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A solution to a problem of Le

by

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ABSTRACT

Let $S_1, S_2, \ldots, S_t$ be pairwise disjoint non-empty stable sets in a graph H. The graph H^* is obtained from H by

- replacing each S_i by a new vertex q_i,
- joining each q_i and q_j, $1 \leq i \neq j \leq t$, and
- joining q_i to all vertices in $H - (S_1 \cup S_2 \cup \cdots \cup S_t)$ which were adjacent to some vertex of S_i.

A *cograph* is a P_4-free graph. A graph G is called a *cograph contraction* if there exist a cograph H and pairwise disjoint non-empty stable sets in H for which $G \simeq H^*$. Solving a problem proposed by Le [2], we give a finite forbidden induced subgraph characterization of cograph contractions.

Keywords: Cograph contractions, perfect graphs, weakly chordal graphs, forbidden induced subgraphs.
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1 Introduction

We use standard terminology of Melnikov, Sarvanov, Tyshkevich, Yemelichev, and Zverovich [3]. As usual, P_n denotes a path of order n. A *cograph* is a P_4-free graph, i.e., a graph without induced subgraph isomorphic to P_4. A 4-path P_4 with $V(P_4) = \{a, b, c, d\}$ and $E(P_4) = \{ab, bc, cd\}$ will be denoted by $P_4 = (a, b, c, d)$. Here a and d are *end-vertices*, b and c are *mid-vertices*, and bc is the *mid-edge* of the P_4. A vertex subset that induces an edgeless subgraph is called a *stable set*.

Definition 1. *Let $S_1, S_2, \ldots, S_t$ be pairwise disjoint non-empty stable sets in a graph H. The graph H^* is obtained from H by*

- *replacing each S_i by a new vertex q_i,*
- *joining each q_i and q_j, $1 \leq i \neq j \leq t$, and*
- *joining q_i to all vertices in $H - (S_1 \cup S_2 \cup \cdots \cup S_t)$ which were adjacent to some vertex of S_i.*

Definition 2. *A graph G is called a* cograph contraction *if there exist a cograph H and pairwise disjoint non-empty stable sets in H for which $G \simeq H^*$.*

Perfect graphs is a well-known class of graphs, see Ramírez Alfonsín and Reed, Eds. [4].

Theorem 1 (Hujter and Tuza [1]). *All cograph contractions are perfect graphs.*

Problem 1 (Hujter and Tuza [1]). *Characterize cograph contractions.*

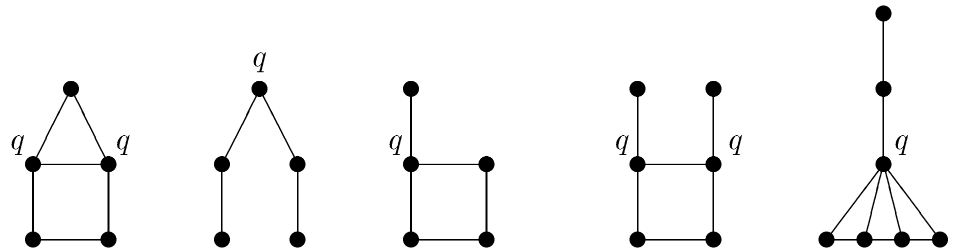
Le [2] has solved Problem cographcont-problem1. A *clique* in a graph G is a complete subgraph of G [not necessarily inclusionwise maximal].

Theorem 2 (Le [2]). *A graph G is a cograph contraction if and only if it has a clique Q such that*

P_4-Condition *each induced P_4 in G has at least one of its mid-vertex in Q, and*

$\overline{P}_5$-Condition *each induced $\overline{P}_5$ in G has both its q-vertices in Q* [q-vertices of $\overline{P}_5$ are shown in Figure 1(a)].

Cograph contractions constitute a hereditary class, i.e., all their induced subgraphs are also cograph contractions.



(a) $H_1 = \overline{P}_5$ (b) $H_2 = P_5$ (c) H_3 (d) H_4 (e) H_5

Figure 1. Graphs $H_1, H_2, H_3, H_4, H_5 \in \mathcal{H}$ and their q-vertices.

We give a solution to the following problem.

Problem 2 (Le [2]). *Find forbidden induced subgraph characterization for cograph contractions.*

2 Results

We denote by $\mathcal{COGRAPHCONT}$ the class of all cograph contractions.

Lemma 1. *The graphs* C_5 [the cycle of order five] *and* $2P_4$ [disjoint union of two 4-paths] *are minimal forbidden induced subgraphs for the class* $\mathcal{COGRAPHCONT}$.

Proof. Each clique in C_5 or $2P_4$ does not satisfy the P_4-Condition. $\square$

Definition 3. *We denote by* $\mathcal{H}$ *the set of graphs shown in* Figure 1*. A* q-vertex *of a graph in* $\mathcal{H}$ *is a vertex marked with 'q' in* Figure 1*. A* q-vertex *of an arbitrary graph* G *is a* q*-vertex of some induced subgraph* $H \in \mathcal{H}$ *of* G*.*

Lemma 2. *Let* $G \in \mathcal{COGRAPHCONT}$ *and let* Q *be a clique of* G *satisfying the* P_4-Condition *and the* $\overline{P}_5$-Condition*. Then* Q *contains all* q*-vertices of* G*.*

Proof. For $H_1 = \overline{P}_5$ the statement coincides with the $\overline{P}_5$-Condition. For H_2, H_3, H_4 and H_5 the statement is easily implied by the P_4-Condition. $\square$

A subgraph of a graph G induced by a set $S \subseteq V(G)$ is denoted by $G(S)$.

Definition 4. *Let Z_1 be the class of all graphs F such that*

- *there exist $X, Y \subseteq V(F)$ with $X \cup Y = V(F)$,*
- *both $F(X)$ and $F(Y)$ are in $\mathcal{H}$, and*
- *there exist distinct non-adjacent q-vertices x in $F(X)$ and y in $F(Y)$.*

Lemma 3. $Z_1 \cap \mathcal{COGRAPHCONT} = \emptyset$.

Proof. If $F \in Z_1 \cap \mathcal{COGRAPHCONT}$ then, by Lemma 2, there exists a clique that contains all q-vertices of F. But x and y are non-adjacent q-vertices, a contradiction. □

Definition 5. *Let Z_2 be the class of all graphs F such that*

- *there exist $X, Y \subseteq V(F)$ with $X \cup Y = V(F)$,*
- *$F(X) \in \mathcal{H}$ and $F(Y) \simeq P_4$, and*
- *there exists a q-vertex of $F(X)$ that is non-adjacent to both mid-vertices of $P_4 = V(Y)$.*

Lemma 4. $Z_2 \cap \mathcal{COGRAPHCONT} = \emptyset$.

Proof. If $F \in Z_2 \cap \mathcal{COGRAPHCONT}$ then, by Lemma 2, there exists a clique Q that contains all q-vertices of F. But then Q does not contain both mid-vertices of $P_4 = V(F)$, a contradiction to the P_4-condition. □

Our main result, Theorem 3, shows that the set $Z = \{C_5, 2P_4\} \cup Z_1 \cup Z_2$ contains all minimal forbidden induced subgraphs. Since $|V(H)| \leq 7$ for each $H \in \mathcal{H}$, $|V(F)| \leq 14$ for every $F \in Z$. Thus, it is a (tedious, but quite possible) routine to extract all minimal forbidden induced subgraphs from Z.

A graph is *Z-free* if it does not contain all graphs in Z as induced subgraphs.

Theorem 3. *A graph G is a cograph contraction if and only if it does not contain any graph of the set $Z = \{C_5, 2P_4\} \cup Z_1 \cup Z_2$ as an induced subgraph.*

Proof. Necessity follows from Lemma 1, Lemma 3, and Lemma 4.

Sufficiency. Let G be a Z-free graph. Let $\mathrm{Mid}(G, P_4)$ be the set of mid-edges of all induced P_4's in G. We define a clique Q of G as follows:

(CLIQ1) Q contains all q-vertices of G [since G is Z_1-free, Lemma 3 implies that q-vertices induces a clique],

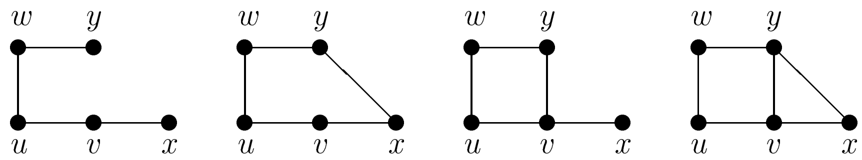
(CLIQ2) the set $\mathrm{Mid}(G - Q) = \mathrm{Mid}(G, P_4) \cap E(G - V(Q))$ is as small as possible, and

(CLIQ3) Q has minimal order among all cliques satisfying (CLIQ1) and (CLIQ2).

By Theorem 2, we need only check the P_4-Condition, since all q-vertices of induced $\overline{P}_5$ are included into Q according to (CLIQ1). If $\mathrm{Mid}(G - Q) = \emptyset$ then the proof is complete. Therefore let uv be the mid-edge of an induced $P_4 = (w, u, v, x)$ in $G - Q$. By (CLIQ1), $G - V(Q)$ has no q-vertices. In particular, u and v are not q-vertices.

Claim 1. $w, x \notin V(Q)$.

Proof. Suppose that $w \in V(Q)$. Since w and x are not adjacent, $x \notin V(Q)$. According to (CLIQ2), $V(Q) \cup \{u\}$ does not induce a clique. Hence there exists a vertex $y \in V(Q)$ that is non-adjacent to u. Then either u is a q-vertex (Figure 2(a)), or G contains a forbidden induced subgraph C_5 (Figure 2(b)), or v is a q-vertex (Figure 2(c)), a contradiction. □



(a) u is a q-vertex (b) induced C_5 (c) v is a q-vertex

Figure 2. An illustration to Claim 1.

Claim 2. *There exists a vertex $a \in V(Q)$ that is non-adjacent to $u, v, w,$ and x.*

Proof. By Claim 1, $\{u, v, w, x\} \cap V(Q) = \emptyset$. By (CLIQ2), $V(Q) \cup \{u\}$ is not a clique, i.e., there exists a vertex $a \in V(Q)$ that is non-adjacent to u. All possible variants for the induced subgraph $G(u, v, w, x, a)$ are shown in Figure 3.

In the case (a) of Figure 3, the proof is complete. The cases (b), (d), (e), (f) and (h) are impossible, since either u is a q-vertex [as in (b)], or v is a q-vertex [as in (d), (e) and (h)], or C_5 is an induced subgraph of G [as in (f)].

It remains to consider the cases (c) and (g) of Figure 3. We find a vertex $b \in V(Q)$ that is non-adjacent to v. As before, there are only two possible variants for the induced subgraph $G(u, v, w, x, b)$, namely,

- $N(b) \cap \{u, v, w, x\} = \{u\}$, or
- $N(b) \cap \{u, v, w, x\} = \{u, w\}$.

As it can be seen, at least one of u, v is a q-vertex, a contradiction. $\square$

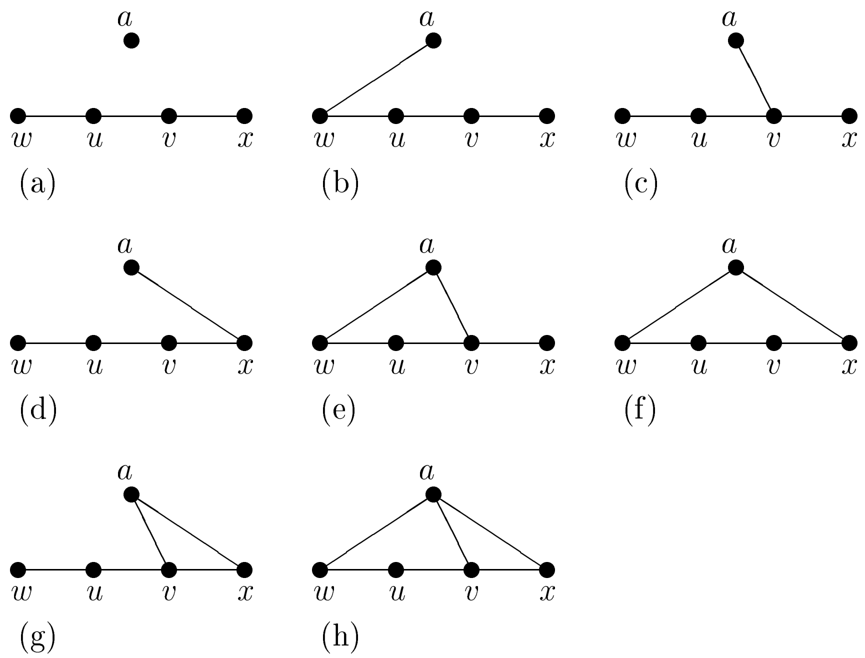


Figure 3. All possible variants for the induced subgraph $G(u, v, w, x, a)$.

The vertex a [see Claim 2] is not a q-vertex. For otherwise, let $H \in \mathcal{H}$ be an induced subgraph that contains a. Then $V(H) \cup \{u, v, w, x\}$ would induce a forbidden subgraph from Z_2.

By the condition (CLIQ3), if we delete a from Q, we obtain an edge ab [where $b \in V(G - Q)$] which is the mid-edge of some induced P_4. Since $b \notin V(Q)$, (CLIQ1) implies that b is not a q-vertex.

Let (c, a, b, d) be an induced P_4. Since a is not adjacent to all u, v, w, x, we have $c, b \notin \{u, v, w, x\}$.

Case 1. $d \in \{w, x\}$.

By symmetry, let $d = x$. Let us consider the induced subgraph $G(a, b, c, d = x, v)$. Figure 4(a) shows that v is adjacent to b and non-adjacent to c. Indeed, otherwise either b is a q-vertex, or there is an induced C_5, a contradiction. Further, Figure 4(b) implies that u is adjacent to b and non-adjacent to c. Similarly, w is adjacent to b and non-adjacent to c. Thus, $\{a, b, c, d = x, u, v, w\}$ induces $H_5 \in \mathcal{H}$ with the q-vertex b, a contradiction.

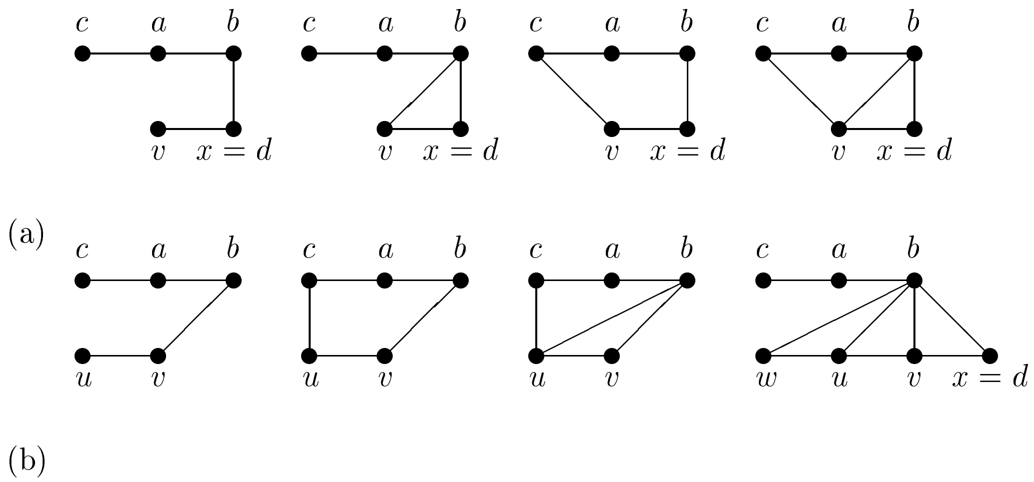


Figure 4. An illustration to Case 1.

Case 2. $d \in \{u, v\}$.

By symmetry, let $d = v$. As in Case 1, b is adjacent to all vertices u, v, w, x, while c is non-adjacent to all of them. Thus, $\{a, b, c, d = v, u, w, x\}$ induces $H_5 \in \mathcal{H}$ with the q-vertex b, a contradiction.

Case 3. $d \notin \{u, v, w, x\}$.

Claim 3. *For any vertex $y \in \{u, v, w, x\}$, either*

- $N(y) \cap \{a, b, c, d\} = \emptyset$ [we say that type$(y) = 0$], *or*
- $N(y) \cap \{a, b, c, d\} = \{b\}$ [we say that type$(y) = 1$], *or*
- $N(y) \cap \{a, b, c, d\} = \{b, d\}$ [we say that type$(y) = 2$].

Proof. Figure 5 shows that otherwise either a is a q-vertex or b is a q-vertex, or C_5 is an induced subgraph of G. □

Claim 4. *For every adjacent vertices y and z in $\{u, v, w, x\}$, we have*

- *if* $\mathrm{type}(y) = 0$ *then* $\mathrm{type}(z) = 0$, *and*
- *if* $\mathrm{type}(y) \geq 1$ *then* $\mathrm{type}(z) \geq 1$.

Proof. It is sufficient to show that the case where $\mathrm{type}(y) = 0$ and $\mathrm{type}(z) \geq 1$ is impossible. The set $\{c, a, b, z, y\}$ induces P_5 with a q-vertex b, a contradiction. □

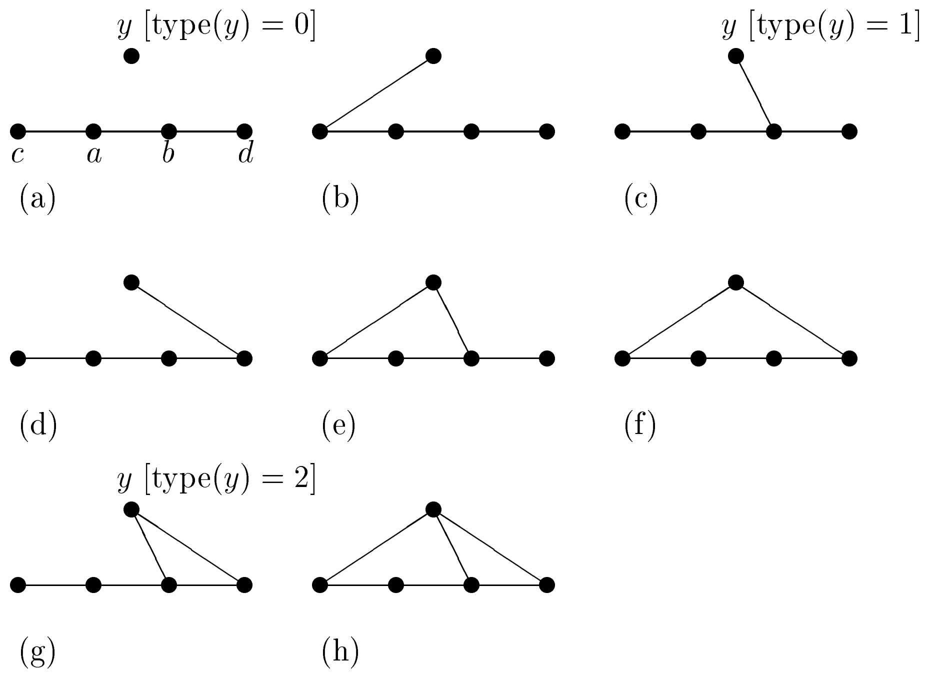


Figure 5. An illustration to Claim 3.

By Claim 4, either

(a) $\mathrm{type}(w) = \mathrm{type}(u) = \mathrm{type}(v) = \mathrm{type}(x) = 0$, or

(b) $\mathrm{type}(w) \geq 1$, $\mathrm{type}(u) \geq 1$, $\mathrm{type}(v) \geq 1$, and $\mathrm{type}(x) \geq 1$.

In (a), $\{u, v, w, x, a, b, c, d\}$ induces $2P_4$, a contradiction. In (b), $\{u, v, w, x, a, b, c\}$ induces H_5 with the q-vertex b, a contradiction. □

Recall that a graph is *weakly chordal* if it does not contain all cycles C_n, $n \geq 5$, and their complements as induced subgraphs. These graphs are perfect. Theorem 1 can be specified as follows.

Corollary 1. *All cograph contractions are weakly chordal graphs.*

Proof. Indeed, $C_5, C_6, \overline{C}_6, \overline{P}_6, P_7 \in Z$ are minimal forbidden induced subgraphs for the class $\mathcal{COGRAPHCONT}$. $\square$

3 About minimal forbidden induced subgraphs

The cycle C_5 is the only minimal forbidden induced subgraph of order $n \leq 5$ for the class $\mathcal{COGRAPHCONT}$. To specify all minimal graphs in Z of order 6, we may add a new vertex x to each of the graphs $H_1, H_2, H_3 \in \mathcal{H}$ (Figure 1), and consider all possible variants for edges between x and $V(H_i)$. Deleting all resulting graphs that either contain an induced C_5 or are not in $Z_1 \cup Z_2$, we obtain the graphs shown in Figure 6.

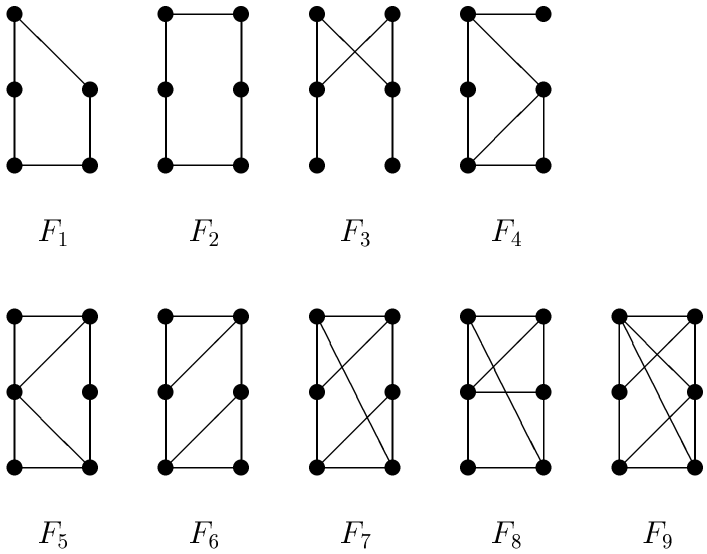


Figure 6. All minimal forbidden induced subgraphs of order $n \leq 6$ for the class $\mathcal{COGRAPHCONT}$.

Of course, it is easy to specify small minimal forbidden induced subgraphs. However we are able to consider the largest case of $n = 14$. Note that a straightforward application of the definition of Z requires to consider 2^{48} cases [some of them are isomorphic]. We shall avoid such time consuming considerations.

Theorem 4. *The are no minimal forbidden induced subgraphs of order $n \geq 14$ for the class $\mathcal{COGRAPHCONT}$.*

Proof. According to Theorem 3, the largest minimal forbidden induced subgraph F may contain exactly 14 vertices, in which case F consists of two disjoint copies of H_5 (Figure 1) with non-adjacent q-vertices.

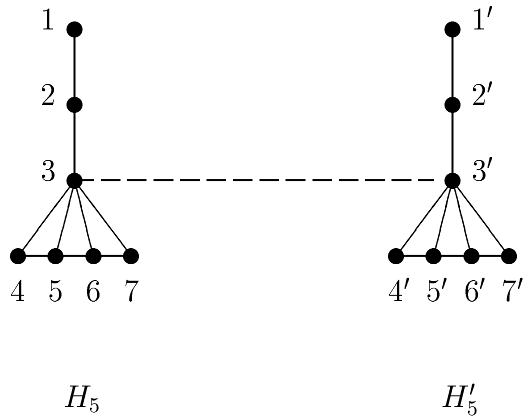


Figure 7. Two disjoint copies of H_5 with non-adjacent q-vertices 3 and $3'$.

We adopt notation of $V(F)$ shown in Figure 7, where two disjoint copies of H_5, namely H_5 and H_5', are induced subgraphs of F. The q-vertex of H_5 is adjacent to the q-vertex of H_5', while the remaining 48 edges between $V(H_5)$ and $V(H_5')$ are undefined.

If the vertex $3'$ is non-adjacent to 2, then we delete the vertices 5, 6 and 7. As a result, we obtain a graph belonging to Z_2, since the set $\{1, 2, 3, 4\}$ induces P_4 with mid-vertices 2 and 3 non-adjacent to the q-vertex $3'$ of H_5'. It follows that the graph is non-minimal, and therefore $3'$ is adjacent to 2.

If $3'$ is non-adjacent to both 5 and 6, then we delete the vertices 1, 2 and 3. We obtain a graph belonging to Z_2, since the set $\{4, 5, 6, 7\}$ induces P_4 with mid-vertices 5 and 6 non-adjacent to the q-vertex $3'$ of H_5'. By symmetry, we may assume that $3'$ is adjacent to 6.

If $3'$ is non-adjacent to 7, then we have a copy of $H_1 = \overline{P}_5$ induced by $\{6, 3, 2, 3', 7\}$. The q-vertex 3 of the H_1 is non-adjacent to the q-vertex $3'$ of H_5'. Thus, we can delete 1, 4 and 5, and obtain a forbidden induced subgraph. It means that $3'$ and 7 are adjacent.

If $3'$ is non-adjacent to 7, then we have a copy of $H_1 = \overline{P}_5$ induced by $\{6, 3, 2, 3', 7\}$. The q-vertex 3 of the H_1 is non-adjacent to the q-vertex $3'$ of H_5'. Thus, we can delete 1, 4 and 5, and obtain a forbidden induced subgraph. It means that $3'$ and 7 are adjacent. In a similar way we can show that $3'$ is adjacent to both 5 and 4.

If $3'$ is adjacent to 1, then the set $\{1, 3', 7, 3, 2\}$ induces H_1. The q-vertex 2 of the H_1 is non-adjacent to both mid-vertices of the P_4 induced by $\{4, 5, 6, 7\}$, i.e., F is not a minimal forbidden induced subgraph.

Finally, if $3'$ is non-adjacent to 1, then we can delete the set $\{1', 2', 4', 5', 6', 7'\}$. As a result, we obtain two copies of H_5 induced by $\{1, 2, 3, 4, 5, 6, 7\}$ and $\{1, 2, 3', 4, 5, 6, 7\}$ with non-adjacent q-vertices 3 and $3'$. As before, F is not minimal. $\square$

Currently the complete list of all minimal forbidden induced subgraphs for the class $\mathcal{COGRAPHCONT}$ is not available. We agree with a referee that it would be interesting to produce such a list.

Acknowledgment

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