DIMACS Technical Report 2003-45
December 31, 2003

The Voter Model with Confidence Levels

by

Stephen Hartke[1,2]
Department of Mathematics
Rutgers University
Hill Center – Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
hartke@math.rutgers.edu

[1]Affiliated Graduate Student Member.
[2]The author gratefully acknowledges support from a National Defense Science and Engineering Graduate Fellowship and from the National Science Foundation under grants NSF SBR 97-09134 and NSF EIA 02-05116.

ABSTRACT

The voter model on a finite connected graph G is a stochastic process where each vertex has an opinion, 0 or 1. As time progresses, each voter's opinion is influenced by its neighbors. We introduce a modification of the voter model that changes how quickly a voter will change its opinion based on its confidence in its opinion. We show that the voter model with confidence levels always results in a uniform opinion, and we determine the probability of each outcome (uniform 1 or 0) based on the initial opinions and the structure of the graph.

Keywords: voter model, interacting particle system
Mathematics Subject Classification: Primary 60K35, Secondary 82C20, 82C22

1 Introduction

The voter model was introduced independently by Clifford and Sudbury [2] and by Holley and Liggett [8]. The voter model can be interpreted as modeling spatial conflicts between two species, the spread of disease through a social network, the configuration of spins in a particle system, or, as the name suggests, the opinions of a group of voters. Our terminology will come from this last interpretation, though our interest in the model is much broader.

In the voter model, there is a group of voters, each of which has an opinion *yes* or *no* on some issue. Each voter has relationships to other neighboring voters, and as time passes, each voter's opinion is influenced by its neighbors. How do the voters' opinions as a whole evolve? Do they ever come to a consensus on the issue? If consensus occurs, which outcome (yes or no) is more likely, given every voter's initial opinion? If the graph G has the voters as its vertices and the relationships between voters as its edges, then how does the structure of G affect the outcome?

In the extensive literature on the voter model, all of these questions are addressed. Unfortunately, the answer to the last question is usually not very interesting. Because of the interaction mechanism usually used for determining how a voter is influenced by its neighbors, the final outcome is based only a simple graph theoretic property: the sum of the degrees of the vertices where the voters' initial opinions are yes. In this paper, we present a different interaction mechanism involving a voter's *confidence level*. In this modified voter model, the structure of G has a large impact on evolution of the voters' opinions. When G is finite and connected, consensus always occurs, and we can explicitly calculate how likely each outcome is. The remarkable feature of the voter model with confidence levels is that determining the likelihood of each outcome remains tractable *despite* this dependence on the structure of G.

The standard reference for the voter model and related interacting particle systems is Liggett [9] and its continuation [10]. Griffeath [6] and Durrett [4] are also widely used sources that provide slightly different perspectives. While the previous authors consider the voter model on infinite graphs, particularly the square lattices in various dimensions, Aldous and Fill [1] is a very readable account of the theory on finite graphs. Also on finite graphs, Donnelly and Welsh [3] consider the probability of each outcome and the time needed to reach consensus. The idea of confidence levels was inspired by the work of Hoffman [7] and Roberts [14] who considered deterministic models of opinion formulation on graphs.

We will first provide a brief discussion of the original voter model to establish terminology and some of the tools that we will be using. In the third section, we will present the voter model with confidence levels and determine the limiting probability of entering the uniform 1 opinion, given the graph and the initial opinions. In the the fourth section, we will present some sample waiting functions and calculate the limiting probability for these functions on a sample graph.

2 The Voter Model

We present here the voter model on arbitrary finite graphs. Let G be a finite connected undirected graph on n vertices labeled v_1 through v_n. Let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of G. We define a continuous-time Markov process $\{Z_t\}_{t\geq 0}$ on the finite state space $\{0,1\}^{V(G)}$ as follows. At every time $t \geq 0$,each vertex v_i has an *opinion* $Z_t(v_i)$ in $\{0,1\}$. Each vertex v_i asynchronously updates its opinion at times $\{T_\ell(v_i)\}_{\ell=1}^\infty$ according to the following rule: for $T_\ell(v_i) < t \leq T_{\ell+1}(v_i)$,

$$Z_t(v_i) = \begin{cases} 0, & \text{with probability } \#\{x \in N_G(v_i) : Z_{T_\ell(v_i)}(x) = 0\}/\#N_G(v_i), \\ 1, & \text{with probability } \#\{x \in N_G(v_i) : Z_{T_\ell(v_i)}(x) = 1\}/\#N_G(v_i). \end{cases}$$

Here, $N_G(v_i)$ denotes the neighborhood of v_i, or set of adjacent vertices, in G. Note that $Z_t(v_i)$ is updated only at the times $\{T_\ell(v_i)\}$[1], and is constant on each interval $(T_\ell(v_i), T_{\ell+1}(v_i)]$. Also observe that the update mechanism is probabilistically the same as v_i adopting the opinion of a neighbor $N_\ell(v_i)$ chosen uniformly at random from $N_G(v_i)$: for $T_\ell(v_i) < t \leq T_{\ell+1}(v_i)$,

$$Z_t(v_i) = Z_{T_\ell(v_i)}(N_\ell(v_i)), \text{ where } N_\ell(v_i) \text{ is chosen uniformly from } N_G(v_i).$$

We will use this view of the update mechanism for analyzing the voter model. The sequence $\{T_\ell(v_i)\}$ is a Poisson process with rate 1; that is, $T_{\ell+1}(v_i) - T_\ell(v_i)$ is exponentially distributed with mean 1 for each ℓ. We set $T_0(v_i) = 0$ for convenience. The $\{T_\ell(v_i)\}$ are also independent for each v_i.

Because G is finite and connected, the only absorbing states for the process $\{Z_t\}_{t\geq 0}$ are the states of uniform opinion where either $Z_t(v_i) = 0$ for all i, or $Z_t(v_i) = 1$ for all i. Since the process will eventually enter one of these absorbing states with probability 1, we wish to know the probability of entering the uniform 1 opinion given the initial configuration Z_0 of opinions. All vertices have the same opinion in the absorbing states, and so this probability is the same as the limiting probability of v_i's opinion being 1, for any vertex v_i. Thus,

$$\Pr\{Z_t \text{ enters the uniform 1 opinion state} \mid Z_0\} = \lim_{t\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\}.$$

This probability can be found by considering another continuous-time Markov process $\{X_s\}$ that is dual to $\{Z_t\}$. The process $\{X_s\}_{s\geq 0}$ is a system of random walks, with one random walk $X_s(v_i)$ starting at each vertex v_i. Formally, the process $\{X_s\}$ is defined on the finite state space $V(G)^{V(G)}$, where at every $s \geq 0$, each vertex has as its state $X_s(v_i)$ a vertex in $V(G)$. The random walk $X_s(v_i)$ updates at the discrete times $\{S_m(v_i)\}_{m=0}^\infty$. We think of the dual process as progressing *backwards* in time relative to $\{Z_t\}$, as shown in Figure 1. Fix a time $t_0 > 0$. The time $t = t_0$ for $\{Z_t\}$ corresponds to the time $s = 0$ for $\{X_s\}$. For each vertex v_i, $X_s(v_i)$ traces where v_i's opinion at time t_0 came from. We start with $X_0(v_i) = v_i$ and $S_0(v_i) = 0$. At some time $T_\ell(v_i)$, where $T_\ell(v_i) < t_0 \leq T_{\ell+1}(v_i)$, v_i adopted the opinion of a neighbor $v_j := N_\ell(v_i)$. We set $S_1(v_i) := t_0 - T_\ell(v_i)$, and update $X_{S_1(v_i)}(v_i)$ to v_j. At some time $T_{\ell'}(v_j)$, where $T_{\ell'}(v_j) < T_\ell(v_i) \leq T_{\ell'+1}(v_j)$, v_j adopted the opinion of a neighbor $v_k := N_{\ell'}(v_j)$. We set $S_2(v_i) := t_0 - T_{\ell'}(v_j)$, and update $X_{S_2(v_i)}(v_i) = v_k$. For those s such that $S_1(v_i) \leq s < S_2(v_i)$, $X_s(v_i)$ is constant. We continue this process, following v_i's opinion backwards in time, for $s \leq t_0$. Since v_i's opinion at time t_0 is the opinion of $X_s(v_i)$ at time $t_0 - s$, we have that

$$Z_{t_0}(v_i) = Z_{t_0-s}(X_s(v_i)).$$

[1]Strictly speaking, this isn't true since v_i's opinion does not change until immediately *after* $T_\ell(v_i)$. However, we will stick to the terminology that "v_i updates at $T_\ell(v_i)$."

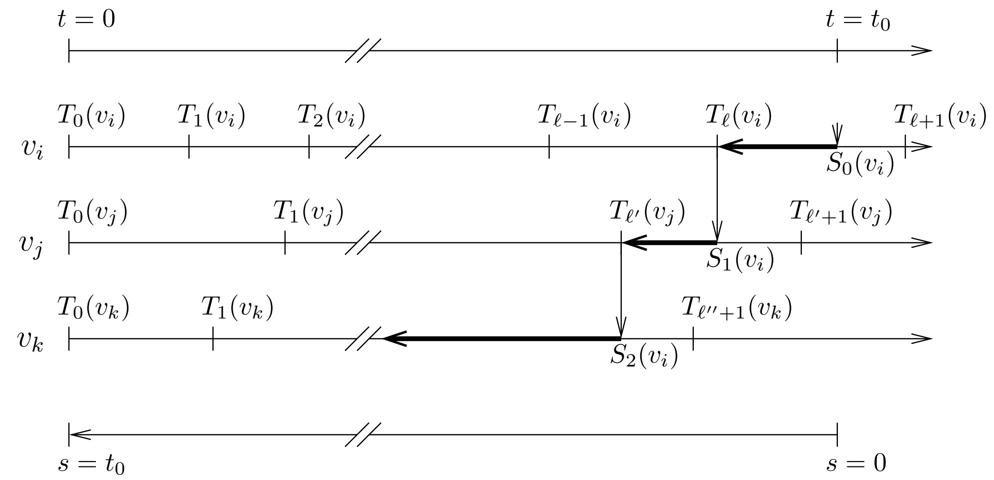

Figure 1: $\{X_s\}$ progresses backwards in time relative to $\{Z_t\}$.

Since these random variables are equal, we also have

$$\Pr\{Z_{t_0}(v_i) = 1 \mid Z_0\} = \Pr\{Z_{t_0-s}(X_s(v_i)) = 1 \mid Z_0\}.$$

By setting $s = t_0$ and taking the limit as $t_0 \to \infty$,

$$\lim_{t_0\to\infty} \Pr\{Z_{t_0}(v_i) = 1 \mid Z_0\} = \lim_{t_0\to\infty} \Pr\{Z_0(X_{t_0}(v_i)) = 1 \mid Z_0\}.$$

The left hand limit is the quantity we are interested in, and thus we have reduced the problem to calculating the limiting distribution of $X_s(v_i)$. The process $X_s(v_i)$ is simply a random walk where $X_s(v_i)$ moves to a neighbor chosen uniformly at random at the discrete times $\{S_m(v_i)\}_{m=0}^{\infty}$. How is $\{S_m(v_i)\}$ distributed? Again, fix $t_0 > 0$, and let m be such that $S_{m+1}(v_i) < t_0$. Set $v_j := X_{S_m(v_i)}(v_i)$, and let ℓ be such that $T_\ell(v_j) < t_0 - S_m(v_i) \leq T_{\ell+1}(v_j)$. Note that $S_{m+1}(v_i) = t_0 - T_\ell(v_j)$. The *length* of the interval $T_{\ell+1}(v_j) - T_\ell(v_j)$ is exponentially distributed with mean 1, and retains no information about the direction of time. Thus, $\mathrm{E}[T_{\ell+1}(v_j) - T_\ell(v_j) \mid T_\ell(v_j)]$ and $\mathrm{E}[T_{\ell+1}(v_j) - T_\ell(v_j) \mid T_{\ell+1}(v_j)]$ are both exponentially distributed with mean 1. Since exponential distributions are memoryless, the distribution of when all the pebbles move from a vertex v_j is independent of when the pebbles arrived at v_j. Hence, $S_{m+1}(v_i) - S_m(v_i)$ is exponentially distributed with mean 1. Since the $\{T_{\ell+1}(v_j) - T_\ell(v_j)\}$ are independent for each v_j and ℓ, the $\{S_{m+1}(v_i) - S_m(v_i)\}$ are also independent for each v_i and m. Therefore, $\{X_s(v_i)\}_{s\geq 0}$ is a standard continuous-time random walk on G; that is, $\{X_s(v_i)\}_{s\geq 0}$ is a random walk that moves to a neighbor chosen uniformly at random according to a Poisson process with rate 1. From the theory of random walks (see, for example, [11]),

$$\lim_{s\to\infty} \Pr\{Z_0(X_s(v_i)) = 1\} = \frac{\displaystyle\sum_{x\in V(G):Z_0(x)=1} \deg_G(x)}{\displaystyle\sum_{x\in V(G)} \deg_G(x)}.$$

Note that the theory of random walks actually gives us more information, in that we can calculate $\lim_{s\to\infty} \Pr\{X_s(v_i) = x\}$. Summing these values over all x such that $Z_0(x) = 1$ gives us the formula above.

The random processes $\{X_s(v_i)\}$ for different vertices v_i are coupled together in the following way. Suppose that $X_s(v_i) = X_s(v_j) = v_k$ for $T_\ell(v_k) < t_0 - s \le T_{\ell+1}(v_k)$. Then $X_{t-T_\ell(v_k)}(v_i) = X_{t-T_\ell(v_k)}(v_j) = N_\ell(v_k)$. In words, if $X_s(v_i)$ and $X_s(v_j)$ are both at v_k, then $X_s(v_i)$ and $X_s(v_j)$ move *together* to the vertex $N_\ell(v_k)$ that is chosen uniformly from v_k's neighborhood. From that point on, $X_s(v_i)$ and $X_s(v_j)$ remain coupled, in the sense that $X_s(v_i) = X_s(v_j)$ for $s \ge t_0 - T_\ell(v_j)$. The random walks $\{X_s\}$ are called *coalescing random walks*, and two random walks $X_s(v_i)$ and $X_s(v_j)$ are said to *coalesce* when the two random walks hit and move together thereafter. With probability 1, all of the random walks coalesce (see [9]); that is, there is some time S such that $X_s(v_i) = X_s(v_j)$ for $s > S$ and for all v_i and v_j. This provides another proof of the fact that the only absorbing states for the process $\{Z_t\}_{t\ge0}$ are the states of uniform opinion.

3 The Voter Model with Confidence Levels

We now extend the voter model to include the *confidence* a voter has in its opinion. Specifically, we include the idea that a voter is less likely to consider the opinions of neighbors the higher its confidence level is. As above, we have a finite connected undirected graph G and a continuous-time stochastic process $\{(Z_t, C_t)\}_{t\ge0}$ defined on the state space $(\{0,1\} \times \mathbb{N})^{V(G)}$. At every time $t \ge 0$,each vertex v_i has an *opinion* $Z_t(v_i)$ in $\{0,1\}$ and a *confidence level* $C_t(v_i)$ in $\mathbb{N}$. A vertex v_i asynchronously updates its opinion at times $\{T_\ell(v_i)\}_{\ell=1}^\infty$ by adopting the opinion of a neighbor $v_j := N_\ell(v_i)$ chosen uniformly at random; that is, for $T_\ell(v_i) < t \le T_{\ell+1}(v_i)$,

$$Z_t(v_i) = Z_{T_\ell(v_i)}(v_j), \text{ where } v_j := N_\ell(v_i) \text{ is chosen uniformly from } N_G(v_i).$$

Vertex v_i's confidence level is updated at the same times $\{T_\ell(v_i)\}$. For $T_\ell(v_i) < t \le T_{\ell+1}(v_i)$, $C_t(v_i)$ is updated as follows:

$$C_t(v_i) = \begin{cases} C_{T_\ell(v_i)}(v_j) + 1 & \text{if } v_j \text{ is a trusted associate of } v_i, \\ 0 & \text{if not.} \end{cases}$$

Notice that v_i increments v_j's confidence level if v_j is a trusted associate, and not v_i's own confidence level. We define v_j to be a *trusted associate* of v_i if and only if v_j's initial opinion $Z_0(v_j)$ agrees with v_i's initial opinion $Z_0(v_i)$. The *trusted component* $K(v_i)$ of v_i is the component containing v_i of the subgraph of G induced by vertices whose initial opinions agree with v_i's.

The length of the time interval $(T_\ell(v_i), T_{\ell+1}(v_i)]$ is dependent purely on the confidence level $C_t(v_i)$ that v_i has during that time interval. Let the *waiting function* $f : \mathbb{N} \to \mathbb{R}^+$ be a positive non-decreasing function. If $c = C_{T_\ell(v_i)}(v_i)$, then $T_{\ell+1}(v_i) - T_\ell(v_i)$ is exponentially distributed with mean $f(c) - f(c-1)$ if c is positive, and exponentially distributed with mean $f(0)$ if $c = 0$. We set $T_0(v_i) = 0$ for convenience.

To motivate this extension of the voter model, we consider a group of voters in a two-party political system. All of the relationships between voters are known and are represented as edges in the graph G. We assume that the two parties have opposing views on some issue, and that the members of each party initially assume their party's position by default before discussion begins. As the discussion progresses, a voter v_i is equally swayed by all of its neighbors in G, whether they are in its party or not. However, members of the same party are trusted associates in the sense that if trusted associate v_j convinces v_i of its opinion, then v_j also convinces v_i of v_j's confidence level in that opinion. In fact, v_i's new confidence level is even one greater than v_j's confidence level $C_t(v_j)$. Not only does v_i accept v_j's confidence in the opinion, but v_j's own credibility is added to v_i's confidence level. When v_i adopts the opinion of a non-trusted associate, then its confidence level simply resets to a level of 0. The effect of a higher confidence level is that it takes longer for any of v_i's neighbors to convince v_i of a new opinion.

To analyze $\{(Z_t, C_t)\}$, we form the dual process $\{(X_s, D_s)\}_{s\geq 0}$, where, as described above, the dual is progressing *backwards* in time relative to $\{(Z_t, C_t)\}$. Fix a time $t_0 > 0$. Again, $\{X_s(v_i)\}_{s\geq 0}^{s\leq t_0}$ traces where v_i's opinion at time t_0 came from, and so we have

$$\begin{aligned} &Z_{t_0}(v_i) = Z_{t_0-s}(X_s(v_i)), \text{ and} \\ &Z_{t_0}(v_i) = Z_0(X_{t_0}(v_i)), \quad \text{by setting } s = t_0. \end{aligned}$$

The process $\{X_s(v_i)\}_{s\geq 0}$ is a random walk on G that moves to a neighbor $N_m(v_i)$ chosen uniformly at random at the discrete times $\{S_m(v_i)\}_{m=0}^{\infty}$. In keeping with our view of the dual progressing backwards in time, the variable $D_s(v_i)$ is defined to be the confidence level $C_{t_0-s}(X_s(v_i))$ of the vertex $X_s(v_i)$. The distribution of $S_{m+1}(v_i) - S_m(v_i)$ is dependent on $c := D_{S_m(v_i)}(v_i)$. As before, since the exponential distribution is independent of the direction of time and is memoryless, $S_{m+1}(v_i) - S_m(v_i)$ is exponentially distributed with mean $f(c) - f(c-1)$ if c is positive, and exponentially distributed with mean $f(0)$ if $c = 0$.

Note that $D_s(v_i)$ updates only when $X_s(v_i)$ updates, and that when $X_s(v_i)$ does not leave the current trusted component $K(X_s(v_i))$, then $D_s(v_i)$ simply decrements by 1. In fact, $X_s(v_i)$ does not leave the trusted component until $D_s(v_i) = 0$. The critical observation is that $D_s(v_i)$ thus counts the number of steps that $X_s(v_i)$ *will take* in the current trusted component before leaving. It is slightly disconcerting that $D_s(v_i)$ is dependent on the future (in the sense of the s time variable). However, since the random variable $N_m(v_i)$ of a randomly chosen neighbor at time $S_m(v_i)$ is independent of $N_{m'}(v_j)$ for any m' and $v_j \neq v_i$, of $N_{m'}(v_i)$ for any $m' \neq m$, and of $S_{m'}(v_j)$ for any m' and v_j, and since $D_s(v_i)$ is dependent only on the choices $\{N_m(v_i)\}$, we will be able to analyze the situation with only a little extra difficulty.

The random walks $\{X_s\}$ are again coupled in that if $X_{s'}(v_i) = X_{s'}(v_j)$ for some s', then $X_s(v_i) = X_s(v_j)$ for all $s \geq s'$. The random variables D_s also are coupled since $D_s(v_i) = D_s(v_j)$ for all $s \geq s'$. The system of random walks $\{X_s\}$ do coalesce into a single random walk. However, this is not immediate, and will be proved in Theorem 3.

From the construction of the dual, we immediately have

Proposition 1 (Duality). *The process $\{(X_s, D_s)\}_{s\geq 0}$ is dual to $\{(Z_t, C_t)\}_{t\geq 0}$ in the sense* *that*

$$\Pr\{Z_{t_0}(v_i) = 1 \mid Z_0\} = \Pr\{Z_0(X_{t_0}(v_i)) = 1 \mid Z_0\}, \text{ and so}$$
$$\lim_{t_0\to\infty} \Pr\{Z_{t_0}(v_i) = 1 \mid Z_0\} = \lim_{s\to\infty} \Pr\{Z_0(X_s(v_i)) = 1 \mid Z_0\},$$

if at least one of the limits exists.

Definition 2. Let $\{\widehat{X}_m(v_i)\}_{m=0}^{\infty}$ be the discrete-time Markov chain given by $\widehat{X}_m(v_i) = X_{S_m(v_i)}(v_i)$. Known as the *embedded chain* or *jump chain* for the process $\{(X_s(v_i), D_s(v_i)\}_{s\geq 0}$, $\{\widehat{X}_m(v_i)\}$ is defined by the state transition probabilities without respect to the holding times at each state. In our case, $\{\widehat{X}_m(v_i)\}$ is a standard discrete-time random walk on G; that is, $\{\widehat{X}_m(v_i)\}_{m=0}^{\infty}$ is a random walk that moves to a neighbor chosen uniformly at random at each discrete time unit.

Theorem 3 (Coalescence). *If Z_0 is not a uniform opinion state, then the system $\{(X_s, D_s)\}_{s\geq 0}$ of random walks almost surely coalesces to a single random walk; that is, with probability* 1 *there exists a time S_c such that for all $s \geq S_c$ and any v_i and v_j, $X_s(v_i) = X_s(v_j)$.*

Proof. For the sake of clarity and simplicity of notation, we will refer to the random variables $X_s(v_i)$, $X_s(v_j)$, and $\widehat{X}_m(v_i)$ as X_s, X'_s, and $\widehat{X}_m$, respectively. For the rest of the proof, v_i and v_j will simply be arbitrary vertices, and not necessarily the starting vertices of the random walks.

If G is connected and non-bipartite, then $\{\widehat{X}_m\}$ is a regular Markov chain. Thus, there exists an integer M such that

$$\Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\} > \varepsilon \tag{1}$$

for any $m \geq M$, any v_i and v_k, and for some positive constant ε (see section 5.5 of [13]). Let $\mathcal{E}_{s'}$ be the event that the number of updates of X_s (or "steps" of $\widehat{X}_m$) that occur between $s = 0$ and $s = s'$ is at least M, given that $X_0 = v_i$ and that $X_{s'} = v_k$. Because Z_0 is not a uniform opinion state, $K(v_i)$ is not all of G. Thus, there exists a sequence of steps of $\widehat{X}_m$ from v_i to v_k that leaves $K(v_i)$, and so the time to take each of these steps has finite expectation. Hence, there exists an $S_{i,k}$ and a positive constant $\delta_{i,k}$, both dependent on v_i and v_k, such that the probability of $\mathcal{E}_{S_{i,k}}$ is greater than $\delta_{i,k}$. Let S be the maximum of $S_{i,k}$ over all i and k, and similarly let δ be the minimum of $\delta_{i,k}$ over all i and k. Then $\mathcal{E}_S$ has probability greater than δ, for any choice of v_i and v_k. Thus,

$$\begin{aligned}
&\Pr\{X_S = v_k \mid X_0 = v_i\} \\
&= \sum_{m=0}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\} \Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\} \\
&\geq \sum_{m=M}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\} \Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\} \\
&> \sum_{m=M}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\}\varepsilon \qquad \text{by (1)}, \\
&> \delta\varepsilon.
\end{aligned} \tag{2}$$

If G is bipartite, we need to consider the parity of the number of steps taken. If v_i and v_k are in the same partite set of G, there exists an even integer M such that

$$\Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\} > \varepsilon \tag{3}$$

for any even $m \geq M$, any v_i and v_k in the same partite set, and for some positive constant ε. Let $\mathcal{E}^{\mathrm{e}}_{s'}$ be the event that the number of updates of X_s (or "steps" of $\widehat{X}_m$) that occur between $s = 0$ and $s = s'$ is even and at least M, given that $X_0 = v_i$ and that $X_{s'} = v_k$. Because Z_0 is not a uniform opinion state, $K(v_i)$ is not all of G. Thus, there exists a sequence of steps of $\widehat{X}_m$ from v_i to v_k that leaves $K(v_i)$, and so the time to take each of these steps has finite expectation. Hence, there exists an $S_{i,k}$ and a positive constant $\delta_{i,k}$, both dependent on v_i and v_k, such that the probability of $\mathcal{E}^{\mathrm{e}}_{S_{i,k}}$ is greater than $\delta_{i,k}$. Let S be the maximum of $S_{i,k}$ over all i and k, and similarly let δ be the minimum of $\delta_{i,k}$ over all i and k. Then $\mathcal{E}_S$ has probability greater than δ, for any choice of v_i and v_k. Thus,

$$\begin{aligned}
&\Pr\{X_S = v_k \mid X_0 = v_i\} \qquad (4)\\
&= \sum_{m=0,\, m \text{ even}}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\} \Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\}\\
&\geq \sum_{m=M,\, m \text{ even}}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\} \Pr\{\widehat{X}_m = v_k \mid \widehat{X}_0 = v_i\}\\
&> \sum_{m=M,\, m \text{ even}}^{\infty} \Pr\{X_s \text{ takes } m \text{ steps during } s \in (0, S]\}\varepsilon \qquad \text{by (3)},\\
&> \delta\varepsilon.
\end{aligned}$$

Similarly, if v_i and v_k are in different partite sets, then the same proof works by considering only an odd number of steps.

Note that these bounds are time translation invariant, in the sense that for any $s > 0$,

$$\Pr\{X_{S+s} = v_k \mid X_s = v_i\} > \delta\varepsilon.$$

This is because X_s is independent of the past.

We now consider a second random walk $\{X'_s\}$. We wish to determine the probability of X_s and X'_s hitting; that is, of there existing a time S_c such that $X_{S_c} = X'_{S_c}$. Let S_h be the first time when $X_s = X'_s$ if X_s and X'_s hit, and let $S_h = \infty$ if X_s and X'_s never hit. Given that $X_0 = v_i$ and $X'_0 = v_j$, the probability that S_h is at most the parameter S from above is exactly equal to the probability to that X_S and X'_S both equal a vertex v_k, for any vertex v_k. Thus, we have

$$\begin{aligned}
\Pr\{S_h \leq S \mid X_0 = v_i, X'_0 = v_j\} &= \sum_{k=1}^{n} \Pr\{X_S = v_k \mid X_0 = v_i\} \Pr\{X'_S = v_k \mid X'_0 = v_j\}\\
&> \sum_{k=1}^{n} (\delta\varepsilon) \Pr\{X'_S = v_k \mid X'_0 = v_j\} \qquad \text{by (4)},\\
&= \delta\varepsilon.
\end{aligned}$$

This result is also time translation invariant, in that

$$\Pr\{S_h \leq S + s \mid X_s = v_i, X_s' = v_j\} > \delta\epsilon$$

for any $s > 0$.

We now consider what happens in q consecutive time intervals each of length S. The probability that X_s and X_s' do not hit in any of the intervals is $(1 - \delta\varepsilon)^q$, and so

$$\Pr\{S_c \leq Sq \mid X_0 = v_i, X_0' = v_j\} \geq 1 - (1 - \delta\varepsilon)^q \to 1 \text{ as } q \to \infty.$$

Thus, with probability 1, the random walks X_s and X_s' hit and so coalesce. Applying this result to all n random walks in the system $\{(X_s, D_s)\}$, we have that with probability 1, the system $\{(X_s, D_s)\}$ coalesces into a single random walk. □

Note that Theorem 3 trivially implies that $\{D_s\}_{s\geq 0}$ also coalesces, *i.e.*, that $D_s(v_i) = D_s(v_j)$ for all $s \geq S_c$. This is true because $D_s(v_i) = D_s(v_j)$ whenever $X_s(v_i) = X_s(v_j)$.

Since the system coalesces, we will now drop the vertex name and only write X_s, D_s, and $\widehat{X}_m$. We wish to calculate $\lim_{s\to\infty} \Pr\{Z_0(X_s) = 1 \mid Z_0\}$; what we will actually calculate is the more detailed $\lim_{s\to\infty} \Pr\{X_s \in K(v_i) \mid Z_0\}$ for vertex v_i. Note that because of our setup it would be very difficult to calculate $\lim_{s\to\infty} \Pr\{X_s = v_i \mid Z_0\}$ as is done in the original voter model.

To analyze the limiting probability distribution of X_s we create a new semi-Markov process $\widetilde{X}_s$ that captures the movement of X_s between different trusted components. Since this only makes sense when there are different trusted components, we henceforth assume that the initial set of opinions Z_0 is not uniform.

Definition 4. The continuous-time process $\{\widetilde{X}_s\}_{s\geq 0}$ is defined on the state space of vertices $v_1, \ldots, v_n$ of G, where we assume that Z_0 is not uniform. Given that $X_s \in K(v_i)$, let $\widetilde{X}_s$ be the vertex v_j in $K(v_i)$ that X_s first visited when entering $K(v_i)$. Formally, let s' be the least time such that $X_{s''} \in K(v_i)$ for all $s' \leq s'' \leq s$. Then let $\widetilde{X}_s = X_{s'}$.

Since X_s is dependent only on its immediate future that is spent in the current trusted component before leaving, $\widetilde{X}_s$ is a semi-Markov process. The transition probability p_{ik} from v_i to v_k is the probability that X_s exits $K(v_i)$ to v_k given that $X_0 = v_i$. Note that $p_{ik} = 0$ if $v_k \in K(v_i)$ or if v_k is not a neighbor to any vertex in $K(v_i)$. The holding time H_i of $\widetilde{X}_s$ at v_i is the time X_s takes to leave $K(v_i)$, given that $X_0 = v_i$.

Lemma 5. *The process $\{\widetilde{X}_s\}_{s\geq 0}$ is dual to $\{X_s\}_{s\geq 0}$ in the sense that*

$$\Pr\{X_s \in K(v_i) \mid Z_0\} = \Pr\{\widetilde{X}_s \in K(v_i) \mid Z_0\}, \text{ and so}$$
$$\lim_{s\to\infty} \Pr\{X_s \in K(v_i) \mid Z_0\} = \lim_{s\to\infty} \Pr\{\widetilde{X}_s \in K(v_i) \mid Z_0\},$$

if at least one of the limits exists.

Proof. Note that if $X_s \in K(v_i)$, then $\widetilde{X}_s = v_j$ for some $v_j \in K(v_i)$. Then

$$\begin{aligned}\Pr\{X_s \in K(v_i) \mid Z_0\} &= \Pr \bigcup_{v_j \in K(v_i)} \{X_s \in K(v_i) \text{ and } \widetilde{X}_s = v_j \mid Z_0\} \\ &= \Pr \bigcup_{v_j \in K(v_i)} \{\widetilde{X}_s = v_j \mid Z_0\} \qquad \text{since } \widetilde{X}_s = v_j \text{ implies } X_s \in K(v_j), \\ &= \Pr\{\widetilde{X}_s \in K(v_i) \mid Z_0\}.\end{aligned}$$

The last statement of the proposition follows by taking the limit as $s \to \infty$ of both sides of the above equation. ☐

If the expected holding time $\mathrm{E}[H_i]$ is finite for each i, then the theory of semi-Markov processes enables us to calculate the limiting probability distribution of $\widetilde{X}_s$ (see [12]). Let $\{\widehat{\widetilde{X}}_r\}_{r=0}^{\infty}$ be the embedded Markov chain for $\widetilde{X}_s$. Then

$$\lim_{s\to\infty} \Pr\{\widetilde{X}_s \in K(v_i) \mid Z_0\} = \frac{\displaystyle\sum_{v_j \in K(v_i)} \left(\mathrm{E}[H_j] \lim_{r\to\infty} \Pr\{\widehat{\widetilde{X}}_r = v_j\} \right)}{\displaystyle\sum_{k=1}^{n} \mathrm{E}[H_k]}. \tag{5}$$

Summing over all trusted components with initial opinion 1, we have

$$\lim_{s\to\infty} \Pr\{Z_0(X_s) = 1 \mid Z_0\} = \sum_{K(v_i):Z_0(v_i)=1} \lim_{s\to\infty} \Pr\{\widetilde{X}_s \in K(v_i) \mid Z_0\},$$

which is our main goal.

Note that $\{\widehat{\widetilde{X}}_r\}$ is defined by the transition probabilities p_{ik}, which in turn can be determined from the Markov chain $\{\widehat{X}_m\}$. Since $\{\widehat{X}_m\}$ ignores waiting times, $\{\widehat{X}_m\}$ and hence $\{\widehat{\widetilde{X}}_r\}$ is independent of the choice of the waiting function f. To explicitly calculate p_{ik}, let A denote the transition matrix of $\widehat{X}_m$ on G, and let A_H denote the restriction of A to a subgraph H of G. We denote the complement in G of a subgraph H by $\overline{H}$. Let $A_{H\times\overline{H}}$ denote the restriction of A to transitions from H to $\overline{H}$. Given that $X_0 = v_i$, we wish to compute the probability that X_s exits $K(v_i)$ to vertex $v_k \in \overline{K(v_i)}$. This probability is exactly the probability that, given that $\widehat{X}_0 = v_i$, $\widehat{X}_m$ leaves $K(v_i)$ by going to vertex v_k. Using Markov chain theory (see, for example, chapter 5 of [13]), the matrix

$$B = (I - A_{K(v_i)})^{-1} A_{K(v_i)\times\overline{K(v_i)}}$$

gives the exit probabilities of $\widehat{X}_m$ from $K(v_i)$. That is, if $B = [b_{jk}]$, then

$$b_{jk} = \Pr\{\widehat{X}_m \text{ exits } K(v_i) \text{ to } v_k \in \overline{K(v_i)} \mid \widehat{X}_0 = v_j \in K(v_i)\}.$$

We also need to calculate the expected holding time $\mathrm{E}[H_i]$, which is just the expected time s when X_s exits $K(v_i)$, given that $X_0 = v_i$. Recall that the ik^{th} entry of $A^d_{K(v_i)} A_{K(v_i)\times\overline{K(v_i)}}$

gives the probability that $\widehat{X}_m$ exits $K(v_i)$ to $v_k \in \overline{K(v_i)}$ in exactly $d+1$ steps, given that $\widehat{X}_0 = v_i$. Thus,

$$\begin{aligned}
&\mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \text{ to } v_k \mid X_0 = v_i \text{ and } X_s \text{ exits to } v_k] \\
&= \sum_{d=0}^{\infty} \Big(\mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \mid \widehat{X}_m \text{ exits to } v_k \text{ in } d+1 \text{ steps and } \widehat{X}_0 = v_i] \\
&\qquad\qquad \Pr\{\widehat{X}_m \text{ exits to } v_k \text{ in } d+1 \text{ steps} \mid \widehat{X}_0 = v_i\} \Big)
\end{aligned}$$

The value of D_s determines the distribution of the holding time of X_s at a vertex. If the vertex is in $K(v_i)$, then D_s is the number of steps that will be taken in $K(v_i)$ before exiting. The holding time of X_s at v_i is thus independent of v_i, v_k, and the sequence of steps taken within $K(v_i)$ from v_i to v_k. Thus,

$$\begin{aligned}
&\mathrm{E}\ \mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \mid \widehat{X}_m \text{ exits in } d+1 \text{ steps to } v_k \text{ and } \widehat{X}_0 = v_i] \\
&\quad = \mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \mid \widehat{X}_m \text{ exits in } d+1 \text{ steps}] \\
&\quad = \sum_{q=0}^{d} \mathrm{E}[\text{holding time for } X_s \text{ at the current vertex} \mid \widehat{X}_m \text{ exits } K(v_i) \text{ in } q+1 \text{ steps } (i.e., D_s = q)] \\
&\quad = f(0) + (f(1) - f(0)) + (f(2) - f(1)) + \ldots + (f(d) - f(d-1)) \\
&\quad = f(d).
\end{aligned}$$

Thus, the expected time until X_s exits $K(v_i)$, given that $X_0 = v_i$ and that $X_s = v_k$, is

$$\begin{aligned}
&\mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \mid \widehat{X}_m \text{ exits to } v_k \text{ and } \widehat{X}_0 = v_i] \\
&= \sum_{d=0}^{\infty} \mathrm{E}[\text{time for } X_s \text{ to exit } K(v_i) \mid \widehat{X}_m \text{ exits in } d+1 \text{ steps to } v_k \text{ and } \widehat{X}_0 = v_i] \\
&\qquad\qquad \Pr[X_s \text{ exits } K(v_i) \text{ in } d+1 \text{ steps} \mid \widehat{X}_m \text{ exits to } v_k \text{ and } \widehat{X}_0 = v_i] \\
&= \sum_{d=0}^{\infty} f(d) \frac{\Pr[X_s \text{ exits } K(v_i) \text{ in } d+1 \text{ steps to } v_k \mid \widehat{X}_0 = v_i]}{\Pr[\widehat{X}_m \text{ exits to } v_k \mid \widehat{X}_0 = v_i]} \\
&= \frac{1}{b_{ik}} \left(ik^{\text{th}} \text{ entry of } \sum_{d=0}^{\infty} f(d) A^d_{K(v_i)} A_{K(v_i) \times \overline{K(v_i)}} \right).
\end{aligned}$$

If the series is convergent, then the expected time is finite, and formula (5) for the limiting distribution of $\widetilde{X}_s$ holds. In the next section, we will consider specific choices for the waiting function f and examine how the process $\{(X_s, D_s)\}$ behaves with this choice.

4 Specific Choices for the Waiting Function f

We first need a result about power series of matrices.

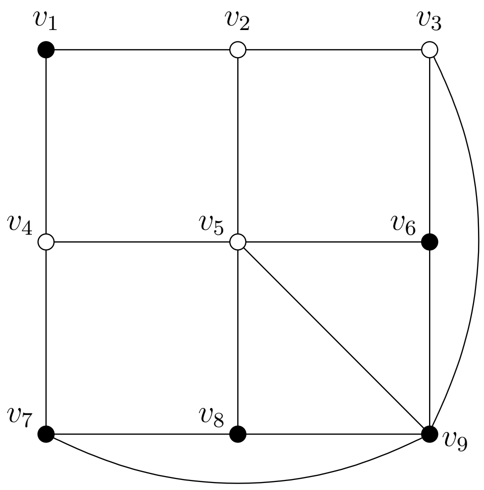

Figure 2: Initial opinion Z_0 on G.

Lemma 6. *Let $\sum_{d=0}^{\infty} f(d)x^d$ be a power series in the complex variable x with radius of convergence R. Then $\sum_{d=0}^{\infty} f(d)M^d$ is a convergent power series in the matrix M if all of the eigenvalues of M have modulus less than R. Furthermore, if $r(x)$ is a rational function in x such that $r(x) = \sum_{d=0}^{\infty} f(d)x^d$ for all $|x| < R$, then $r(M) = \sum_{d=0}^{\infty} f(d)M^d$ if all of the eigenvalues of M have modulus less than R.*

This result follows from the fact that a power series formula r clearly holds for diagonalizable matrices, and the diagonalizable matrices are dense in the space of all matrices.

Definition 7. Let G^1 denote the subgraph induced by vertices v_i whose initial opinions $Z_0(v_i)$ are 1, and similarly define G^0. Let A denote the transition matrix of $\widehat{X}_m$ on G, and let A_H denote the restriction of A to a subgraph H of G. For each connected component K of G^1 or G^0, let $e(K)$ denote the largest modulus of eigenvalues of A_K. Define $\mu(G, Z_0)$ to be the maximum $e(K)$ taken over all components K of G^1 and G^0. If every component that attains the maximum $\mu(G, Z_0)$ has the same initial opinion, then define $\mu'(G, Z_0)$ to be the maximum $e(K)$ taken over all components K that have the opposite opinion.

Lemma 8. *For any non-uniform set of initial opinions Z_0, $\mu(G, Z_0) < 1$.*

Proof. This statement follows immediately from the fact that $\widehat{X}_m$ will leave each trusted component with probability 1. □

We now consider several specific waiting functions, and show how the voter model with confidence levels behaves with these choices. To make the examples concrete, we calculate $\lim_{s\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\}$ for the graph G shown in Figure 2 and for various parameters of the waiting functions. In the figure, vertices with initial opinion $Z_0(v_i) = 1$ are marked with black dots, while vertices with initial opinion 0 are marked with hollow dots. We will refer to the three trusted components as $K(v_1)$, $K(v_2)$, and $K(v_6)$. The transition matrix A

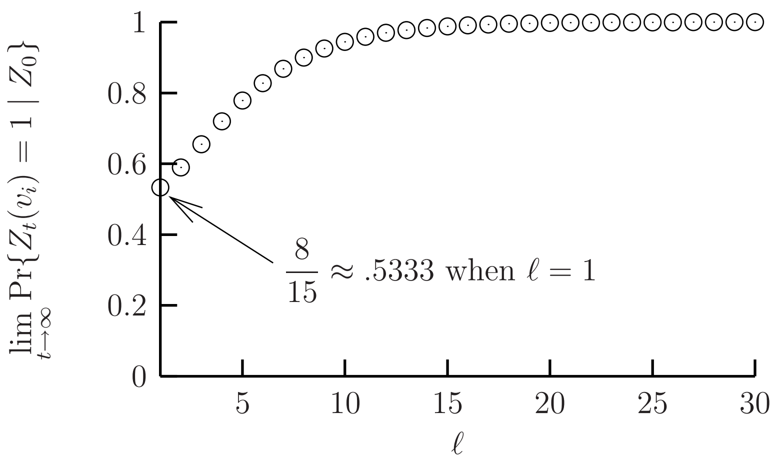

Figure 3: $\Pr\{Z_t$ enters the uniform 1 opinion state $\mid Z_0\}$ as a function of ℓ.

of $\widehat{X}_m$ on G is

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1/5 & 0 & 1/5 & 0 & 1/5 & 0 & 1/5 & 1/5 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1/5 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \end{bmatrix}.$$

It is interesting to note the behavior of the classic voter model, where $\lim_{t\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\} = 8/15 \approx .5333$.

Lemma 6 enables us to calculate the limiting probability for any polynomial waiting function. We present one specific class of polynomials here.

Example 9. Let f be the polynomial $f(d) = \binom{d+\ell}{\ell}$, where ℓ is a positive integer. Note that

$$\sum_{d=0}^{\infty} \binom{d+\ell}{\ell} x^d = \frac{1}{(1-x)^{\ell+1}}$$

by integrating the power series ℓ times. This series is convergent for $|x| < 1$, and so $\sum_{d=0}^{\infty} \binom{d+\ell}{\ell} A_{K(v_i)}^d = (I - A_{K(v_i)})^{\ell+1}$ holds for any component $K(v_i)$. Figure 3 shows the limiting distribution for $\ell = 1, \ldots, 30$ for our example graph G. Note that when $\ell = 1$, $f(d) = d + 1$, and the voter model with confidence levels reduces to the special case of the classic voter model. Thus, at $\ell = 1$, $\lim_{s\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\} = 8/15$.

Example 10. Let f be the exponential function $f(d) = \lambda^d$, where $\lambda > 1$. If $\lambda < 1/\mu(G, Z_0)$, then all of the eigenvalues of $\lambda A_{K(v_i)}$ are less than 1 in modulus for any component $K(v_i)$.

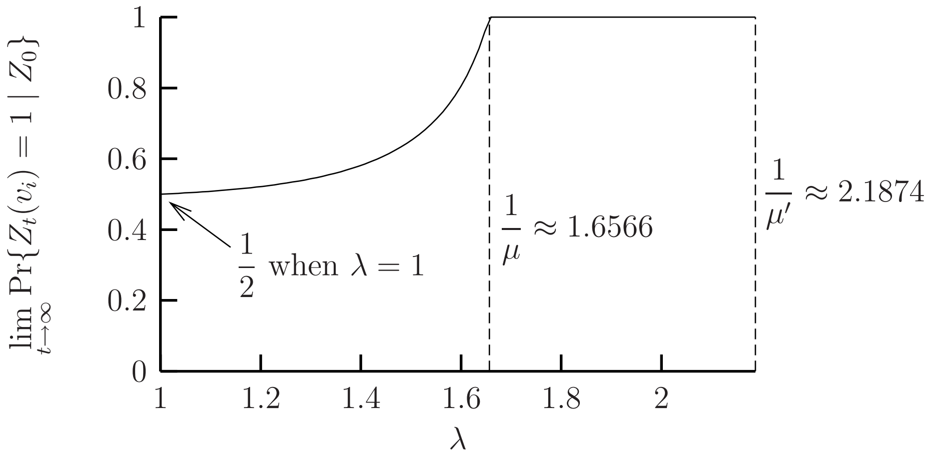

Figure 4: $\Pr\{Z_t$ enters the uniform 1 opinion state $\mid Z_0\}$ as a function of λ.

Thus, $\sum_{d=0}^{\infty} \lambda^d A_{K(v_i)}^d$ is a convergent power series, and, using the formula for a convergent geometric series, is equal to $(I - \lambda A_{K(v_i)})^{-1} A_{K(v_i)\times\overline{K(v_i)}}$. If $\mu'(G, Z_0) \neq \mu(G, Z_0)$, then for $1/\mu(G, Z_0) \leq \lambda < 1/\mu'(G, Z_0)$ the limiting probability is 0 or 1, depending on the initial opinion of the components K that attain the maximum $e(K)$. This is because the random walk X_s essentially spends all of its time in these components. For larger values of λ, we conjecture that the probability actually remains constant:

Conjecture 11. *For $\lambda > 1/\mu(G, Z_0)$, the limiting probability is* 0 *or* 1*, depending on the initial opinion of the components K that attain the maximum $e(K)$.*

It is an open question to determine what occurs when $\mu'(G, Z_0) = \mu(G, Z_0)$, though we make the following

Conjecture 12. *If $\mu'(G, Z_0) = \mu(G, Z_0)$, then for $\lambda > 1/\mu(G, Z_0)$, the limiting probability is constant and equals* $\lim_{\lambda\to 1/\mu(G,Z_0)} \lim_{t\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\}$.

The existence of the critical threshold $\lambda = 1/\mu(G, Z_0)$ where the behavior of the system changes is quite interesting. Viewed from a physicist's perspective, we say that a *phase transition* occurs at $\lambda = 1/\mu(G, Z_0)$. For $\mu'(G, Z_0) \neq \mu(G, Z_0)$, the limiting probability is continuous as a function of λ, so we have a first order phase transition (*e.g.*, see [5]).

When $\lambda = 1$, $f(d)$ is a constant function. Thus, the expected time for X_s to leave a trusted component is 1, regardless of the number of steps taken. The limiting probability reduces to the stationary distribution of the embedded Markov chain $\widehat{\widehat{X}}_r$ of the semi-Markov process $\widetilde{X}_s$. Since there are two opinions, the underlying graph of $\widehat{\widehat{X}}_r$ is always a bipartite graph, and hence $\lim_{t\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\} = 1/2$.

For our example graph G, the largest eigenvalue in modulus of $A_{K(v_1)}$, $A_{K(v_2)}$, and $A_{K(v_6)}$ is $\mu(G, Z_0) \approx .6037$ from $A_{K(v_6)}$, and $\mu'(G, Z_0) \approx .4572$ from $A_{K(v_2)}$. We can thus calculate $\lim_{s\to\infty} \Pr\{Z_t(v_i) = 1 \mid Z_0\}$ for $1 < \lambda < 1/\mu(G, Z_0) \approx 1.6566$ using the geometric series formula. For $1/\mu(G, Z_0) \leq \lambda < 1/\mu'(G, Z_0) \approx 2.1874$, the limiting probability is 1. Figure 4 shows the limiting probability as a function of λ.

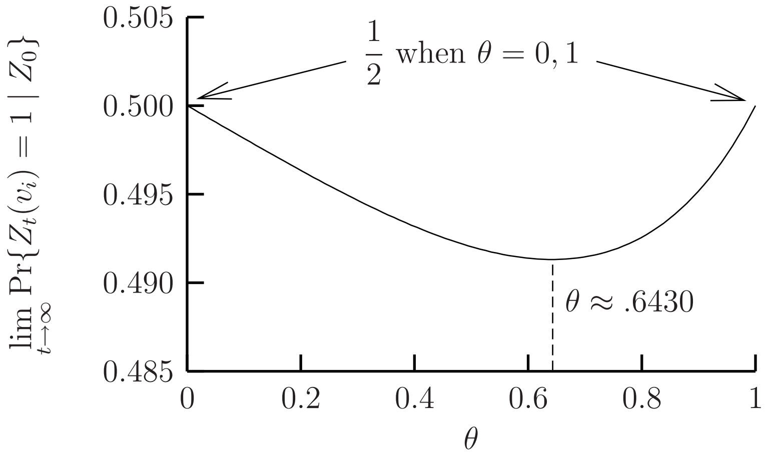

Figure 5: $\Pr\{Z_t$ enters the uniform 1 opinion state $\mid Z_0\}$ as a function of θ.

We can also create a waiting function that is increasing with d yet bounded.

Example 13. Let f be the function $f(d) = 2 - \theta^d$, for $0 < \theta < 1$. Here 2 is an arbitrary constant chosen for concreteness. Since

$$\sum_{d=0}^{\infty}(2-\theta^d)x^d = \frac{2}{1-x} - \frac{1}{1-\theta x}$$

for $|x| < 1$, $\sum_{d=0}^{\infty}(2-\theta^d)A^d_{K(v_i)}$ is convergent for any component $K(v_i)$. The limiting probability as a function of θ is shown in Figure 5. When θ is 0 or 1, $f(d)$ is constant, and, as seen above, the limiting probability is 1/2. Interestingly, the limiting probability is less than 1/2 for $0 < \theta < 1$, attaining a minimum at $\theta \approx .6430$. However, because the waiting function is bounded, high confidence levels do not significantly affect the limiting probability, and the limiting probability varies little for θ in the range $0 < \theta < 1$.

5 Conclusion

As with the classic voter model, the results we have presented on the voter model with confidence levels hold in slightly more general situations, particularly where there are multiple opinions and weights on edges, digraphs, and loops when determining a neighbor's influence. Also trusted components can be any connected induced subgraphs as long as the initial opinions are uniform. Unfortunately, if certain features of the model are weakened, then the type of arguments we have made no longer work. The independence of the confidence level and the opinion held (0 or 1) is absolutely crucial for the dual to be tractable. Similarly, the 1 added to the confidence level when adopting a trusted associate's opinion must remain constant for all vertices and trusted associates within a trusted component so that the expected time for X_s to leave a trusted component can be calculated.

As mentioned in the introduction, our main motivation was to modify the voter model to produce a system with nontrivial dependence on the structure of G but which still remained tractable. It would be very interesting to construct other modifications of the voter model that also have this property.

Acknowledgements

The author would like to thank Fred Roberts, Eva Curry, Nick Weininger, and Pieter Blue for helpful discussions and encouragement; Yehuda Vardi for the suggestion of generalizing the waiting function; and Michael Weingart for pointing out the simple proof of Lemma 6.

References

[1] D. J. Aldous and J. A. Fill, *Reversible Markov Chains and Random Walks on Graphs*, monograph in preparation, available from `http://www.stat.berkeley.edu/users/aldous`.

[2] P. Clifford and A. Sudbury, A Model for Spatial Conflict, *Biometrika*, 60 (1973), pp. 581-588.

[3] P. Donnelly and D. Welsh, Finite Particle Systems and Infection Models, *Math. Proc. Comb. Phil. Soc.*, 94 (1983), pp. 167-182.

[4] R. Durrett, *Lecture Notes on Particle Systems and Percolation*, Wadsworth and Brooks/Cole, Pacific Grove, California, 1988.

[5] C. B. P. Finn, *Thermal Physics*, second edition, Chapman and Hall, London, 1993, pp. 176-177.

[6] D. Griffeath, *Additive and Cancellative Interacting Particle Systems*, Lecture Notes in Mathematics, vol. 724, Springer-Verlag, Berlin, 1979.

[7] F. Hoffman, presentation at the Twenty-Sixth Southeastern International Conference on Combinatorics, Graph Theory, and Computing. Held at Florida Atlantic University in Boca Raton, Florida, March 6-10, 1995.

[8] R. Holley and T. M. Liggett, Ergodic Theorems for Weakly Interacting Systems and the Voter Model, *Ann. Probab.*, 3 (1975), no. 4, pp. 643-663.

[9] T. M. Liggett, *Interacting Particle Systems*, Grundlehren der mathematicshen Wissenschaften, no. 276, Springer-Verlag, New York, 1985.

[10] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Grundlehren der mathematicshen Wissenschaften, no. 324, Springer-Verlag, Berlin, 1999.

[11] L. Lovász, “Random Walks on Graphs: A Survey,” in: *Combinatorics, Paul Erdős is Eighty*, vol. 2, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest, 1996.

[12] R. G. Gallager, *Discrete Stochastic Processes*, Kluwer Academic Publishers, Boston, 1996, pp. 174-178.

[13] F. S. Roberts, *Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems*, Prentice-Hall, Upper Saddle River, New Jersey, 1976, pp. 258-363.

[14] F. S. Roberts, unpublished notes, 1999.