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Perfect interval filament graphs

by

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ABSTRACT

Let I be a family of intervals on a line L. In the plane, above L, construct to each interval $i \in I$ a curve f_i connecting its two endpoints, such that if two intervals are disjoint, their curves do not intersect; $FI=\{ f_i \mid i \in I\}$ is a *family of interval filaments* and its intersection graph is an *interval filament graph*. The interval filament graphs contain the polygon-circle graphs, the circle graphs, the chordal graphs and the cocomparability graphs. Similar families of intersection graphs of filaments can be defined using families of circular-arcs of a circle and families of subtrees of a tree or of a cactus.

In the present paper we describe polynomial time algorithms to find holes and antiholes of a given parity in interval filament graphs, circular-arc filament graphs and subtree filament graphs on a tree and on a cactus; assuming that Berge's SPGC is true, these algorithms give polynomial time algorithms to test for perfectness. In addition we describe polynomial time algorithms to find minimum coverings by cliques for their subfamilies of perfect graphs.

1 Introduction

In the present paper we consider only finite graphs $G(V,E)$ with no parallel edges and no self-loops, where V is the set of vertices, E the set of edges and $|V|=n$, $|E|=m$. For $U \subseteq V$, $G(U)$ denotes the vertex subgraph defined by U. Two vertices connected by an edge $u \text{—} v$ are called *adjacent*; a directed edge from u to v is denoted $u \rightarrow v$. We also denote $N(v)=\{u \mid u \text{—} v \in E\}$ and $N[v]=N(v) \cup \{v\}$. The complement of G is denoted coG. By a path p we always mean a simple path; its vertex set is also denoted by p. A *chord* of a path or a cycle is an edge connecting two nonconsecutive vertices. A *hole* is a chordless cycle with four or more vertices. An *antihole* is the complement of a hole. A path p is an *induced path* if it has no chords. A subset of V is a *clique* (an *independent set*) if every two of its vertices are adjacent (are not adjacent, respectively). The size of a maximum independent set of G is denoted $\alpha(G)$ and the size of a minimum covering by cliques of G is denoted $\gamma(G)$.

An oriented graph $G(V,E)$ is called *transitive* if it is acyclic and for every three vertices $u,v,w \in V$, $u \rightarrow v, v \rightarrow w \in E$ implies $u \rightarrow w \in E$ [GRU]. The underlying undirected graph of a transitive graph is called a *comparability graph*.

Two subsets of a set *intersect* if they have a non-empty intersection. A family of sets is called *Helly* if every subfamily of mutually intersecting sets has a non-empty intersection. A graph G is an *intersection graph* of a family S of distinct subsets of a set if there is a one-to-one correspondence between the vertices of G and the subsets in S such that two subsets intersect if and only if their corresponding vertices are adjacent; S is called a *representation* of G. Intersection graphs are of great interest in various domains such as computer science, genetics, archaeology and ecology[RO].

Consider a line L in a plane P and a family I of intervals on L; w.l.o.g. we assume that no two intervals have a common endpoint.. Two distinct intervals b,d on L *intersect properly* if $b \cap d \neq \phi$, $b \not\subset d$ and $d \not\subset b$. A sequence s of intervals $a_1,\ldots,a_k$ is a *sequence of properly intersecting intervals* if every two consecutive intervals a_{i-1}, a_i intersect properly and every two non-consecutive intervals are disjoint; we denote $previous_s(a_i)=a_{i-1}$. The parity of s is the parity of its number of intervals. As known, a family of intervals on a line has the Helly property, i.e., a set of mutually intersecting intervals has a non-empty intersection. In the plane P, above L, we connect the two endpoints of each interval $a \in I$ by a curve f_a, such that if two intervals are disjoint, their curves do not intersect. Let $FI=\{f_a \mid a \in I\}$; FI is called *a family of interval filaments* and its intersection graph is called an *interval filament graph*. Gavril[GA4] described polynomial time algorithms to find maximum weight cliques and independent sets for interval filament graphs, and proved that this family of graphs contains the circular-arc graphs[GA2], the polygon-circle graphs[JK], the trapezoid graphs[FMW], the circle graphs, the chordal graphs[GA1] and the

cactus-subtree graphs[GA3]. Gavril[GA5] described polynomial time algorithms to find maximum weight induced paths for this family of graphs.

Consider a family of arcs *AR* on a circle *CR* such that no two arcs cover *CR*. A family of arcs on a circle may not have the Helly property, since three arcs covering the circle must have an empty intersection, otherwise two of them cover *CR*. Consider a set *CL* of mutually intersecting arcs. If $\cup\{a|a\in CL\}$ does not cover *CR* then they are like a family of intervals on a line and have a non-empty intersection by the Helly property. If $\cup\{a|a\in CL\}$ covers *CR* then $\cap\{a|a\in CL\}=\phi$ otherwise *CL* contains two arcs covering *CR*. Consider a point $x\in CR$ and $X=\{a|\ a\in CL,\ x\in a\}$, for which $|X|$ is maximum. Consider a point $y\in CR-\cup\{a|a\in X\}$. For every $b\in AR-X$, *b* must contain $CR-\cup\{a|a\in X\}$, otherwise $X\cup\{b\}$ does not cover *CR* and by the Helly property their intersection is non-empty, contradicting the maximality of $|X|$; thus *b* contains *y* but not *x*. Therefore, there are two points *x,y* in *CR* such that each arc in *CL* contains either *x* or *y* but not both.

In the plane containing *CR* we connect the endpoints of every arc $a\in AR$ with a curve f_a inside the circle, such that curves of disjoint arcs do not intersect. Let $FAR=\{f_a|\ a\in AR\}$; *FAR* is called *a family of circular-arc filaments* and its intersection graph is called a *circular-arc filament graph*[GA4]. Similar families of filaments and corresponding intersection graphs[GA4] can be defined using families of subtrees of a tree or of a cactus. For example, consider a tree *T* in a plane *PL*, a family *ST* of subtrees of *T* , every two intersecting subtrees having a branch in common, and a surface perpendicular to *PL* whose intersection with *PL* is exactly *T*. On this surface, above *T*, consider a family $FT=\{f_t|\ t\in ST\}$ of curves, each f_t connecting all the endpoints of *t*, such that if two subtrees overlap, their curves intersect, if two subtrees are disjoint, their curves do not intersect, and if two subtrees are contained one in another, their curves may or may not intersect. The family *FT* is *a family of subtree filaments*.

Given the intersection graph *G(V,E)* of a family *F* of filaments, we denote the filament corresponding to a vertex *v* of *G* also by *v*.

Let *G(V,E)* be the intersection graph of a family *FI* of interval filaments based on a family of intervals $I=\{a(v)|\ v\in V\}$ on a line *L*. For every open segment *d* on *L*, let $X_d=\{v|\ v\in V,\ d\subseteq a(v)\}$. Every two non-adjacent vertices $v,u\in X_d$, have either $a(v)\subset a(u)$ or $a(u)\subset a(v)$, since $u\cap v=\phi$ and $d\subseteq a(v)\cap a(u)$. If $a(v)\subset a(u)$, we orient in $coG(X_d)$ the edge v—u as $v\rightarrow u$. It is easy to see that we obtain a transitive orientation of $coG(X_d)$, implying that $G(X_d)$ is a cocomparability graph. If the left and right endpoints of *d* are left and right endpoints of intervals in *I*, and *d* contains no endpoints of intervals in *I*, then $G(X_d)$ is a maximal cocomparability graph among those defined by the segments of *L*; we call *d* a *maxcocomp segment* of *L*.

Similarly, let *G(V,E)* be the intersection graph of a family *FAR* of circular-arc filaments based on a family of arcs *AR* on a circle *CR*. For two points *x,y* on *CR*, let $X_{x,y}$ be the subset of vertices *v* in *V* such that the filament *v* has its endpoints on opposite sides of the diagonal *x,y*. For every two non-adjacent vertices $v,u\in X_{x,y}$, we orient in $coG(X_{x,y})$ the edge v—u as $v\rightarrow u$ whenever filament *v* is between filament *u* and the point *x* of *CR*. It is easy to see that we obtain a transitive

orientation of $coG(X_{x,y})$, implying that $G(X_{x,y})$ is a cocomparability graph. When x,y are taken in open arcs d_1,d_2 of CR whose left and right endpoints are left and right endpoints of arcs in AR and d_1,d_2 do not contain endpoints of arcs in AR, then $G(X_{x,y})$ is a maximal cocomparability graph among those defined by the diagonals of CR; we call d_1,d_2 *maxcocomp arcs* of CR.

A graph G is called *perfect* if in every subgraph H of G $\alpha(H)=\gamma(H)$. Berge's Strong Perfect Graph Conjecture(SPGC) states that a graph is perfect iff it has no odd holes and no odd antiholes. The yet unpublished manuscript [CRST] by Chudnovsky, Robertson, Seymour and Thomas contains a proof of SPGC making it a Theorem. In the present paper we describe polynomial time algorithms to find holes and antiholes of a given parity in interval filament graphs, circular-arc filament graphs and subtree filament graphs on a tree and on a cactus. These algorithms can be adjusted to find odd holes or antiholes through a given vertex, problems which are NP-complete for general graphs[BI]. The algorithm described in [GA5] for maximum weight induced paths can also be used to find holes of a given parity in interval filament graphs. Assuming that SPGC is true, these algorithms give polynomial time algorithms to test for perfectness interval filament graphs, circular-arc filament graphs and subtree filament graphs on a tree and on a cactus. In addition we describe polynomial time algorithms to find minimum coverings by cliques for their subfamilies of perfect graphs. Polynomial time algorithms to test for perfectness other families of graphs are described in [GA3] for the Helly cactus subtree graphs and in [GLW] for the normal fraternally oriented or perfectly oriented graphs.

2 Holes of Given Parity in Filament Graphs

Let $G(V,E)$ be the intersection graph of a family FI of interval filaments based on a family of intervals $I=\{a(v)|\ v\in V\}$ on a line L. Clearly, if $u\cap v=\phi$ and $a(u)\cap a(v)\neq\phi$, then either $a(u)\subset a(v)$ or $a(v)\subset a(u)$. Also, if two intervals $a(u),a(v)$ intersect properly then $u\cap v\neq\phi$. For an interval a we denote its left endpoint by $l(a)$ and its right endpoint by $r(a)$.

Consider an induced path $p=v_1—..—v_k$ in G. For $1\leq r\leq t\leq k$ we denote $A_p(r,t)=a(v_r)\cup a(v_{r+1})\cup..\cup a(v_t)$; $A_p(r,t)$ is an interval on L, since $v_i—v_{i+1}$ implies $a(v_i)\cap a(v_{i+1})\neq\phi$.

Lemma 1. Let $p=v_1—..—v_k$ be an induced path in G and consider some i, $1\leq i\leq k-2$. If $a(v_i)\subset a(v_k)$, then $A_p(1,k-2)\subset a(v_k)$. If $a(v_i)\cap a(v_k)=\phi$ and $a(v_k)\not\subset a(v_j)$ for every j, $1\leq j\leq k-2$, then $A_p(1,k-2)\cap a(v_k)=\phi$.

Proof. $A_p(1,k-2)$ is a connected interval, and no filament v_i, $1\leq i\leq k-2$ intersects v_k. Thus either $A_p(1,k-2)\subset a(v_k)$ or $A_p(1,k-2)\cap a(v_k)=\phi$ or $a(v_k)\subset a(v_j)$ for some $1\leq j\leq k-2$. Consider some i, $1\leq i\leq k-2$. If $a(v_i)\subset a(v_k)$, then $A_p(1,k-2)\subset a(v_k)$. If $a(v_i)\cap a(v_k)=\phi$ then $A_p(1,k-2)\cap a(v_k)=\phi$ or

$a(v_k) \subset a(v_j)$ for some $1 \le j \le k-2$. Thus, if $a(v_i) \cap a(v_k) = \phi$ and $a(v_k) \not\subset a(v_j)$ for every $1 \le j \le k-2$, then $A_p(1,k-2) \cap a(v_k) = \phi$.

Lemma 2. Any hole $h = v_1 — v_2 — \ldots — v_k — v_1$ of G has two non-adjacent vertices v_i, v_j such that $a(v_i) \subset a(v_j)$.

Proof. If $a(v_1) \subset a(v_j)$ for some $3 \le j \le k-1$, the Lemma is true. Assume that $a(v_1) \not\subset a(v_j)$ for every $3 \le j \le k-1$. The filaments v_1, v_{k-1} do not intersect, thus $a(v_{k-1}) \subset a(v_1)$ or $a(v_1) \cap a(v_{k-1}) = \phi$. Assume that $a(v_1) \cap a(v_{k-1}) = \phi$, by Lemma 1, $A_h(3,k-1) \cap a(v_1) = \phi$. Let d be the segment separating the intervals $a(v_1)$ and $A_h(3,k-1)$. The intervals v_2, v_k must both contain d since both intersect $a(v_1)$ and $A_h(3,k-1)$. Hence, $d \subseteq a(v_2) \cap a(v_k) \neq \phi$, implying that $a(v_2) \subset a(v_k)$ or $a(v_k) \subset a(v_2)$, since $v_2 \cap v_k = \phi$.

Lemma 3. Consider a hole $h = v_1 — v_2 — \ldots — v_k — v_1$ of G with $k \ge 5$. There are two adjacent vertices in h which we denote $v_1 — v_2$ such that:

a) $a(v_1)$ is maximal in $\{a(v_i) \mid 1 \le i \le k\}$ and $A_h(3,k-1) \subset a(v_1)$.
b) $a(v_2) \not\subset a(v_k)$, and $A_h(3,k-2) \cap a(v_k) = \phi$.
c) There are no two intervals $a(v_s), a(v_t)$, $3 \le s,t \le k-1$, such that $a(v_s) \subset a(v_t)$, except for $s=3$ and $t=4$.
d) If $a(v_2), a(v_3)$ are properly intersecting intervals or $a(v_2) \subset a(v_3)$, then $a(v_3), \ldots, a(v_{k-1})$ is a sequence of properly intersecting intervals; if $a(v_3) \subset a(v_2)$, then $a(v_4), \ldots, a(v_{k-1})$ is a sequence of properly intersecting intervals and $a(v_4) \not\subset a(v_3)$.

Proof. a) By Lemma 2, h has two non-adjacent vertices v_i, v_j such that $a(v_i) \subset a(v_j)$, thus, by Lemma 1, in the path $v_{j+2} — v_{j+3} — \ldots — v_{j-2} — v_{j-1} — v_j$ we have $A_h(j+2,j-2) \subset a(v_j)$. If $a(v_j)$ is maximal in $\{a(v_i) \mid 1 \le i \le k\}$ we rename v_j as v_1. Otherwise, let $a(v_s)$ be a maximal interval in $\{a(v_i) \mid 1 \le i \le k\}$ containing $a(v_j)$ and rename v_s as v_1. If v_j does not intersect v_1, then, by Lemma 1, in the path $v_3 — v_4 — \ldots — v_{k-1} — v_k — v_1$ we obtain $A_h(3,k-1) \subset a(v_1)$. If v_j intersects v_1, then $j=2$ or $j=k$. Since $A_h(j+2,j-2) \subset a(v_j)$, it follows that $a(v_4) \subset a(v_2) \subset a(v_1)$ or $a(v_{k-2}) \subset a(v_k) \subset a(v_1)$, correspondingly. Thus, by Lemma 1 in the path $v_3 — v_4 — \ldots — v_{k-1} — v_k — v_1$ we obtain $A_h(3,k-1) \subset a(v_1)$.

b) If $a(v_2) \subset a(v_k)$, then we rename v_k as v_2 and rename the other vertices correspondingly, obtaining that $a(v_2) \not\subset a(v_k)$. If $A_h(3,k-2) \cap a(v_k) \neq \phi$, then some $a(v_j)$, $3 \le j \le k-2$, intersects $a(v_k)$, and since $v_k \cap v_j = \phi$, it follows that $a(v_j) \subset a(v_k)$ or $a(v_k) \subset a(v_j)$. If $a(v_j) \subset a(v_k)$, then by Lemma 1 in the path $v_2 — v_3 — \ldots — v_{k-2} — v_{k-1} — v_k$ we obtain $A_h(2,k-2) \subset a(v_k)$, contradicting the fact that $a(v_2) \not\subset a(v_k)$. If $a(v_k) \subset a(v_j)$, then by Lemma 1 on the path $v_j — v_{j+1} — \ldots — v_k — v_1$ we obtain $A_h(j+2,1) \subset a(v_j)$, contradicting the maximality of $a(v_1)$. Therefore $A_h(3,k-2) \cap a(v_k) = \phi$.

c) Assume that there are two intervals $a(v_s), a(v_t)$, $3 \le s,t \le k-1$, such that $a(v_s) \subset a(v_t)$. If $s > t+1$ or $s < t-1$, then by Lemma 1 in the paths $v_t — v_{t+1} — v_{t+2} — \ldots — v_k — v_1$ and $v_1 — \ldots — v_{t-2} — v_{t-1} — v_t$ we

obtain $A_h(t+2,1) \subset a(v_t)$ or $A_h(1,t-2) \subset a(v_t)$, respectively, contradicting the maximality of $a(v_1)$. Therefore $s=t+1$ or $s=t-1$.

Assume that $s=t+1$, i.e., $a(v_{t+1}) \subset a(v_t)$ and $3 \le t \le k-2$. Since $a(v_{t+1}) \cap a(v_{t+2}) \neq \phi$ and $a(v_{t+1}) \subset a(v_t)$, it follows that $a(v_{t+2}) \cap a(v_t) \neq \phi$. Since $v_t \cap v_{t+2} = \phi$, it follows that either $a(v_{t+2}) \subset a(v_t)$ or $a(v_t) \subset a(v_{t+2})$. If $a(v_{t+2}) \subset a(v_t)$, then, by Lemma 1 in the path v_t—v_{t+1}—v_{t+2}—... —v_k—v_1 we obtain $A_h(t+2,1) \subset a(v_t)$, contradicting the maximality of $a(v_1)$. If $a(v_t) \subset a(v_{t+2})$, then, by Lemma 1 in the path v_2—…—v_t—v_{t+1}—v_{t+2} we obtain $A_h(2,t) \subset a(v_{t+2})$ implying that $a(v_2) \subset a(v_{t+2})$. If $t+2=k$, this contradicts the condition $a(v_2) \not\subset a(v_k)$ in (b) above. If $t+2<k$, that is $t<k-2$, then $A_h(1,t) \subset a(v_{t+2})$, by Lemma 1 in the path v_1—…—v_t—v_{t+1}—v_{t+2}, implying $a(v_1) \subset a(v_{t+2})$ in contradiction to the maximality of $a(v_1)$. Therefore $s \neq t+1$.

Assume that $s=t-1$, i.e., $a(v_{t-1}) \subset a(v_t)$ and $5 \le t \le k-1$. Since $a(v_{t-2}) \cap a(v_{t-1}) \neq \phi$ and $a(v_{t-1}) \subset a(v_t)$, it follows that $a(v_t) \cap a(v_{t-2}) \neq \phi$. Since $v_t \cap v_{t-2} = \phi$, it follows that either $a(v_{t-2}) \subset a(v_t)$ or $a(v_t) \subset a(v_{t-2})$. If $a(v_{t-2}) \subset a(v_t)$, then, by Lemma 1 in the path v_1—…—v_{t-2}—v_{t-1}—v_t, we obtain $A_h(1,t-2) \subset a(v_t)$, contradicting the maximality of $a(v_1)$. If $a(v_t) \subset a(v_{t-2})$, then, by Lemma 1 in the path v_{t-2}—v_{t-1}—v_t—…—v_1—v_k we obtain $A_h(t,1) \subset a(v_{t-2})$ contradicting the maximality of $a(v_1)$.

Therefore, in h there are no two intervals $a(v_s)$, $a(v_t)$, $3 \le s,t \le k-1$, such that $a(v_s) \subset a(v_t)$, except for $s=3$ and $t=4$.

d) Assume that $a(v_2)$, $a(v_3)$ are properly intersecting intervals, or $a(v_2) \subset a(v_3)$. If $a(v_3) \subset a(v_4)$, then $a(v_2) \subset a(v_4)$, thus $a(v_1) \subset a(v_4)$, contradicting the maximality of $a(v_1)$. Therefore $a(v_3) \not\subset a(v_4)$ and by (c), $a(v_3),\ldots,a(v_{k-1})$ is a sequence of properly intersecting intervals.

Assume that $a(v_3) \subset a(v_2)$. Then, by (c), $a(v_4),\ldots,a(v_{k-1})$ is a sequence of properly intersecting intervals and $a(v_4) \not\subset a(v_3)$.

Consider a family I of intervals on a line L. Let a_1 denote the interval having $l(a_1)$ first from the left on L. For every other interval a_j we wont to find two sequences s,t of properly intersecting intervals from a_1 to a_j with the two parities, if such sequences exist, such that $r(previous_s(a_j))$, $r(previous_t(a_j))$ are leftmost; we label a_j with the parities and insert pointers to $previous_s(a_j)$ and $previous_t(a_j)$ to recover s and t. This can be done as follows:

Algorithm Interval Sequence. We start at $l(a_1)$ and go right in order of left endpoints. Assume that we labeled a_j with the parities of two sequences s,t of properly intersecting intervals from a_1 to a_j; hence $r(previous_s(a_j)), r(previous_t(a_j)) \in a_j$. From $r(previous_s(a_j))$ we go right on a_j to find the leftpoints of intervals a which properly intersect a_j and we label a with the parity of $|s|+1$, if a is not already labeled with this parity; similarly with t and $r(previous_t(a_j))$. We insert pointers *previous* from a to a_j as needed. The way we label a ensures that the right endpoints of $previous_s(a_j)$ and $previous_t(a_j)$ are the leftmost on L among all sequences with these parities of properly intersecting intervals from a_1 to a. By induction, if there is a sequence of properly intersecting intervals from a_1 to a_j of a given parity, the above algorithm will find such a

sequence. The algorithm works in time $O(|I|)$. A similar algorithm works from the right to the left of L.

The algorithm to find a hole $h=v_1—v_2—... —v_k—v_1$ of a given parity in an interval filament graph G, given its representation FI as a set of interval filaments on a line L, works as follows:

Algorithm Hole(Interval filament graph). By Lemma 3, h has three vertices $v_k—v_1—v_2$ such that $A_h(3,k-1) \subset a(v_1)$, $a(v_1)$ is maximal in $\{a(v_i) | 1 \le i \le k\}$, $a(v_2) \not\subset a(v_k)$, and $A_h(3,k-2) \cap a(v_k) = \phi$. We consider every vertex of G as candidate for v_1 of a hole h, and every vertex in $N(v_1)$ as candidate for v_2. We consider every $v \in N(v_2)-N[v_1]$, $a(v) \subset a(v_1)$ as candidate for v_3. Cases:

Case 1: $a(v_2),a(v_3)$ are properly intersecting intervals or $a(v_2) \subset a(v_3)$. Starting with $a(v_3)$, we apply Algorithm Sequence to the set of intervals $X=\{a(v) | v \notin N[v_1] \cup N[v_2], a(v) \subset a(v_1)\}$ to find sequences with both parities of properly intersecting intervals from $a(v_3)$ to every interval in X; we apply Algorithm Sequence both to the intervals at the right of $a(v_3)$ and to those at the left of $a(v_3)$. Consider a newly visited interval $a(v) \in X$. Assume that $a(v)$ is at the right of $a(v_3)$. Then for every $u \in (N(v_1) \cap N(v))-N[v_2]$, such that $a(v_1),a(v_2) \not\subset a(u)$, we take u as candidate for v_k: if $a(v_k)$ does not intersect $previous(v)$ and is at the right of the right endpoint of $previous(v)$, we obtain a sequence of properly intersecting intervals $v_3—... —previous(v)—v—v_k$, label v_k with the corresponding parity plus one and insert a pointer to v. Considering the parities on v_k, we obtain holes of both parities through v_1, if such holes exist.

Case 2: $a(v_3) \subset a(v_2)$. In this case we consider every vertex in $\{v | v \in N(v_3), v \notin N[v_1] \cup N[v_2], a(v_4) \not\subset a(v_3), a(v_4) \subset a(v_1) \cap a(v_2)\}$ as candidate for $a(v_4)$ and apply Algorithm Interval Sequence from $a(v_4)$ with the additional labeling for the candidates to v_k as described in Case 1.

Algorithm Hole(Chordal filament graph). In this case, the filaments are based on subtrees of a tree T. It is easy to see that Lemmas 1and 2 are true also for tree-filaments and any hole h has a vertex v_1 such that $A_h(3,k-1) \subset a(v_1)$. Thus, we consider every vertex of G as candidate for v_1 of a hole h, every two non-adjacent vertices in $N(v_1)$ as v_2,v_k and apply Algorithm Hole(Interval filament graph) to the interval filaments on the unique subpath connecting $a(v_2),a(v_k)$ in the subtree $a(v_1)$.

Algorithm Hole(Circular-arc filament graph). In this case, the filaments are based on arcs on a circle CR such that no two arcs cover CR. It is easy to see that Lemma 1 is true also for circular-arc filaments but Lemma 2 is not, since in h we can have either

Condition (a): h has two non-adjacent vertices v_i,v_1 such that $a(v_i) \subset a(v_1)$, or

Condition (b): for every two non-adjacent vertices v_i,v_j there exists $a(v_i) \cap a(v_j) = \phi$.

To find holes h fulfilling Condition (a), we consider arc $a(v_1)$ and the intersections of the other arcs with $a(v_1)$ as a family of intervals on a line defined by $a(v_1)$: thus Lemma 3 applies, and

we can use the above Algorithm Hole(Interval filament graph). Consider now holes h fulfilling Condition (b). For two adjacent vertices v_1,v_2 we cannot have $a(v_2) \subset a(v_1)$ since this would imply $a(v_3) \subset a(v_1)$ contradicting Condition (b). Thus, $a(v_1),a(v_2)$ are properly intersecting intervals. Therefore $h=v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$ is a sequence of properly intersecting intervals. To find such holes h, we take a vertex as v_1, we take two non adjacent vertices in $N(v_1)$ as v_2,v_k and on the arc $CR\text{-}a(v_1)$ we find sequences of properly intersecting intervals from v_2 to v_k using Algorithm Interval Sequence.

Algorithm Hole(Cactus-filament graph). In this case, the filaments are based on subtrees of a cactus. It is easy to see that Lemma 1 and Conditions (a),(b) above remain true. To find holes h, we take a vertex as v_1 and take two non adjacent vertices in $N(v_1)$ as v_2,v_k. For holes fulfilling Condition (a), we consider subtree $a(v_1)$ and the intersections of the other subtrees with the unique subpath connecting $a(v_2),a(v_k)$ in the subtree $a(v_1)$; then, we use the above Algorithm Hole(Interval filament graph). For holes fulfilling Condition (b), we cannot have $a(v_2) \subset a(v_1)$, as in the case of the circular-arcs on a circle. Therefore h is a sequence of properly intersecting intervals. To find such holes h, we apply an algorithm similar to Algorithm Sequence to the subtrees on the cactus, taking care, when we enter a circle of the cactus, to go both ways around the circle.

All the above algorithms for holes of a given parity work in time $O(|V|^6)$.

To find a hole h of a given parity through a given vertex v, we consider v as candidate for v_1,v_2,v_3,v_4 of h, and if holes of the requested parity are not found, we continue by considering v an inner vertex of h. We consider every vertex w having $w \cap v = \phi$, $a(v) \subset a(w)$ as candidate for v_1 of h. Then, in Algorithm Sequence we consider only sequences (of properly intersecting intervals) passing through v.

3 Antiholes of Given Parity in Filament Graphs

Arikati and Peled[AP] present a polynomial time algorithm to find an induced path of a given parity in perfectly orientable graphs and thus in transitive graphs. The directions of the edges in an induced path of a transitive graph are alternating, otherwise a chord appears in the path. Thus, the algorithm to find an induced path of a given parity in a transitive graph $G(V,E)$ works as follows: Let v_1 be the starting vertex. We label with path parity even every vertex v_2 adjacent to v_1 by an edge $v_1 \rightarrow v_2$ or $v_1 \leftarrow v_2$ and insert a pointer from v_2 to v_1. We continue labeling the vertices which are not labeled with both parities as follows. Assume that we found an induced path p of a given parity from v_1 to a vertex v and we have pointers from v along p to v_1, to recover p. If the last edge on p is $x \rightarrow v$, we continue on the edges $v \leftarrow w$ to vertices w who are not labeled

yet with the parity of $|p|+1$. From v we backtrack on p to find the vertex u closest to v_1 and adjacent to w. The edge between u and w is oriented $w \rightarrow u$, because $u \rightarrow w$ and $v \leftarrow w$ would imply $u \rightarrow v$. Therefore, the path q obtained from p by going from v_1 to u on p and then directly to w has the same parity as $|p|+1$, since both end with backward edges; we label w with the parity of $|p|+1$ and insert a pointer from w to u. If the last edge on p is $x \leftarrow v$, we continue on the edges $v \rightarrow w$ to vertices w who are not labeled yet with the parity of $|p|+1$. From v we backtrack on p to find the vertex u closest to v_1 and adjacent to w. The edge between u and w is oriented $u \rightarrow w$, because $u \leftarrow w$ and $v \rightarrow w$ would imply $v \rightarrow u$. Therefore, the path q obtained from p by going from v_1 to u on p and then directly to w has the same parity as $|p|+1$, since both end with forward edges; we label w with the parity of $|p|+1$ and insert a pointer from w to u. This algorithm requires $O(|V|^3)$ time for each starting vertex v_1 and $O(|V|^4)$ overall.

Let $G(V,E)$ be the intersection graph of a family FI of interval filaments based on a family of intervals $I=\{a(v)|\ v \in V\}$ on a line L. Let $h=v_1 - v_2 - \ldots - v_k - v_1$ be a hole in coG and let coh be the corresponding antihole in G. For every two non-adjacent vertices v_i, v_{i-1} of coh we have either $a(v_i) \subset a(v_{i-1})$ or $a(v_{i-1}) \subset a(v_i)$ or $a(v_i) \cap a(v_{i-1}) = \phi$. Cases:

Case 1. There is no i such that $a(v_i) \cap a(v_{i-1}) = \phi$. Then, every two intervals corresponding to vertices of coh have a non-empty intersection, therefore, by the Helly property $d = a(v_1) \cap a(v_2) \cap \ldots \cap a(v_k) \neq \phi$. Let $X_d = \{v|\ v \in V,\ d \subseteq a(v)\}$; as explained in the Introduction, $G(X_d)$ is a cocomparability graph and the antihole coh is contained in $G(X_d)$, implying that coh is an even antihole, since cocomparability graphs are perfect. Therefore, this situation happens only for even antiholes.

Case 2. There is an i such that $a(v_i) \cap a(v_{i-1}) = \phi$ and let d be the segment separating $a(v_i)$ and $a(v_{i-1})$. Consider the pair $a(v_i)$, $a(v_{i-1})$ having a maximal d among all such pairs in coh. W.l.o.g. assume that $i=1$, $i-1=k$ and $a(v_k)$ is at the right of $a(v_1)$. Then, the interval $a(v_2) \cup a(v_{k-1})$ intersects both $a(v_1)$ and $a(v_k)$, and contains d. If one of $a(v_2)$, $a(v_{k-1})$, say $a(v_2)$, does not intersect d, then, $a(v_2)$ is further apart from $a(v_1)$ than $a(v_k)$, since its left endpoint must be contained in $a(v_k)$, contradicting the maximality of d. Therefore $d' = a(v_2) \cap a(v_{k-1}) \cap d \neq \phi$, and let $X_{d'} = \{a(v)|\ v \in V,\ d' \subseteq a(v)\}$. As explained in the Introduction, $G(X_{d'})$ is a cocomparability graph and $\{v_2, v_3, \ldots, v_{k-1}\} \subseteq X_{d'}$, since every v_i, $3 \leq i \leq k-2$, intersects both $a(v_1)$ and $a(v_k)$ and thus contains d'. Therefore $v_2 - v_3 - \ldots - v_{k-1}$ is an induced path in the comparability graph $coG(X_{d'})$.

Algorithm Antihole(Interval filament graph). In both Cases 1 and 2 the problem to find antiholes of a given parity in G reduces to finding an induced path of a given parity in a comparability graph coG, and this can be done by the algorithm described at the beginning of this Section. The algorithm works as follows: To find antiholes fulfilling Case 1, for every maxcocomp segment d on L, we find the set X_d. In X_d we take a vertex as v_1 and two non-adjacent vertices v_2, v_k in $V-N[v_1]$. Then, in $coG((X_d - N[v_1]) \cup \{v_2, v_k\})$ we find an odd induced path p between v_2 and v_k; this path together with v_1 gives an even antihole in G. To find antiholes

fulfilling Case 2, for every two vertices v_1,v_k with disjoint intervals (with segment d' separating them) and every two adjacent vertices v_2,v_{k-1} such that $v_2 \in N(v_k)-N[v_1]$, $v_{k-1} \in N(v_1)-N[v_k]$ and $d=a(v_2) \cap a(v_{k-1}) \cap d' \neq \phi$, we find X_d. In $coG((X_d-N[v_1]-N[v_k]) \cup \{v_2,v_{k-1}\})$ we find an induced path p, from v_2 to v_{k-1} , of the same parity as the parity requested for the antihole; this path together with v_1, v_k gives an antihole of the requested parity in G. The algorithm works in time $O(|V|^8)$.

Algorithm Antihole(Chordal-filament graph). A similar algorithm works for chordal-filament graphs, Case 1 existing as is, and Case 2 existing on the unique path between $a(v_1)$ and $a(v_k)$.

Algorithm Antihole(Circular-arc filament graph). A similar algorithm works for circular-arc filament graphs, only now, in Case 1 there is no Helly property, but, as mentioned in the Introduction, there are two points x,y on the circle such that every arc in coH has its endpoints on both sides of the diagonal x,y. The antihole is a subgraph of the cocomparability graph defined by the set $X_{x,y}$ of vertices whose arcs have the endpoints on opposite sides of the diagonal. In Case 2, arcs corresponding to vertices intersecting both v_2,v_{k-1} may not contain d but instead contain the arc f between v_1,v_k on the other side of the circle. Again, we take in f a point y, and we have the previous situation.

Algorithm Antihole(Cactus-filament graph). For cactus-subtree filament graphs the intersections are either a segment d or two arcs containing two points x,y on one circle: if the points x,y are on two different circles, then we can replace one of the points with a point on the unique path connecting the two circles on the cactus, obtaining two points on one circle. Thus, the algorithm is similar to the one for circular arc graphs.

To find an antihole coh of a given parity through a given vertex v, we consider v as candidate for v_1,v_2,v_{k-1},v_k of coh, and if antiholes of the requested parity are not found, we continue by considering v an inner vertex of coh. Then, in the corresponding comparability graph, we consider only induced paths through v.

4 Minimum Covering by Cliques of Perfect Filament Graphs

Let $G(V,E)$ be the intersection graph of a family FI of interval filaments based on a family of intervals $I=\{a(v) \mid v \in V\}$ on a line L and assume that G is a perfect graph. In a perfect graph and any of its subgraphs, the size α of a maximum independent set is equal to the size γ of a minimum covering by cliques. Thus, any clique CL in a minimum covering by cliques of G, must intersect all maximum independent sets of G. The converse is also true: if a clique CL intersects all

maximum independent sets, then it is a clique of a minimum covering by cliques, and $\alpha(G(V\text{-}CL))=\alpha(G(V))\text{-}1=\gamma(G(V))\text{-}1$. We can evaluate $\alpha(G)$ in time $O(|V|^3)$ by the algorithm described in [GA4]. Our purpose is to describe an algorithm to find a clique CL which intersects all maximum independent sets of G; we then delete CL from G and continue on the remaining graph $G(V\text{-}CL)$. By the Helly property, the intervals corresponding to the vertices in CL have a non-empty intersection and contain a segment d; let $X_d=\{v|\ v\in V,\ d\subseteq a(v)\}$. For every maximum independent set IND of G, all the intervals corresponding to the vertices in $IND\cap X_d$ contain d and thus form a set of intervals one contained in another $a(v_1)\subset a(v_2)\subset\ldots\subset a(v_k)$; we call $a(v_1)$ the *innermost interval* of IND in X_d. *Let* $CON(X_d,v_1)=\{u|\ u\in X_d,\ a(u)\subset a(v_1)\}$.

Lemma 4. An interval $a(v)$, $v\in X_d$, is innermost interval in X_d for some maximum independent set IND if and only if $\alpha(G(V\text{-}N(v)\text{-}CON(X_d,v)))=\alpha(G(V))$.

Proof. If $a(v)$, $v\in X_d$, is innermost interval in X_d for some maximum independent set IND, then, clearly, $\alpha(G(V\text{-}N(v)\text{-}CON(X_d,v))=\alpha(G(V))=|IND|$.

Conversely, assume that $\alpha(G(V\text{-}N(v)\text{-}CON(X_d,v))=\alpha(G(V))$ for some $a(v)$, $v\in X_d$. Since v has no adjacent vertices in $G(V\text{-}N(v)\text{-}CON(X_d,v))$, every maximum independent set IND of $G(V\text{-}N(v)\text{-}CON(X_d,v))$ contains v and contains no u having $a(u)\subset a(v)$. Thus, $a(v)$ is innermost interval of IND in X_d.

Algorithm Clique-MAXIND(Interval filament graph). The algorithm to find a clique CL which intersects all maximum independent sets of G works as follows. Going from right to left on L, we consider every maxcocomp segment d of L and find X_d. If $\alpha(G(V\text{-}X))=\alpha(G(V))$, X_d contains no such clique CL and we go to the next maxcocomp segment at the left of d on L. Assume that $\alpha(G(V\text{-}X_d))<\alpha(G(V))$. We wont to find a clique CL in X_d, such that $\alpha(G(V\text{-}CL))<\alpha(G(V))$. Let $Y_d=\{v|\ v\in X_d$ *and* $\alpha(G(V\text{-}N(v)\text{-}CON(X_d,v)))=\alpha(G(V))\}$; Y_d is the subset of vertices in X_d whose intervals are innermost intervals of the maximum independent sets. There are two cases:

Case 1: Y_d is a clique. By the construction, every maximum independent set intersects X_d and the vertex corresponding to its innermost interval is in Y_d. Therefore Y_d is a clique intersecting all the maximum independent sets of G.

Case 2: Y_d is not a clique. Consider $u,v\in Y_d$, $a(u)$ innermost for IND_u and $a(v)$ innermost for IND_v, such that $u\cap v=\phi$ and $a(u)\subset a(v)$. For every $w\in IND_v\cap X_d$ we have $a(u)\subset a(v)\subset a(w)$ and $w\cap u=\phi$, thus $u\notin CL$, otherwise $CL\cap IND_v=\phi$. We delete every such u from X_d and from Y_d. If the new X_d fulfils $\alpha(G(V\text{-}X_d))=\alpha(G(V))$, then X_d does not contain a clique CL intersecting all the maximum independent sets of G and we go to the next segment at the left of d on L. If the new X_d fulfils $\alpha(G(V\text{-}X_d))<\alpha(G(V))$, we look in $X_d\text{-}Y_d$ for the vertices which replace in Y_d the deleted vertices as innermost; we do this by adding to Y_d those vertices $v\in X_d\text{-}Y_d$ fulfilling $\alpha(G(V\text{-}N(v)\text{-}CON(X_d,v)))=\alpha(G(V))$.

We continue in this way, until for some d and X_d we have $\alpha(G(V-X_d))<\alpha(G(V))$ and Y_d is a clique. Since G is perfect, such a clique Y_d always exists. The algorithm works in time $O(|V|^5)$.

Algorithm Clique-MAXIND(Chordal filament graph). The above algorithm works also for chordal filament graphs, since families of subtrees of a tree have the Helly property and [GA4] presents an $O(|V|^3)$ time algorithm to find $\alpha(G)$ for a chordal filament graph G. By the Helly property, the subtrees corresponding to the vertices in a clique CL, which intersects all maximum independent sets, have a non-empty intersection and contain a subtree d. Let $X_d=\{v|\ v\in V, d\subseteq a(v)\}$; CL is contained in X_d. The method is again to consider maxcocomp subtrees d of the base tree T and to find X_d and Y_d.

Algorithm Clique-MAXIND(Circular-arc filament graph). A similar algorithm can also be described for circular-arc filament graphs G. Consider a clique CL of G. As explained in the Introduction, there are two points x,y on CR such that every $a(v)$, $v\in CL$, has its endpoints on both sides of the diagonal x,y of CR. Let $X_{x,y}$ be the set of vertices v in V such that $a(v)$ has its endpoints on both sides of the diagonal x,y of CR; CL is contained in $X_{x,y}$. Therefore, to find CL we consider every two maxcocomp arcs of CR, in them we take two points x,y and find the set $X_{x,y}$ of vertices whose arcs have the endpoints on both sides of the diagonal x,y. When $\alpha(G(V-X_{x,y}))<\alpha(G(V))$, every maximum independent set of G intersects $X_{x,y}$ in a sequence of mutually non-intersecting filaments, cut by the diagonal x,y. We find the subset $Y_{x,y}$, $Y_{x,y}\subseteq X_{x,y}$, of filaments which are closest (innermost) to x for some maximum independent set. If $Y_{x,y}$ is a clique, then it is the clique which intersects all maximum independent sets of G. The algorithm works in time $O(|V|^7)$.

Algorithm Clique-MAXIND(Cactus filament graph). The above algorithm works also for cactus filament graphs G, since for a clique CL, there are two points x,y on one circle of the cactus such that every $a(v)$, $v\in CL$, has its endpoints, on both sides of the diagonal x,y. Therefore, as above we consider every two maxcocomp arcs on every circle, take $X_{x,y}$, and proceed as above. In addition we consider every maxcocomp segment on the subgraph connecting the circles of the cactus and proceed as in the algorithm for interval filament graphs.

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