f}{4}$ is an increasing function of f.
- $\zeta(2,p+1,j) - \zeta(2,p,j) = 1 + \frac{j^2}{4}\left(\frac{1}{p+1} - \frac{1}{p}\right) = 1 - \frac{j^2}{4p(p+1)} > \frac{3}{4}$ for all $p \ge 1$.
- $$\begin{aligned} S(2,f) &= \left(\sum_{j=0}^{f-2}\mathbf{e}^{-\zeta(2,f-1,j)}\cdot\mathbf{e}^{\zeta(2,f-1,j)-\zeta(2,f,j)}\right) + \mathbf{e}^{-\zeta(2,f,f-1)} \\ &< \mathbf{e}^{-3/4}S(2,f-1) + \mathbf{e}^{-\zeta(2,f,f-1)} < \tfrac{1}{2}S(2,f-1) + e^{-\zeta(2,f,f-1)} \end{aligned}$$

  Hence

  $$\begin{aligned} & S(2,f-1) \le S(2,2) \bigwedge \mathbf{e}^{-\zeta(2,f,f-1)} < \tfrac{1}{2}S(2,2) \Longrightarrow S(2,f) < S(2,2) \\ \equiv \\ & S(2,f-1) \le S(2,2) \bigwedge \zeta(2,f,f-1) > -\ln S(2,2) + \ln 2 \Longrightarrow S(2,f) < S(2,2) \end{aligned}$$

- $S(2,3) < 0.44 < S(2,2)$.
- $\zeta(2,4,3) > 1.56 > 0.78 + 0.694 > -\ln S(2,2) + \ln 2$. Moreover, since $\zeta(2,f,f-1)$ increases with f, this implies $\zeta(2,f,f-1) > -\ln S(2,2) + \ln 2$ for all $f \ge 4$. Thus, $S(2,f) < S(2,2)$ for all $f \ge 4$. ❑

Now we are able to complete our proof on the expected performance ratios of our algorithm as follows:

- If $n/k \leq 5$ and $c = 2$ then $E[|\mathcal{C}_0 \cup \mathcal{C}_1|] \leq 2 \cdot \mathrm{OPT}$ and $E[|\mathcal{C}_2|] \leq \left(\mathbf{e}^{-2} + \mathbf{e}^{-9/8}\right) \cdot n = \left(\mathbf{e}^{-2} + \mathbf{e}^{-9/8}\right) \cdot \frac{n}{k} \cdot \mathrm{OPT}$ by Lemma 5(b).
- If $n/k < 9$ and $c = 3$ then $E[|\mathcal{C}_0 \cup \mathcal{C}_1|] \leq 3 \cdot \mathrm{OPT}$ and $E[|\mathcal{C}_2|] < \zeta(3,1,1) = \mathbf{e}^{-3/2} n \approx 0.223n = 0.223 \cdot \frac{n}{k} \cdot \mathrm{OPT}$ by Lemma 5(a).
- If $n/k \geq 7$ and $c = 2\ln(\frac{n}{2k})$ then $E[|\mathcal{C}_0 \cup \mathcal{C}_1|] \leq 2\ln(\frac{n}{2k}) \cdot \mathrm{OPT}$. Also, since $n/k \geq 7$ implies $c = 2\ln(\frac{n}{2k}) \geq \frac{5}{2}$, by Lemma 5(a) we have $E[|\mathcal{C}_2|] \leq \mathbf{e}^{-\ln(\frac{n}{2k})} n = \frac{2k}{n} \cdot n = 2k$.

❑

Remark 2 *Since* **CP** *is equivalent to* $\mathbf{SC}_{n-1}$*, our expected performance ratio for* **CP** *is about* $2.46 + \frac{0.46}{n-1}$.

Remark 3 *By repeating the randomized algorithm a polynomial number of times and taking the best solution, the performance ratio can be made to be close to its expected value with high probability.*

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