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L(2,1)-Labelings of Products of Two Cycles

by

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ABSTRACT

An L(2,1)-labeling of a graph is an assignment of nonnegative integers to its vertices so that adjacent vertices get labels at least two apart and vertices at distance two get distinct labels. The λ-number of a graph G, denoted by $\lambda(G)$, is the minimum range of labels taken over all of its L(2,1)-labelings. We show that the λ-number of the Cartesian product of any two cycles is 6, 7 or 8. In addition, we provide complete characterizations for the products of two cycles with λ-number exactly equal to each one of these values.

1 Introduction

Behind the apparent simplicity of the L(2,1)-labelings lies a surprising complexity that has provided challenging research topics for more than a decade since Griggs and Yeh, motivated by the problem of assigning channels to interfering radio and TV transmitters, introduced this graph coloring variation in [13]. We refer the reader to [1—23] for a comprehensive overview on L(2,1)-labelings. An *L(2,1)-labeling* of a graph G is an assignment of nonnegative integers to its vertices so that adjacent vertices get integers at least two apart and vertices at distance two get integers at least one apart. If an L(2,1)-labeling uses labels in the set $\{0, 1, ..., k\}$ it will be called a *k-labeling*. The minimum k so that G has a k-labeling is called the *λ-number* of G and will be denoted by $\lambda(G)$. The long standing conjecture by Griggs and Yeh, which states that the λ-number of a graph cannot exceed the square of its maximum degree, has motivated the study of the λ-number of particular classes of graphs.

The λ-number of the Cartesian products of graphs was investigated in [11,12,14,15,16, 23]. The *Cartesian product* of two graphs G and H (or simply product), denoted by $G \times H$, is defined as the graph with vertex set given by the Cartesian product of the vertex set of G and the vertex set of H, where two vertices (u, v) and (w, z) are adjacent if and only if either $[u, w$ are adjacent in G and $v = z]$ or $[v, z$ are adjacent in H and $u = w]$. The following are the known results on the λ-number of the product of pairs of graphs involving complete graphs, paths and cycles.

Theorem 1 ([11,12]) *If $n, m \geq 2$ then*

$$\lambda(K_n \times K_m) = \begin{cases} 4, & \text{if } n = m = 2 \\ nm - 1, & \text{otherwise.} \end{cases} \bullet$$

Theorem 2 ([23]) *If $n, m \geq 2$ then*

$$\lambda(P_n \times P_m) = \begin{cases} 5, & \text{if } n = 2 \text{ and } m \geq 4 \\ 6, & \text{if } n, m \geq 4 \text{ or } [n \geq 3 \text{ and } m \geq 5]. \end{cases} \bullet$$

Theorem 3 ([16]) *If $n \geq 4$ and $m \geq 3$ then*

(i) $\lambda(P_2 \times C_m) = \begin{cases} 5, & \text{if } m \equiv 0 \text{ mod } 3 \\ 6, & \text{otherwise;} \end{cases}$

(ii) $\lambda(P_3 \times C_m) = \begin{cases} 7, & \text{if } m = 4 \text{ or } 5 \\ 6, & \text{otherwise;} \end{cases}$

(iii) $\lambda(P_n \times C_m) = \begin{cases} 6, & \text{if } m \equiv 0 \text{ mod } 7 \\ 7, & \text{otherwise.} \end{cases} \bullet$

Theorem 4 ([15]) *If $n, m \geq 3$ then*

$$\lambda(C_n \times C_m) = 6, \text{ if } n, m \equiv 0 \text{ mod } 7.$$

In addition,

$\lambda(C_n \times C_m) \leq 7$, *if* $[n \equiv 0 \ mod\ 4 \ and\ m \geq 4]$ *or* $[n \equiv 0 \ mod\ 3 \ and\ m \equiv 0 \ mod\ 6]$. •

We will present a complete result on the λ-number of the product of two cycles, namely

Theorem 5 *If* $n, m \geq 3$ *then*

$$\lambda(C_n \times C_m) = \begin{cases} 6, & \text{if } n, m \equiv 0 \ mod\ 7 \\ 8, & \text{if } \{n, m\} \in A \\ 7, & \text{otherwise,} \end{cases}$$

where $A = \{\{3, i\} : i \geq 3, i \ odd\ or\ i = 4, 10\} \cup \{\{5, i\} : i = 5, 6, 9, 10, 13, 17\} \cup$ $\{\{6, 7\}, \{6, 11\}, \{7, 9\}, \{9, 10\}\}$. •

The lower bound $\lambda(C_n \times C_m) \geq 6$ for all $n, m \geq 3$ follows from the following Lemma by observing that $C_n \times C_m$ is a 4-regular graph for $n, m \geq 3$.

Lemma 1 ([13]) *If a graph contains three vertices with maximum degree* $\Delta \geq 2$ *and one of them is adjacent to the other two vertices, then its* λ*-number is at least* $\Delta + 2$. •

Theorem 3 could be used to establish the better lower bound $\lambda(C_n \times C_m) \geq 7$ when $[n = 3$ and $m = 4$ or $5]$ or when $[n \geq 4$ and m is not a multiple of $7]$, since $P_n \times C_m$ is a subgraph of $C_n \times C_m$. The proof of this theorem relies on extensive case checking done by computer programs that are not easily verifiable without their codes. For instance, in the proof of item (iii), a computer program generates a directed graph of 1248 vertices, 912 edges and searches for all its cycles. We chose not to use this theorem in our proofs, providing arguments that do not require the aide of computer programs whenever possible. Our results can actually be adapted to provide a "computer-free" proof for item (iii) in Theorem 3 (see Section 5).

In Section 2, we establish a general upper bound $\lambda(C_n \times C_m) \leq 8$ for all $n, m \geq 3$. We also show that $\lambda(C_n \times C_m) \leq 7$ for selected values of n and m. In particular, $\lambda(C_n \times C_m) \leq 7$ if n and m are sufficiently large, namely, if $n, m \geq 28$. In Sections 3 and 4, we present exact values for $\lambda(C_3 \times C_m)$ and for $\lambda(C_4 \times C_m)$, respectively, for any $m \geq 3$. In Section 5, we completely characterize the product of cycles with λ-number exactly 6. We conclude by computing $\lambda(C_n \times C_m)$ when $n, m \geq 5$ in Section 6, where computer programs were used to generate 7-labelings or to show that they do not exist.

Throughout, each vertex in $C_n \times C_m$ will be represented by an ordered pair (i, j) with $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$ so that two vertices are adjacent if their pair representations satisfy exactly one of the following two conditions:

1. Both pairs agree on the first coordinate and differ in absolute value by exactly 1 or $m - 1$ on the second coordinate.

2. Both pairs agree on the second coordinate and differ in absolute value by exactly 1 or $n - 1$ on the first coordinate.

Notice that if we present the vertices of $C_n \times C_m$ as an n-by-m matrix with vertex (i, j) placed in the entry in row i, column j, the subgraph induced by each row is isomorphic to C_m and the subgraph induced by each column is isomorphic to C_n. For convenience, k-labelings of the vertices in $C_n \times C_m$ will also be represented as n-by-m matrices where the entry on the ith row, jth column will be the label of vertex (i, j).

2 Upper Bounds for $\lambda(C_n \times C_m)$

In this section, we establish upper bounds for the λ-number of products of two cycles. We begin by presenting two auxiliary Lemmas that will be used to construct k-labelings of products of two cycles from k-labelings of products of pairs of smaller cycles. Lemma 2 is a straightforward result that takes advantage of the symmetry of products of two cycles. Lemma 3 is a number theory result that identifies consecutive nonnegative integers that can be written as nonnegative combinations of two given relatively prime positive integers different from one. Their proofs are elementary and will be omitted.

If M is a matrix and α, β are nonnegative integers we define $\alpha M \beta$ as the block matrix obtained by arranging α rows of β copies of matrix M per row.

Lemma 2 *Let the matrices X and Y be k-labelings of $C_p \times C_r$ and $C_p \times C_s$, respectively, where $p, r, s \geq 3$. If the block matrix $[X|Y]$ obtained by arranging a copy of X followed by a copy of Y in a row is a k-labeling of $C_p \times C_{(r+s)}$, then for every nonnegative integers α and β, the block matrix $[1X\alpha|1Y\beta]$ is a k-labeling of $C_p \times C_{(\alpha r+\beta s)}$.* •

Given two integers r and s, the set of all nonnegative integer combinations of r and s will be denoted by $S(r, s)$, that is, $S(r, s) = \{\alpha r + \beta s : \alpha, \beta \text{ nonnegative integers}\}$.

Lemma 3 *If r and s are relatively prime integers greater than one, then $t \in S(r, s)$ for all $t \geq (r-1)(s-1)$, and $(r-1)(s-1)-1 \notin S(r, s)$.* •

Theorem 6 *If $n, m \geq 3$ then $\lambda(C_n \times C_m) \leq 8$.*

Proof: Consider the graph $C_n \times C_m$ with $n, m \geq 3$. Let us first consider the case where $n = 5$ and show that $C_5 \times C_m$ has an 8-labeling for all $m \geq 3$. The matrices X, Y, and Z in Figure 1 plus the block matrix $[X|Y]$ are 8-labelings of $C_5 \times C_3$, $C_5 \times C_4$, $C_5 \times C_5$, and $C_5 \times C_7$, respectively.

$$X = \begin{pmatrix} 0 & 2 & 5 \\ 3 & 6 & 1 \\ 8 & 0 & 4 \\ 1 & 5 & 7 \\ 6 & 8 & 3 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 2 & 4 & 7 \\ 3 & 5 & 8 & 1 \\ 8 & 0 & 2 & 6 \\ 1 & 3 & 7 & 4 \\ 6 & 8 & 0 & 2 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & 2 & 4 & 6 & 8 \\ 6 & 8 & 0 & 2 & 4 \\ 2 & 4 & 6 & 8 & 0 \\ 8 & 0 & 2 & 4 & 6 \\ 4 & 6 & 8 & 0 & 2 \end{pmatrix}$$

Figure 1:

If $m \geq 6$, then Lemma 3 implies that there are nonnegative integers α and β so that $m = 3\alpha + 4\beta$, and Lemma 2 with the given matrices X and Y shows that $C_5 \times C_m$ has an 8-labeling. The case $m = 5$ and $n \geq 3$ follows since the product of two graphs is commutative.

Let us then focus on the case where $n \neq 5$ and $m \neq 5$. We will construct an 8-labeling of $C_n \times C_m$ using block combinations of the matrices in Figure 2. Each of these 4 matrices and

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 3 & 5 & 7 \\ 6 & 8 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 6 & 4 \\ 3 & 5 & 0 & 7 \\ 6 & 8 & 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 2 & 5 \\ 8 & 6 & 3 \\ 2 & 0 & 7 \\ 6 & 8 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 2 & 7 & 5 \\ 8 & 4 & 1 & 3 \\ 2 & 0 & 5 & 7 \\ 6 & 8 & 3 & 1 \end{pmatrix}$$

Figure 2:

each of the block matrices in Figure 3 is, respectively, an 8-labeling of the product of the cycle with length equal to the number of rows in the matrix and the cycle with length equal to the number of columns in the matrix. Since $n, m \geq 3$, $n \neq 5$ and $m \neq 5$, from Lemma 3

$$\begin{pmatrix} A & B \end{pmatrix} \quad \begin{pmatrix} C & D \end{pmatrix} \quad \begin{pmatrix} A \\ C \end{pmatrix} \quad \begin{pmatrix} B \\ D \end{pmatrix} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Figure 3:

we conclude that there exist nonnegative integers k, l, p, q so that $n = 3k + 4p$, $m = 3l + 4q$. By multiple applications of Lemma 2 we conclude that the block matrix in Figure 4 is an 8-labeling of $C_n \times C_m$. •

$$\begin{pmatrix} kAl & kBq \\ pCl & pDq \end{pmatrix}$$

Figure 4:

Theorem 7 *If $n, m \geq 3$, $n \in S(4, 7)$ and $m \in S(5, 8)$, then $\lambda(C_n \times C_m) \leq 7$. In particular, if $n \geq 18$ and $m \geq 28$, then $\lambda(C_n \times C_m) \leq 7$.*

Proof: Let us assume $n, m \geq 3$, $n \in S(4, 7)$ and $m \in S(5, 8)$. We will construct a 7-labeling of $C_n \times C_m$ using block combinations of the matrices in Figure 5. Each of these 4 matrices and each of the block matrices in Figure 3 is, respectively, a 7-labeling of the product of the cycle with length equal to the number of rows in the matrix and the cycle with length equal to the number of columns in the matrix. Since $n \in S(4, 7)$ and $m \in S(5, 8)$, Lemma 3 implies that there exist nonnegative integers k, l, p, q so that $n = 4k + 7p$, $m = 5l + 8q$. By multiple applications of Lemma 2 we conclude that the block matrix in Figure 4 is a 7-labeling of $C_n \times C_m$. Finally, Lemma 3 shows that if $n \geq 18$ and $m \geq 28$, then $\lambda(C_n \times C_m) \leq 7$. •

For future reference (see Section 6), we combine the bounds provided by Lemma 1, Theorems 6 and 7 in Table 1 as follows. A symbol * in row number n and column number

$$A = \begin{pmatrix} 0 & 3 & 6 & 1 & 5 \\ 2 & 5 & 0 & 3 & 7 \\ 4 & 7 & 2 & 5 & 1 \\ 6 & 1 & 4 & 7 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 2 & 5 & 0 & 3 & 6 & 1 & 4 & 7 \\ 4 & 7 & 2 & 5 & 0 & 3 & 6 & 1 \\ 6 & 1 & 4 & 7 & 2 & 5 & 0 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 3 & 6 & 1 & 5 \\ 2 & 5 & 0 & 3 & 7 \\ 4 & 7 & 2 & 5 & 1 \\ 6 & 1 & 4 & 0 & 3 \\ 0 & 3 & 6 & 2 & 7 \\ 4 & 7 & 0 & 5 & 1 \\ 6 & 1 & 4 & 7 & 3 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 2 & 5 & 0 & 3 & 6 & 1 & 4 & 7 \\ 4 & 7 & 2 & 5 & 0 & 3 & 6 & 1 \\ 6 & 1 & 4 & 7 & 2 & 5 & 0 & 3 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 4 & 7 & 2 & 5 & 0 & 3 & 6 & 1 \\ 6 & 1 & 4 & 7 & 2 & 5 & 0 & 3 \end{pmatrix}$$

Figure 5:

m, where $n \leq m$, indicates that $6 \leq \lambda(C_n \times C_m) \leq 7$, while a blank entry indicates that $6 \leq \lambda(C_n \times C_m) \leq 8$. The *'s below the main diagonal are not shown since the table is symmetric. Rows 28 and above are not shown because they contain only *'s above the main diagonal. Furthermore, columns 29 and above are not shown since, for each row, the entry in column 28 and above are all the same.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
3																										
4			*			*		*			*		*	*		*		*	*		*	*	*	*		*
5					*	*			*	*		*	*	*		*	*	*	*	*	*	*	*	*	*	*
6																										
7						*		*			*		*	*		*		*	*		*	*	*	*		*
8						*		*	*	*	*	*	*	*		*	*	*	*	*	*	*	*	*	*	*
9																										
10									*	*		*	*	*		*	*	*	*	*	*	*	*	*	*	*
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24																						*	*	*	*	*
25																							*	*	*	*
26																								*	*	*
27																										*

Table 1:

3 The λ-number of $C_3 \times C_m$

The final goal of this section is to show that Theorem 8 holds. This result provides an infinite family of products of cycles where the upper bound presented in Theorem 6 is tight.

Theorem 8 *If $m \geq 3$ then*

$$\lambda(C_3 \times C_m) = \begin{cases} 7, & \textit{if } m \textit{ even and } m \neq 4, 10 \\ 8, & \textit{otherwise.} \end{cases}$$

We will first show that for any $m \geq 3$, $C_3 \times C_m$ does not have a 6-labeling, establishing 7 as a lower bound for its λ-number. For convenience, we will say that two rows (resp., columns) are adjacent in a matrix representation of an L(2,1)-labeling if they are consecutive rows (resp., columns) where we assume that the matrix "wraps-around" the rows (resp., columns), that is, the last and the first rows (resp., columns) are also considered consecutive. In this and subsequent proofs we will use the concept of duality to reduce the case discussion. The *dual* of a k-labeling of a graph is the labeling that assigns to each vertex the difference between k and its current label. It is not difficult to see that the dual of a k-labeling is also a k-labeling.

Lemma 4 *If $m \geq 3$ then $\lambda(C_3 \times C_m) \geq 7$.*

Proof. Suppose by contradiction that there exists $m \geq 3$ so that $C_3 \times C_m$ has a 6-labeling given by a 3-by-m matrix representation. Let S be the set containing all possible subsets of three distinct pairwise nonconsecutive labels in $\{0, 1, ..., 6\}$ that could be in the same column of the given matrix representation. Throughout, we will be using the following fact: $X \in S$ if and only if $X^* \in S$, where $X^* = \{6 - x : x \in X\}$, the *dual* of set X.

We will show first that if $X \in S$ then all labels in X must have the same parity. The only subsets of three distinct pairwise nonconsecutive labels in $\{0, 1, ..., 6\}$ containing both even and odd labels are $\{0, 2, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{0, 3, 5\}$, and $\{0, 3, 6\}$. The subset $\{0, 3, 6\}$ is not in S since a column adjacent to a column with labels in $\{0, 3, 6\}$ would have labels in $\{1, 2, 4, 5\}$ which does not contain three distinct pairwise nonconsecutive labels. Notice that the 3^{rd} and 4^{th} subsets on the list above are duals of the first two subsets on the same list, respectively. Therefore, it is enough to show that $\{0, 2, 5\}$ and $\{1, 3, 6\}$ do not belong to S. Suppose to the contrary that $\{0, 2, 5\}$ are the labels in a column C. The two columns adjacent to C must have labels in $\{1, 3, 4, 6\}$, which forces label 1 in both of these columns. Furthermore, these two labels equaling 1 must be in the same row as label 5 in C, but this is not possible since the corresponding vertices are at distance two. The proof that $\{1, 3, 6\}$ is not in S is similar. We conclude that S contains only subsets that do not mix parities.

Two adjacent columns cannot have all labels of the same parity, since the subgraph induced by their six vertices has diameter two, forcing all six labels to be different, and $\{0, 1, ..., 6\}$ has only 4 even and 3 odd distinct labels. Therefore consecutive columns must alternate parities and since $m \geq 3$, there are two columns of odd labels adjacent to the same column with three distinct even labels. But this is not possible since there are only two even labels, namely 0 and 6, that can be assigned to a vertex adjacent to two vertices with odd labels. •

The next result combined with the previous lemma proves the first equality in Theorem 8, more specifically, it shows $\lambda(C_3 \times C_m) = 7$ if $m \geq 3$, m even and $m \neq 4, 10$.

Lemma 5 *If $m \geq 3$, m even and $m \neq 4, 10$, then $\lambda(C_3 \times C_m) \leq 7$.*

Proof. The matrices X and Y in Figure 6, and the block matrix $[X|Y]$ are 7-labelings of $C_3 \times C_6$, $C_3 \times C_8$, and $C_3 \times C_{14}$, respectively.

$$X = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 7 \\ 4 & 7 & 0 & 3 & 6 & 1 \\ 6 & 1 & 4 & 7 & 0 & 3 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ 4 & 7 & 2 & 5 & 0 & 3 & 6 & 1 \\ 6 & 1 & 4 & 7 & 2 & 5 & 0 & 3 \end{pmatrix}$$

Figure 6:

Since any $m > 10$, m even can be written as $6k$, $6k+2$, or $6k+4$, for some integer $k \geq 2$, m can also be written as $6k+8(0)$, $6(k-1)+8(1)$, or $6(k-2)+8(2)$, for some integer $k \geq 2$. Therefore by Lemma 2 with the matrices X and Y in Figure 6, we conclude that $C_3 \times C_m$ has a 7-labeling for any even $m \geq 3$, except for $m = 4, 10$. •

The following Lemma combined with Theorem 6 shows that $\lambda(C_3 \times C_m) = 8$ for $m \geq 3$, m odd.

Lemma 6 *If $m \geq 3$, m odd, then $\lambda(C_3 \times C_m) \geq 8$.*

Proof. Let $m \geq 3$ so that $C_3 \times C_m$ has a 7-labeling given by a 3-by-m matrix representation. To establish the Lemma we must show that m is even. Let X and Y be the sets of labels of two adjacent columns in the matrix, respectively. Since the subgraph induced by the six vertices on these two columns has diameter two, we must have $X \cap Y = \emptyset$. Let e_X be the number of even labels in X and e_Y be the number of even labels in Y. Since there are only 4 even labels and 4 odd labels available in $\{0, 1, ..., 7\}$, we must have $2 \leq (e_X + e_Y) \leq 4$ and the pair (e_X, e_Y) cannot be in the set $\{(3,3), (0,0), (3,2), (2,3), (0,1), (1,0)\}$.

Suppose $(e_X, e_Y) = (2,2)$. Thus $X \cap \{0,2,4,6\}$ and $Y \cap \{0,2,4,6\}$ partition the set $\{0,2,4,6\}$. Without loss of generality we may assume that $0 \in X$. We have three cases to examine.

Case 1: $X \cap \{0,2,4,6\} = \{0,2\}$. Therefore $X = \{0,2,5\}$ or $\{0,2,7\}$, and $Y = \{4,6,1\}$ with label 1 in the same row as labels 5 or 7 in X. Let Z be the set of labels in the other column adjacent to the column with labels in X. Then $Z \cap \{0,1,2,5\} = \emptyset$ or $Z \cap \{0,1,2,7\} = \emptyset$, which imply $Z \subseteq \{3,4,6,7\}$ or $Z \subseteq \{3,4,5,6\}$, which is impossible since Z is a set of three distinct pairwise nonconsecutive integers.

Case 2: $X \cap \{0,2,4,6\} = \{0,4\}$. Therefore $Y \cap \{0,2,4,6\} = \{2,6\}$, leaving no choice for the third odd label in Y.

Case 3: $X \cap \{0,2,4,6\} = \{0,6\}$. Therefore $X = \{0,3,6\}$ and $Y = \{2,4,7\}$, with label 3 in X and 7 in Y in the same row. Let W be the set of labels in the other column adjacent to the column with labels in Y. Then $W \cap \{2,3,4,7\} = \emptyset$, which implies $W \subseteq \{0,1,5,6\}$, which is impossible since W is a set of three distinct pairwise nonconsecutive integers.

We conclude that (e_X, e_Y) cannot be equal to $(2,2)$. Replacing each label l with its dual $7 - l$ in the previous argument, we show that (e_X, e_Y) cannot be equal to $(1,1)$ either.

Combining these facts with our initial observations on the forbidden possibilities for (e_X, e_Y), we have that this pair must be in the set $\{(0,3),(1,3),(0,2),(1,2),(3,0),(3,1),(2,0),(2,1)\}$. So the number of even labels in each column must alternate between an integer in $\{0,1\}$ and an integer in $\{2,3\}$ as we follow the sequence of consecutive columns in the matrix. Since these columns must "wrap around," m is forced to be even. •

In view of Theorem 6, it remains to be shown that $C_3 \times C_m$ does not have a 7-labeling if $m = 4$ or 10 to complete the proof of Theorem 8. Although this could be easily accomplished by a computer program, we provide in the sequel a proof based on an interesting property of certain 7-labelings of $C_3 \times C_m$.

Lemma 7 *If a 7-labeling of $C_3 \times C_m$ uses labels of both parities in a column of a matrix representation, then m is a multiple of 6.*

Proof. Consider a matrix representation of a 7-labeling of $C_3 \times C_m$ for some $m \geq 3$. Let S be the set containing all possible subsets of three distinct pairwise nonconsecutive labels in $\{0, 1, ..., 7\}$ that could be in the same column of the given matrix representation. Throughout, we will be using the following fact: $X \in S$ if and only if $X^* \in S$, where $X^* = \{7-x : x \in X\}$, the dual of set X.

Let us first show that $\{0,2,5\}, \{0,2,7\}, \{0,3,5\}$ are not in S, and consequently, by taking duals, $\{2,5,7\}, \{0,5,7\}, \{2,4,7\}$ are not in S. Assume to the contrary that $\{0,2,5\}$ are labels in a column C. The two columns adjacent to C must have labels in $\{1,3,4,6,7\}$ forcing the label 1 to appear in both of these columns. Furthermore, these two labels equaling 1 must be in the same column as the label 5 in C, so the two corresponding vertices labeled 1 are at distance two, a contradiction. The proofs that the sets $\{0,2,7\}$ and $\{0,3,5\}$ are not in S are similar and the details are omitted. Therefore, if $X \in S$ and X contains labels of both parities, then $X \in \{\{0,3,7\}, \{1,4,6\}, \{0,3,6\}, \{0,4,7\}, \{1,3,6\}, \{1,4,7\}\}$. We may assume that X is one of the first three sets on this list since the last three sets are duals of the first three and similar arguments using dual labels can be applied to reach the conclusions.

Let us begin our case discussion by assuming $X = \{0,3,7\}$. Up to rotating or reversing the order of rows in the matrix we may assume that $(0,3,7)^T$ (where ^{T} denotes the transposition operation for matrices) is a column of the matrix representation. Its adjacent columns must have labels $2,4,5,$ or 6 in the 1^{st} row, labels $1,5,$ or 6 in the 2^{nd} row, and labels $1,2,4,$ or 5 in the 3^{rd} row. So we must have one of the four possibilities in Figure 7 for two consecutive columns where the first one uses labels in X.

0	6	0	4	0	4	0	2
3	1	3	6	3	6	3	6
7	4	7	1	7	2	7	4
(A1)		(A2)		(A3)		(A4)	

Figure 7:

Case (A4) does not occur since the only possible labels for the column following $(2,6,4)^T$ in the 2^{nd} and 3^{rd} rows are 0 and 1. Cases (A2) and (A3) reduce to case (A1) since in both cases the column preceding the $(0,3,7)^T$ column must be $(6,1,4)^T$ and by reversing the order of columns in (A2) and (A3), the new second and third columns are exactly the columns in (A1). Let us then focus on case (A1). The following column must have labels 2 or 3 in the 1^{st} row, labels 5 or 7 in the 2^{nd} row, and labels 0 or 2 in the 3^{rd} row. So the possible labelings for this column are $(2,5,0)^T$, $(2,7,0)^T$, $(3,5,0)^T$, and $(3,7,0)^T$. But since $\{0,2,5\}$, $\{0,2,7\}$, and $\{0,3,5\}$ are not in S, as we proved previously, this column must be $(3,7,0)^T$. The column following the latter must have labels 1 or 5 in the 1^{st} row, labels $2,4$, or 5 in the 2^{nd} row, and labels $2,5$, or 6 in the 3^{rd} row. Therefore the only possible labeling for this column is $(1,4,6)^T$. Case (A1) forces so far the first four columns of the submatrix in Figure 8. Notice that the third and fourth columns on this submatrix are row

$$\begin{pmatrix} 0 & 6 & 3 & 1 & 7 & 4 \\ 3 & 1 & 7 & 4 & 0 & 6 \\ 7 & 4 & 0 & 6 & 3 & 1 \end{pmatrix}$$

Figure 8:

rotations of the first two. Consequently, similar arguments show that case (A1) forces the entire submatrix in Figure 8, after which the pattern of these six consecutive columns is forced to repeat. Observe that this pattern "wraps-around" more than two columns if and only if the number of columns is a multiple of 6. In conclusion, if $X = \{0,3,7\}$ then m is a multiple of 6 (notice that the case (A3) never "wraps-around").

If $X = \{1,4,6\}$ (resp., $X = \{0,3,6\}$) a similar case discussion also shows that m is a multiple of 6 since the pattern in the submatrix of Figure 8 (resp., Figure 9) is forced to repeat up to rotating or reversing the order of rows in the matrix, which concludes our proof. •

$$\begin{pmatrix} 0 & 4 & 6 & 1 & 3 & 7 \\ 3 & 7 & 0 & 4 & 6 & 1 \\ 6 & 1 & 3 & 7 & 0 & 4 \end{pmatrix}$$

Figure 9:

Lemma 8 *If $m = 4$ or 10, then $\lambda(C_3 \times C_m) \geq 8$.*

Proof. Suppose by contradiction that there exists a 7-labeling of $C_3 \times C_m$ for $m = 4$ or 10 given by its matrix representation. From Lemma 7 we have that the labels in each column must have the same parity. Moreover, since the labels in two consecutive columns must be distinct and since there are only 4 even and 4 odd labels in $\{0,1,...,7\}$, the column parities must alternate as we follow the sequence of consecutive columns in the matrix and consequently each row alternates odd and even labels. So there is a natural one-to-one correspondence between the possible rows in the matrix and the sequence of vertex labels of circuits with m vertices in the graph of Figure 10, where incident edges in the circuits are distinct.

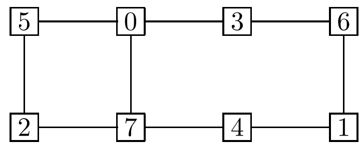

Figure 10:

Up to rotation and order reversal, the only such sequences of 4 and 10 vertices are $(5, 0, 7, 2)$ and $(5, 0, 7, 4, 1, 6, 3, 0, 7, 2)$, respectively. If $m = 4$, then all three rows in the matrix must be equal to $(5, 0, 7, 2)$, up to rotation and order reversal, which implies that the label 0 appears three times in the matrix. But the matrix has four columns and any pair of adjacent columns cannot contain the same label more than once, forcing each label to appear at most twice, a contradiction. On the other hand, if $m = 10$, then all three rows in the matrix must be equal to $(5, 0, 7, 4, 1, 6, 3, 0, 7, 2)$, up to rotation and order reversal, which implies that the label 0 appears six times in the matrix. But the matrix has ten columns and any pair of adjacent columns cannot contain the same label more than once, forcing each label to appear at most five times, a contradiction. •

4 The λ-number of $C_4 \times C_m$

The main objective of this section is to compute $\lambda(C_4 \times C_m)$ for all $m \geq 4$. Lemma 9 will be instrumental in the calculation. It follows easily from the definition of 6-labelings, observing that the set of labels $\{0, 1, ..., 6\}$ contains 4 even and 3 odd labels.

Lemma 9 *Let G be a graph with a 6-labeling. If G contains a subgraph isomorphic to*

(i) C_4 *or* $K_{1,3}$*, then the labels in this subgraph cannot be all odd;*

(ii) C_5 *or* $K_{1,4}$*, then the labels in this subgraph cannot be all even;*

(iii) $K_{1,3}$*, then the labels for the leaves in this subgraph cannot all have parities different from the label of the center vertex;*

(iv) C_4*, then the labels in this subgraph cannot alternate parities around the cycle;*

(v) $K_{1,2}$ *with an even label assigned to the center vertex and odd labels to the leaves, then either* $(0, 3, 5)$ *or* $(6, 1, 3)$ *are the labels assigned to the center vertex and the two leaves, respectively.*

(vi) $K_{1,2}$ *with an odd label assigned to the center vertex and even labels to the leaves, then* $(5, 0, 2)$*,* $(1, 4, 6)$ *or* $(3, 0, 6)$ *are the labels assigned to the center vertex and the two leaves, respectively.* •

Theorem 9 *If $m \geq 3$ then*

$$\lambda(C_4 \times C_m) = \begin{cases} 7, & \text{if } m \neq 3 \\ 8, & \text{otherwise.} \end{cases}$$

Proof: Theorem 4 shows that $\lambda(C_4 \times C_m) \leq 7$ for $m \geq 4$, and Theorem 8 shows that $\lambda(C_4 \times C_3) = 8$. To complete the proof, it remains to be shown that $\lambda(C_4 \times C_m) \geq 7$ for $m \geq 4$. Suppose by contradiction that this is not the case for some $m \geq 4$ and consider a matrix representation of a 6-labeling of $C_4 \times C_m$. Let us examine the parities of the labels in a column C of this matrix. From items (i) and (iv) in Lemma 9, we have that the labels in C cannot be simultaneously odd, or alternate parities around the cycle.

Let us show that C has more than one even label. Suppose to the contrary that C has exactly one even label. From item (v) of Lemma 9, C must be either $(3, 0, 5, 1)^T$ or $(1, 6, 3, 5)^T$, up to rotating or reversing the order of rows in the matrix. It is enough to show that $(3, 0, 5, 1)^T$ is impossible since $(1, 6, 3, 5)^T$ is its dual, up to rotation and order reversal. A column adjacent to $(3, 0, 5, 1)^T$ must have labels 5 or 6 in the 1^{st} row, 2, 4, or 6 in the 2^{nd} row, 2 or 3 in the 3^{rd} row, and 4 or 6 in the 4^{th} row. Notice that choosing the label 2 for the 2^{nd} row would leave no choice for the 3^{rd} row. Also, choosing label 6 for the 2^{nd} or 4^{th} rows would leave no choice for the 1^{st} row. So, label 4 must be in the 2^{nd} and 4^{th} rows, a contradiction.

Now, let us show that C cannot have exactly one odd label. Suppose to the contrary that C has exactly one odd label. From item (vi) of Lemma 9, C must be $(0, 5, 2, 4)^T$, $(0, 5, 2, 6)^T$, $(0, 3, 6, 2)^T$, $(4, 1, 6, 2)^T$, $(4, 1, 6, 0)^T$ or $(0, 3, 6, 4)^T$, up to rotating or reversing the order of rows in the matrix. Again, it is enough to show that the first three possibilities cannot occur since the last three are duals of the first three, respectively, up to rotation and order reversal. We will only verify that $(0, 5, 2, 4)^T$ is impossible since the other two cases are similar. A column adjacent to $(0, 5, 2, 4)^T$ must have labels 2, 3, or 6 in the 1^{st} row, 1 or 3 in the 2^{nd} row, 0 or 6 in the 3^{rd} row, and 1 or 6 in the 4^{th} row. Notice that choosing the label 6 for the 1^{st} row would force label 0 or 1 for both the 3^{rd} and 4^{th} rows, which is not possible. Choosing the label 2 for the 1^{st} row would leave no choice for the 2^{nd} row. The remaining choice for the 1^{st} row is label 3 which would force label 1 in the 2^{nd} row, leaving only label 6 for the 3^{rd} and 4^{th} rows, which is not possible either.

Our discussion so far implies that any column in the given matrix representation must have all even labels or two consecutive even labels followed by two odd labels, up to row rotation. Let C and C' be two adjacent columns in the matrix. We discuss the following three cases:

<u>Case 1</u>: Both C and C' have even labels only. So $C = (a, b, c, d)^T$ and $C' = (c, d, a, b)^T$ where $\{a, b, c, d\} = \{0, 2, 4, 6\}$. Let C'' be the other column adjacent to C'. From item (i) in Lemma 9 we have that one of the labels in C'' must be even. But this is impossible since each vertex corresponding to a label in C'' is within distance 2 of a vertex labeled 0, 2, 4, and 6 in C or C'.

<u>Case 2</u>: C has only even labels and C' has two consecutive even labels followed by two odd labels, up to row rotation. We must have $C = (a, b, c, d)^T$ and $C' = (c, d, x, y)^T$ where

$\{a, b, c, d\} = \{0, 2, 4, 6\}$ and $\{x, y\} \subseteq \{1, 3, 5\}$, up to rotating rows in the matrix. By item (vi) of Lemma 9, since the vertices labeled x and y are center vertices of two subgraphs isomorphic to $K_{1,2}$ with leaves labeled c and d, we must have $x = y$, a contradiction.

Case 3: C and C' have each two consecutive even labels followed by two odd labels, up to row rotation. First notice that the two odd labels in C cannot be in the same rows as the two odd labels in C', since we would have a subgraph isomorphic to C_4 with all odd labels, contradicting item (i) of Lemma 9. In addition, the two odd labels in C cannot be in the same rows as the two even labels in C' since we would have a subgraph isomorphic to C_4 alternating odd and even labels, contradicting item (iv) of Lemma 9. We must then have exactly one even label in C in the same row as one even label in C'. Up to rotating rows in the matrix, we must have $C = (a, b, x, y)^T$ and $C' = (x, c, a, z)^T$ or $(x, c, d, z)^T$ where $\{a, b, c, d\} = \{0, 2, 4, 6\}$ and $\{x, y, z\} = \{1, 3, 5\}$. By item (vi) of Lemma 9, since the two vertices labeled x are the center vertices of two subgraphs isomorphic to $K_{1,2}$, one with leaves labeled c and d, and the other with leaves labeled [a and b] or [b and d], we must have that $\{a, c\} = \{a, b\}$ or $\{a, c\} = \{b, d\}$, a contradiction.

Since all three cases led to contradictions, we conclude that $\lambda(C_4 \times C_m) \geq 7$ for $m \geq 4$. •

5 Products of Two Cycles with λ-number 6

We completely characterize the family $C_n \times C_m$ with λ-number exactly equal to the general lower bound 6. Notice that item (iii) of Theorem 3 could be used to prove Theorem 10 below, since for $n \geq 4$ and $m \geq 3$, $\lambda(C_n \times C_m) = 6$ implies $\lambda(P_n \times C_m) = 6 = \lambda(C_n \times P_m)$ and consequently n, m are both multiples of 7. Nevertheless, as pointed out in the introduction, we will not use this approach, presenting instead an alternative proof that does not rely on the aide of a computer program.

Theorem 10 $\lambda(C_n \times C_m) = 6$ *if and only if n and m are multiples of 7.*

Proof: In view of Theorem 4 we only need to show that if $\lambda(C_n \times C_m) = 6$ then n and m are multiples of 7. Suppose $\lambda(C_n \times C_m) = 6$ and let M be a matrix representation of a 6-labeling of $C_n \times C_m$. Notice that from Theorems 8 and 9 we must have $n, m \geq 5$. From items (i) and (ii) in Lemma 9, M must contain labels of both parities, so there is a row or a column in M containing both even and odd labels. This row or column must contain a sequence of k consecutive odd labels immediately preceded and followed by even labels. Fact 2 below implies that $1 \leq k \leq 3$, and Facts 3, 4, and 5 show that n and m are multiples of 7 when $k = 3, 2, 1$, respectively.

For convenience, we say that an ordered sequence of labels $(l_1, l_2, ..., l_i)$ *appears in M* if the labels $l_1, l_2, ..., l_i$ are consecutive entries all in the same column or in the same row of M (recall that rows and columns "wrap around," so the last and first entries in a row or column are also considered consecutive). Fact 1 will be instrumental in reducing the case discussion.

Fact 1: The following sequences of labels and the corresponding reverses cannot appear in M:

$$\begin{array}{ccccc}(3,6,1), & (3,1,4), & (5,1,6), & (0,3,1), & (0,3,5),\\ (3,0,5), & (3,5,2), & (1,5,0), & (6,3,5), & (6,3,1).\end{array}$$

A straightforward case discussion shows that the first five sequences above cannot appear in M and consequently the last five sequences cannot appear in M either since they are duals of the first five, respectively. The details are omitted for the sake of brevity.

Fact 2: A sequence of four or more consecutive odd labels immediately preceded or followed by an even label cannot appear in M.

Verification: The only sequences of four odd labels that could appear in M are, up to taking reversals: $(1,3,5,1)$, $(3,1,5,3)$, $(5,3,1,5)$. To prove Fact 2, it is enough to show that the first two sequences when immediately preceded or followed by an even label cannot appear in M since the last one is the dual of the first sequence.

Let us consider the second sequence, $(3,1,5,3)$. By adding an even label to the beginning or the end of this sequence we obtain the following possible sequences: $(0,\underline{3,1,5,3})$, $(6,\underline{3,1,5,3})$, $(\underline{3,1,5,3},0)$, $(\underline{3,1,5,3},6)$. They contain, respectively, $(0,3,1)$, $(6,3,1)$, $(5,3,0)$, $(5,3,6)$, and by Fact 1 we conclude that none can appear in M.

Finally, let us focus on the first sequence, $(1,3,5,1)$. By adding an even label to the beginning or the end of this sequence we obtain the following possible sequences: $(4,\underline{1,3,5,1})$, $(6,\underline{1,3,5,1})$, $(\underline{1,3,5,1},4)$, $(\underline{1,3,5,1},6)$. The first and last sequences contain, respectively, $(4,1,3)$ and $(5,1,6)$, and by Fact 1 we conclude that they cannot appear in M. The second and third sequences $(6,1,3,5,1)$ and $(1,3,5,1,4)$ force in M the submatrices in Figure 11, respectively, up to reversing the order of rows/columns or transposing rows and columns, where the symbol $*$ represents a label that is not relevant to the discussion, and $\#$ is an entry for which there are no possible labels. (For convenience, whenever we say that a submatrix is forced in M, it will be assumed that it is forced up to reversing the order of rows/columns or transposing rows and columns.) The last row of the first submatrix contains the sequence

$$\begin{pmatrix} * & * & \mathbf{6} & * \\ 0 & 4 & \mathbf{1} & 5 \\ 2 & 6 & \mathbf{3} & 0 \\ 4 & 0 & \mathbf{5} & 2 \\ 6 & 3 & \mathbf{1} & * \end{pmatrix} \quad \begin{pmatrix} * & * & \mathbf{1} & * \\ * & 0 & \mathbf{3} & 6 \\ 4 & 2 & \mathbf{5} & 0 \\ 0 & 6 & \mathbf{1} & 3 \\ * & \# & \mathbf{4} & * \end{pmatrix}$$

Figure 11:

$(6,3,1)$, which is forbidden by Fact 1 and we conclude that $(6,1,3,5,1)$ cannot appear in M. The second submatrix shows that $(1,3,5,1,4)$ cannot appear in M either. $\diamondsuit$

Fact 3: If a sequence of three consecutive odd labels immediately preceded and followed by even labels appears in M, then it must be $(6,1,3,5,0)$, up to taking reversals, forcing n and m to be multiples of 7.

Verification: The only sequences of three odd labels that could appear in M are, up to taking reversals, $(1,3,5)$, $(1,5,3)$, $(3,1,5)$. It is enough to examine the first two sequences since the reverse of the third is the dual of the second.

By adding an even label to the end of the second sequence $(1,5,3)$ we get $(\underline{1,5,3},0)$ or $(\underline{1,5,3},6)$, which would contain $(5,3,0)$ and $(5,3,6)$, respectively, contradicting Fact 1. Therefore $(1,5,3)$ followed by an even label cannot appear in M.

Let us add one even label at the beginning and one at the end of the first sequence $(1,3,5)$. The label 4 cannot precede $(1,3,5)$ since otherwise $(4,1,3)$ would appear in M, contradicting Fact 1. Similarly, the label 2 cannot follow $(1,3,5)$ since otherwise $(3,5,2)$ would appear in M, again contradicting Fact 1. So the only possible sequence, up to taking reversals, is $(6,\underline{1,3,5},0)$. This sequence (in bold) forces in M the submatrix M' in Figure 12. Let us expand M' by examining the frame of two rows and two columns around it. The

$$M' = \begin{pmatrix} 5 & 2 & \mathbf{6} & 3 & 0 \\ 0 & 4 & \mathbf{1} & 5 & 2 \\ 2 & 6 & \mathbf{3} & 0 & 4 \\ 4 & 1 & \mathbf{5} & 2 & 6 \\ 6 & 3 & \mathbf{0} & 4 & 1 \end{pmatrix}$$

Figure 12:

submatrix M'' in Figure 13 is forced in M.

$$M'' = \left(\begin{array}{c|ccccccc|c} 0 & 4 & 1 & 5 & 2 & 6 & 3 & 0 & 4 \\ \hline 2 & \mathbf{6} & 3 & 0 & 4 & 1 & 5 & 2 & 6 \\ 4 & \mathbf{1} & 5 & 2 & \mathbf{6} & 3 & 0 & 4 & 1 \\ 6 & \mathbf{3} & 0 & 4 & \mathbf{1} & 5 & 2 & \mathbf{6} & 3 \\ 1 & \mathbf{5} & 2 & 6 & \mathbf{3} & 0 & 4 & \mathbf{1} & 5 \\ 3 & \mathbf{0} & 4 & 1 & \mathbf{5} & 2 & 6 & \mathbf{3} & 0 \\ 5 & 2 & 6 & 3 & \mathbf{0} & 4 & 1 & \mathbf{5} & 2 \\ 0 & 4 & 1 & 5 & 2 & 6 & 3 & \mathbf{0} & 4 \\ \hline 2 & 6 & 3 & 0 & 4 & 1 & 5 & 2 & 6 \end{array}\right)$$

Figure 13:

The submatrix N of M'', obtained by deleting the first and last rows and columns of M'', will repeat across the columns and down the rows as we use the same arguments starting at a different occurrence of the $(6,1,3,5,0)$ sequence (in bold). Notice that no submatrix of N with 7 rows and k columns, $3 \leq k \leq 6$, "wrap around" the columns, and no submatrix of N with 7 columns and p rows, $3 \leq p \leq 6$, "wrap around" the rows. Therefore, up to reversing the order of rows/columns or transposing rows and columns, M must be of the form $\alpha N \beta$ where α and β are positive integers (recall the matrix notation introduced at the beginning of Section 2), and consequently, n and m are multiples of 7. $\diamondsuit$

<u>Fact 4</u>: If a sequence of two consecutive odd labels immediately preceded and followed by even labels appears in M, then it must be $(4,1,5,2)$, up to taking reversals, forcing n and m to be multiples of 7.

<u>Verification</u>: The only sequences of two odd labels that could appear in M are, up to taking reversals, $(1,3)$, $(1,5)$, $(5,3)$. It is enough to examine the first two sequences since the third is the dual of the first.

By adding an even label to the end of the first sequence $(1,3)$ we get either $(1,3,0)$ or $(1,3,6)$, contradicting Fact 1. Therefore $(1,3)$ followed by an even label cannot appear in M. Let us add an even label to the beginning and to the end of the second sequence $(1,5)$. The label 6 cannot precede $(1,5)$ since $(6,1,5)$ cannot appear in M by Fact 1. The label 0 cannot follow $(1,5)$ since $(1,5,0)$ cannot appear in M by Fact 1. So the only possibility is $(4,\underline{1,5},2)$ which forces in M the submatrix in Figure 14, where Y is 0 or 1. If $Y=1$ then $(6,1,3,5,1)$ would appear in M contradicting Fact 2. So $Y=0$ forcing $(6,1,3,5,0)$ to appear in M and by Fact 3 we conclude that n and m are multiples of 7. $\diamondsuit$

$$\begin{pmatrix} * & \mathbf{4} & 6 & * & * \\ 6 & \mathbf{1} & 3 & 5 & Y \\ 3 & \mathbf{5} & 0 & 2 & * \\ * & \mathbf{2} & * & * & * \end{pmatrix}$$

Figure 14:

<u>Fact 5</u>: If a sequence of one odd label immediately preceded and followed by even labels appears in M, then it must be $(0,3,6)$, up to taking reversals, forcing n and m to be positive multiples of 7.
<u>Verification</u>: The only possible sequences of one odd label preceded and followed by even labels are, up to taking reversals, $(4,1,6)$, $(0,3,6)$, $(2,5,0)$. It is enough to examine the first two sequences since the third is the dual of the first.

Let us first examine $(4,1,6)$. Label 6 cannot precede $(4,1,6)$ since otherwise the submatrix (i) in Figure 14 would be forced in M. Label 3 cannot follow $(4,1,6)$ since otherwise $(1,6,3)$ would appear in M, contradicting Fact 1. Label 0 cannot follow $(4,1,6)$ since otherwise the submatrix (ii) in Figure 15 would be forced in M. So, by adding a label preceding

$$\begin{pmatrix} * & \mathbf{6} & * \\ 0 & \mathbf{4} & 2 \\ 3 & \mathbf{1} & 5 \\ \# & \mathbf{6} & * \end{pmatrix} \qquad \begin{pmatrix} * & \mathbf{4} & * \\ 3 & \mathbf{1} & 5 \\ \# & \mathbf{6} & * \\ * & \mathbf{0} & * \end{pmatrix}$$

(i) (ii)

Figure 15:

and following $(4,1,6)$ we are left with the sequences $(0,\underline{4,1,6},2)$, $(2,\underline{4,1,6},2)$, $(0,\underline{4,1,6},4)$, $(2,\underline{4,1,6},4)$ and they force in M the submatrices in Figure 16, respectively.

$$\begin{pmatrix} * & \mathbf{0} & * \\ 6 & \mathbf{4} & 2 \\ 3 & \mathbf{1} & 5 \\ 0 & \mathbf{6} & 3 \\ * & \mathbf{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} * & \mathbf{2} & * \\ 6 & \mathbf{4} & 0 \\ 3 & \mathbf{1} & 5 \\ 0 & \mathbf{6} & 3 \\ * & \mathbf{2} & 0 \end{pmatrix} \qquad \begin{pmatrix} * & * & \mathbf{0} & * \\ * & 6 & \mathbf{4} & 2 \\ 5 & 3 & \mathbf{1} & 5 \\ \# & 0 & \mathbf{6} & 3 \\ * & 2 & \mathbf{4} & * \end{pmatrix} \qquad \begin{pmatrix} * & * & \mathbf{2} & * \\ * & 6 & \mathbf{4} & 0 \\ 5 & 3 & \mathbf{1} & 5 \\ \# & 0 & \mathbf{6} & * \\ * & 2 & \mathbf{4} & * \end{pmatrix}$$

Figure 16:

The 3^{rd} column in the first two submatrices contain a sequence of two consecutive odd labels immediately preceded and followed by even labels different, up to reversal, from $(4, 1, 5, 2)$, so by Fact 4, the corresponding cases are not possible. Therefore we conclude that $(4, 1, 6)$ cannot appear in M.

We next examine the sequence $(0, 3, 6)$. Label 2 cannot precede $(0, 3, 6)$ since otherwise the submatrix (i) in Figure 17 would be forced in M. Label 5 cannot precede $(0, 3, 6)$ since otherwise $(5, 0, 3)$ would appear in M, contradicting Fact 1. If label 4 precedes $(0, 3, 6)$ then the submatrix (ii) in Figure 17 is forced in M where Y is 5 or 6, and Z is 0 or 1.

$$\begin{pmatrix} * & \mathbf{2} & * \\ * & \mathbf{0} & \# \\ 1 & \mathbf{3} & 5 \\ * & \mathbf{6} & * \end{pmatrix} \qquad \begin{pmatrix} * & * & \mathbf{4} & * & * \\ * & * & \mathbf{0} & 2 & * \\ Y & 1 & \mathbf{3} & 5 & Z \\ * & 4 & \mathbf{6} & * & * \end{pmatrix}$$

(i) (ii)

Figure 17:

By Fact 2 we must have that $Y = 6$ and $Z = 0$. Therefore since the third row of this submatrix is $(6, 1, 3, 5, 0)$, Fact 3 implies that n and m are multiples of 7. Label 1 cannot follow $(0, 3, 6)$ since otherwise $(3, 6, 1)$ would appear in M, contradicting Fact 1. The only remaining sequences when adding a label preceding and following $(0, 3, 6)$ are $(6, \underline{0, 3, 6}, 0)$, $(6, \underline{0, 3, 6}, 2)$, $(6, \underline{0, 3, 6}, 4)$. The last two sequences are the reverse of the duals of $(4, \underline{0, 3, 6}, 0)$ and $(2, 0, 3, 6, 0)$, respectively, which contain $(4, 0, 3, 6)$ and $(2, 0, 3, 6)$, respectively, cases that were previously discussed. The sequence $(6, \underline{0, 3, 6}, 0)$ forces in M the submatrix in Figure 18.

$$\begin{pmatrix} * & * & \mathbf{6} & * & * \\ * & 5 & \mathbf{0} & 2 & * \\ 6 & 1 & \mathbf{3} & 5 & 0 \\ * & 4 & \mathbf{6} & 1 & * \\ * & * & \mathbf{0} & * & * \end{pmatrix}$$

Figure 18:

Since the third row of this submatrix is $(6, 1, 3, 5, 0)$, Fact 3 implies that n and m are multiples of 7. $\diamondsuit$ $\bullet$

Theorem 10 is used to show the following two results.

Theorem 11 *If $n, m \geq 3$, $n \in S(4, 7)$ and $m \in S(5, 8)$ then*

$$\lambda(C_n \times C_m) = \begin{cases} 6, & \textit{if } n \textit{ and } m \textit{ are multiples of 7} \\ 7, & \textit{otherwise.} \end{cases}$$

Proof: This result is an immediate consequence of Theorems 7 and 10. $\bullet$

Theorem 12 *Let $n, m \geq 6$ so that n and m are not simultaneously multiples of 7. If either*

(i) n is a multiple of 3 and m is an even number different from 4 and 10, or

(ii) n is a multiple of 4,

then $\lambda(C_n \times C_m) = 7$.

Proof: From Theorem 10, we have that $\lambda(C_n \times C_m) \geq 7$. If (i) (resp., (ii)) holds, then from Theorem 8 (resp., Theorem 9), $C_3 \times C_m$ (resp., $C_4 \times C_m$) has a 7-labeling given by a matrix M. Therefore by Lemma 2, if $n = 3k$ (resp., $n = 4k$), $k = 1, 2, ...$, the block matrix $kM1$ is a 7-labeling of $C_n \times C_m$ and therefore $\lambda(C_n \times C_m) = 7$. •

6 The Remaining Cases

If $n, m \geq 3$, Theorems 8, 9, 11 and 12, provide complete results on $\lambda(C_n \times C_m)$ except when $n \in \{5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 19, 22, 27\}$. In this section we compute $\lambda(C_n \times C_m)$ for these remaining cases in Lemmas 10—22, respectively, finally establishing Theorem 5. Unlike previous sections, we use computer programs to either generate 7-labelings or to show that they do not exist for specific values of n and m.

Recall Table 1 in Section 2. For convenience we define the set $T(n)$ as the set of all integers $i \geq n$ so that the square in row number n and column number i in Table 1 does not contain the symbol *. Since in Theorem 11 we calculated $\lambda(C_n \times C_m)$ for $n, m \geq 3$ so that $m \notin T(n)$, we will focus on the cases where $m \in T(n)$ and n or m is not a multiple of 7 in the remaining Lemmas. For such cases, to show that $\lambda(C_n \times C_m) = 7$ or 8, it is enough to exhibit a 7-labeling of $C_n \times C_m$ or to show that one does not exist, respectively. For the sake of brevity and clarity, some of the matrix representations of 7-labelings of $C_n \times C_m$ generated by a computer program are provided in the Appendix at the end of this paper and will be simply referred to by $M(n, m)$.

Lemma 10 *If $m \geq 5$ then*

$$\lambda(C_5 \times C_m) = \begin{cases} 8, & \text{if } m \in \{5, 6, 9, 10, 13, 17\} \\ 7, & \text{otherwise.} \end{cases}$$

Proof: First notice that $T(5) = \{5, 6, 9, 10, 13, 17\}$. Using a computer program we verified that $C_5 \times C_m$ does not have a 7-labeling if $m \in T(5)$. •

Lemma 11 *If $m \geq 6$ then*

$$\lambda(C_6 \times C_m) = \begin{cases} 8, & \text{if } m \in \{7, 11\} \\ 7, & \text{otherwise.} \end{cases}$$

$$X = \begin{pmatrix} 0 & 4 & 7 \\ 3 & 6 & 1 \\ 7 & 0 & 4 \\ 1 & 3 & 6 \\ 4 & 7 & 0 \\ 6 & 1 & 3 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 4 & 7 & 2 & 6 & 1 & 5 & 0 & 3 & 5 \\ 3 & 6 & 1 & 5 & 0 & 3 & 7 & 2 & 6 & 1 \\ 7 & 2 & 4 & 7 & 2 & 6 & 1 & 5 & 0 & 4 \\ 1 & 5 & 0 & 3 & 5 & 0 & 3 & 7 & 2 & 6 \\ 4 & 7 & 2 & 6 & 1 & 4 & 6 & 1 & 5 & 0 \\ 6 & 1 & 5 & 0 & 3 & 7 & 2 & 4 & 7 & 2 \end{pmatrix}$$

Figure 19:

Proof: The matrices X and Y in Figure 19 plus the block matrix $[X|Y]$ are 7-labelings of $C_6 \times C_3$, $C_6 \times C_{10}$, and $C_6 \times C_{13}$, respectively.

If $m \in S(3, 10)$ and $m \geq 6$, then there are nonnegative integers α and β so that $m = 3\alpha + 10\beta$, and Lemma 3 with the given matrices X and Y shows that $C_6 \times C_m$ has a 7-labeling.

Notice that the only values $m \geq 6$ that are not in $S(3, 10)$ belong to $\{7, 8, 11, 14, 17\}$. We have that $\lambda(C_6 \times C_m) = 7$ for $m = 8, 14$ by item (i) of Theorem 12. A computer program verified that $C_6 \times C_m$ does not have a 7-labeling if $m \in \{7, 11\}$. $M(6, 17)$ completes the proof. •

Lemma 12 *If* $m \geq 7$ *then*

$$\lambda(C_7 \times C_m) = \begin{cases} 8, & \textit{if } m = 9 \\ 6, & \textit{if } m \textit{ is a multiple of } 7 \\ 7, & \textit{otherwise.} \end{cases}$$

Proof: If m is a multiple of 7 then $\lambda(C_7 \times C_m) = 6$, from Theorem 10. Let us assume m is not a multiple of 7. The matrices X and Y in Figure 20 plus the block matrix $[X|Y]$ are 7-labelings of $C_7 \times C_4$, $C_7 \times C_7$, and $C_7 \times C_{11}$, respectively.

$$X = \begin{pmatrix} 0 & 6 & 4 & 2 \\ 3 & 1 & 7 & 5 \\ 6 & 4 & 2 & 0 \\ 2 & 0 & 6 & 4 \\ 5 & 3 & 1 & 7 \\ 1 & 7 & 5 & 3 \\ 4 & 2 & 0 & 6 \end{pmatrix} \quad Y = \begin{pmatrix} 7 & 5 & 2 & 0 & 6 & 4 & 2 \\ 3 & 0 & 6 & 3 & 1 & 7 & 5 \\ 6 & 4 & 1 & 7 & 4 & 2 & 0 \\ 1 & 7 & 5 & 2 & 0 & 6 & 4 \\ 5 & 2 & 0 & 6 & 3 & 1 & 7 \\ 0 & 6 & 3 & 1 & 7 & 5 & 3 \\ 4 & 1 & 7 & 4 & 2 & 0 & 6 \end{pmatrix}$$

Figure 20:

If $m \in S(4, 7)$ and $m \geq 7$, then there are nonnegative integers α and β so that $m = 4\alpha + 7\beta$, and Lemma 2 with the given matrices X and Y shows that $C_7 \times C_m$ has a 7-labeling.

Notice that the only values $m \geq 7$ that are not in $S(4, 7)$ belong to $\{9, 10, 13, 17\}$. We have that $\lambda(C_7 \times C_{10}) = \lambda(C_7 \times C_{13}) = 7$ since $10, 13 \notin T(7)$. A computer program verified that $C_7 \times C_9$ does not have a 7-labeling. $M(7, 17)$ completes the proof. •

Lemma 13 *If* $m \geq 9$ *then*

$$\lambda(C_9 \times C_m) = \begin{cases} 8, & \textit{if } m = 10 \\ 7, & \textit{otherwise.} \end{cases}$$

Proof: If $m \geq 9$ is even and different from 10, then the result follows from item (i) in Theorem 12.

Since any $m \geq 9$, m odd can be written as $8k+1$, $8k+3$, $8k+5$ or $8k+7$, for some integer $k \geq 1$, m can also be written as $8(k-4)+3(11)$, if $k \geq 4$, or $8(k-1)+11$, $8(k-1)+13$, $8(k-1)+15$, respectively, if $k \geq 1$, except for $m \in \{9, 17, 25\}$. For each $i = 11, 13, 15$, the pair of matrices $X(i)$ and $Y(i)$ in Figure 21, and the block matrix $[X(i)|Y(i)]$ are 7-labelings of $\mathrm{C}_9\times\mathrm{C}_8$, $\mathrm{C}_9\times\mathrm{C}_i$, and $C_9 \times C_{8+i}$, respectively. Therefore by Lemma 2, we can conclude that $\lambda(C_9 \times C_m) = 7$ if $m \geq 9$, m odd and $m \notin \{9, 17, 25\}$.

$$X(11) = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 7 & 1 & 4 \\ 2 & 5 & 0 & 3 & 6 & 0 & 3 & 6 \\ 4 & 7 & 2 & 5 & 1 & 4 & 7 & 1 \\ 0 & 3 & 6 & 0 & 3 & 6 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 1 & 4 & 7 \\ 4 & 0 & 3 & 6 & 0 & 3 & 6 & 1 \\ 6 & 2 & 5 & 1 & 4 & 7 & 0 & 3 \\ 1 & 4 & 0 & 3 & 6 & 1 & 4 & 7 \\ 5 & 7 & 2 & 5 & 0 & 3 & 6 & 2 \end{pmatrix} \quad Y(11) = \begin{pmatrix} 0 & 3 & 6 & 1 & 3 & 6 & 1 & 4 & 7 & 1 & 4 \\ 2 & 5 & 0 & 4 & 7 & 0 & 3 & 6 & 0 & 3 & 6 \\ 4 & 7 & 2 & 6 & 1 & 4 & 7 & 1 & 4 & 7 & 1 \\ 0 & 3 & 5 & 0 & 3 & 6 & 0 & 3 & 6 & 2 & 5 \\ 2 & 6 & 1 & 4 & 7 & 1 & 4 & 7 & 0 & 4 & 7 \\ 4 & 0 & 3 & 6 & 0 & 3 & 6 & 1 & 3 & 6 & 1 \\ 6 & 2 & 5 & 1 & 4 & 7 & 0 & 4 & 7 & 0 & 3 \\ 1 & 4 & 0 & 3 & 6 & 1 & 3 & 6 & 1 & 4 & 7 \\ 5 & 7 & 2 & 5 & 0 & 4 & 7 & 0 & 3 & 6 & 2 \end{pmatrix}$$

$$X(13) = \begin{pmatrix} 0 & 3 & 6 & 1 & 4 & 6 & 1 & 4 \\ 2 & 5 & 0 & 3 & 7 & 0 & 3 & 6 \\ 4 & 7 & 2 & 5 & 1 & 4 & 7 & 1 \\ 0 & 3 & 6 & 0 & 3 & 6 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 1 & 4 & 7 \\ 4 & 0 & 3 & 6 & 0 & 3 & 6 & 1 \\ 7 & 2 & 5 & 1 & 4 & 7 & 0 & 3 \\ 1 & 4 & 0 & 3 & 6 & 1 & 4 & 6 \\ 5 & 7 & 2 & 5 & 0 & 3 & 7 & 2 \end{pmatrix} \quad Y(13) = \begin{pmatrix} 7 & 3 & 6 & 1 & 4 & 7 & 1 & 4 & 7 & 2 & 6 & 1 & 4 \\ 2 & 5 & 0 & 3 & 6 & 2 & 5 & 0 & 3 & 5 & 0 & 3 & 6 \\ 4 & 7 & 2 & 5 & 1 & 4 & 7 & 2 & 6 & 1 & 4 & 7 & 1 \\ 0 & 3 & 6 & 0 & 3 & 6 & 1 & 5 & 0 & 3 & 6 & 2 & 5 \\ 2 & 5 & 1 & 4 & 7 & 0 & 4 & 7 & 2 & 5 & 1 & 4 & 7 \\ 4 & 0 & 3 & 6 & 1 & 3 & 6 & 1 & 4 & 0 & 3 & 6 & 1 \\ 7 & 2 & 5 & 0 & 4 & 7 & 2 & 5 & 7 & 2 & 5 & 0 & 3 \\ 1 & 4 & 7 & 3 & 6 & 1 & 4 & 0 & 3 & 6 & 1 & 4 & 6 \\ 5 & 0 & 2 & 5 & 0 & 3 & 6 & 2 & 5 & 0 & 3 & 7 & 2 \end{pmatrix}$$

$$X(15) = \begin{pmatrix} 0 & 4 & 1 & 6 & 4 & 1 & 6 & 3 \\ 2 & 7 & 3 & 0 & 7 & 3 & 0 & 5 \\ 4 & 1 & 6 & 4 & 1 & 6 & 2 & 7 \\ 0 & 5 & 2 & 7 & 3 & 0 & 5 & 3 \\ 2 & 7 & 4 & 1 & 6 & 4 & 1 & 6 \\ 4 & 1 & 6 & 3 & 0 & 7 & 3 & 0 \\ 7 & 5 & 2 & 7 & 4 & 1 & 6 & 2 \\ 3 & 0 & 4 & 1 & 6 & 3 & 0 & 5 \\ 6 & 2 & 7 & 3 & 0 & 7 & 4 & 1 \end{pmatrix} \quad Y(15) = \begin{pmatrix} 0 & 4 & 1 & 6 & 2 & 0 & 4 & 2 & 5 & 1 & 6 & 3 & 1 & 6 & 3 \\ 2 & 6 & 3 & 0 & 5 & 3 & 6 & 0 & 7 & 4 & 2 & 7 & 4 & 0 & 5 \\ 4 & 1 & 7 & 4 & 1 & 7 & 2 & 5 & 3 & 0 & 5 & 1 & 6 & 2 & 7 \\ 0 & 5 & 2 & 6 & 3 & 0 & 4 & 1 & 6 & 2 & 7 & 3 & 0 & 5 & 3 \\ 2 & 7 & 4 & 0 & 5 & 2 & 6 & 3 & 0 & 4 & 1 & 6 & 4 & 1 & 6 \\ 5 & 1 & 6 & 3 & 7 & 4 & 0 & 5 & 2 & 7 & 5 & 2 & 7 & 3 & 0 \\ 7 & 4 & 2 & 5 & 0 & 6 & 3 & 1 & 6 & 3 & 0 & 4 & 1 & 6 & 2 \\ 3 & 0 & 7 & 1 & 4 & 2 & 7 & 4 & 0 & 5 & 2 & 6 & 3 & 0 & 5 \\ 6 & 2 & 5 & 3 & 7 & 5 & 1 & 6 & 3 & 7 & 4 & 0 & 7 & 4 & 1 \end{pmatrix}$$

Figure 21:

A computer program verified that $C_9 \times C_{10}$ does not have a 7-labeling. $M(9, j)$ for $j \in \{9, 17, 25\}$ complete the proof. •

Lemma 14 *If $m \geq 10$ then $\lambda(C_{10} \times C_m) = 7$.*

Proof: First notice that $T(10) = \{10, 13, 17\}$. The proof follows from $M(10, 10)$, $M(10, 13)$, and $M(10, 17)$. •

Lemma 15 *If $m \geq 11$ then $\lambda(C_{11} \times C_m) = 7$.*

Proof: First notice that $T(11) = \{11, 12, 14, 17, 19, 22, 27\}$. From Lemma 12 and Lemma 13, $C_7 \times C_{11}$ and $C_9 \times C_{11}$ have 7-labelings given by matrices X and Y, respectively. Therefore, the block matrices $1X^T2$, $1M(11, 11)2$, and $1Y^T3$ are 7-labelings of $C_{11} \times C_{14}$, $C_{11} \times C_{22}$, and $C_{11} \times C_{27}$, respectively. Item (ii) of Theorem 12 implies that $\lambda(C_{11} \times C_{12}) = 7$. The proof is complete with $M(11, 17)$ and $M(11, 19)$. •

Lemma 16 *If $m \geq 13$ then $\lambda(C_{13} \times C_m) = 7$.*

Proof: First notice that $T(13) = \{13, 17\}$. The proof follows from $M(13, 13)$ and $M(13, 17)$. •

Lemma 17 *If $m \geq 14$ then*

$$\lambda(C_{14} \times C_m) = \begin{cases} 6, & \textit{if } m \textit{ is a multiple of 7} \\ 7, & \textit{otherwise.} \end{cases}$$

Proof: From Theorem 10, $\lambda(C_{14} \times C_m) = 6$ if m is a multiple of 7. Notice that $T(14) = \{14, 17, 19, 22, 27\}$. By Lemma 12, if m is in $\{17, 19, 22, 27\}$ then $C_7 \times C_m$ has a 7-labeling given by a matrix X and the block matrix $2X1$ is a 7-labeling of $C_{14} \times C_m$. •

Lemma 18 *If $m \geq 15$ then $\lambda(C_{15} \times C_m) = 7$.*

Proof: First notice that $T(15) = \{17\}$. The block matrix $[X|Y]^T$ with the matrices X and Y given in the proof of Lemma 19 below is a 7-labeling of $C_{15} \times C_{17}$. •

Lemma 19 *If $m \geq 17$ then $\lambda(C_{17} \times C_m) = 7$.*

Proof: The matrices X and Y given in Figure 22 plus the block matrix $[X|Y]$ are 7-labelings of $C_{17} \times C_4$, $C_{17} \times C_{11}$, and $C_{17} \times C_{15}$, respectively.

$$X = \begin{pmatrix} 0 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 \\ 6 & 0 & 2 & 4 \\ 1 & 3 & 5 & 7 \\ 4 & 6 & 0 & 2 \\ 0 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 \\ 6 & 0 & 2 & 4 \\ 2 & 4 & 6 & 0 \\ 5 & 7 & 1 & 3 \\ 0 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 \\ 6 & 0 & 2 & 4 \\ 2 & 4 & 6 & 0 \\ 5 & 7 & 1 & 3 \\ 1 & 3 & 5 & 7 \\ 4 & 6 & 0 & 2 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 3 & 5 \\ 3 & 5 & 7 & 1 & 3 & 5 & 7 & 1 & 4 & 6 & 1 \\ 6 & 0 & 2 & 4 & 6 & 0 & 2 & 5 & 7 & 2 & 4 \\ 1 & 3 & 5 & 7 & 1 & 3 & 6 & 0 & 3 & 5 & 7 \\ 4 & 6 & 0 & 2 & 4 & 7 & 1 & 4 & 6 & 0 & 2 \\ 0 & 2 & 4 & 6 & 0 & 2 & 5 & 7 & 1 & 3 & 5 \\ 3 & 5 & 7 & 1 & 3 & 6 & 0 & 2 & 4 & 6 & 1 \\ 6 & 0 & 2 & 4 & 7 & 1 & 3 & 5 & 7 & 0 & 4 \\ 2 & 4 & 6 & 0 & 2 & 4 & 6 & 0 & 3 & 5 & 7 \\ 5 & 7 & 1 & 3 & 5 & 7 & 1 & 4 & 6 & 1 & 3 \\ 0 & 2 & 4 & 6 & 0 & 2 & 5 & 7 & 2 & 4 & 6 \\ 3 & 5 & 7 & 1 & 3 & 6 & 0 & 3 & 5 & 7 & 1 \\ 6 & 0 & 2 & 4 & 7 & 1 & 4 & 6 & 0 & 2 & 4 \\ 2 & 4 & 6 & 0 & 2 & 5 & 7 & 1 & 3 & 5 & 0 \\ 5 & 7 & 1 & 3 & 6 & 0 & 2 & 4 & 6 & 1 & 3 \\ 1 & 3 & 5 & 7 & 1 & 3 & 5 & 0 & 2 & 4 & 6 \\ 4 & 6 & 0 & 2 & 4 & 6 & 1 & 3 & 5 & 7 & 2 \end{pmatrix}$$

Figure 22:

If $m \in S(4, 11)$ and $m \geq 17$, then there are nonnegative integers α and β so that $m = 4\alpha + 11\beta$, and Lemma 2 with the given matrices X and Y shows that $C_{17} \times C_m$ has a 7-labeling.

Notice that the only values of $m \geq 17$ that are not in $S(4,11)$ are the ones in $\{17, 18, 21, 25, 29\}$. The block matrix $1M(7,17)^T3$ is a 7-labeling of $C_{17} \times C_{21}$. The block matrix $1M(9,17)^T2$ is a 7-labeling of $C_{17} \times C_{18}$. $M(17,17)$, $M(17,25)$, and $M(17,29)$ complete the proof. •

Lemma 20 *If* $m \geq 19$ *then* $\lambda(C_{19} \times C_m) = 7$.

Proof: Notice first that $T(19) = \{19, 22, 27\}$. By Lemma 13, $C_9 \times C_{19}$ has a 7-labeling given by a matrix X, so the block matrix $1X^T3$ is a 7-labeling of $C_{19} \times C_{27}$. The block matrix $1M(11,19)^T2$ is a 7-labeling of $C_{19} \times C_{22}$. $M(19,19)$ completes the proof. •

Lemma 21 *If* $m \geq 22$ *then* $\lambda(C_{22} \times C_m) = 7$.

Proof: First notice that $T(22) = \{22, 27\}$. By Lemma 13 and Lemma 15, $C_9 \times C_{22}$ and $C_{11} \times C_{22}$ have 7-labelings given by matrices X and Y, respectively, so the block matrices $1X^T3$ and $1Y^T2$ are 7-labelings of $C_{22} \times C_{27}$ and $C_{22} \times C_{22}$, respectively. •

Lemma 22 *If* $m \geq 27$ *then* $\lambda(C_{27} \times C_m) = 7$.

Proof: First notice that $T(27) = \{27\}$. By Lemma 13, $C_9 \times C_{27}$ has a 7-labeling given by a matrix X, so the block matrix $1X^T3$ is a 7-labeling of $C_{27} \times C_{27}$. •

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Appendix

$$M(6,17) = \begin{pmatrix} 0 & 3 & 6 & 1 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 \\ 2 & 5 & 0 & 4 & 7 & 1 & 4 & 2 & 7 & 0 & 2 & 5 & 1 & 4 & 7 & 3 & 6 \\ 4 & 7 & 3 & 6 & 2 & 5 & 7 & 0 & 5 & 3 & 6 & 0 & 3 & 6 & 0 & 5 & 1 \\ 0 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 7 & 1 & 4 & 7 & 1 & 4 & 2 & 7 \\ 3 & 6 & 0 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 5 & 7 & 0 & 5 \\ 7 & 1 & 4 & 7 & 0 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 \end{pmatrix}$$

$$M(7,17) = \begin{pmatrix} 0 & 3 & 7 & 2 & 4 & 7 & 0 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 5 \\ 2 & 5 & 0 & 6 & 1 & 5 & 2 & 4 & 0 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 7 \\ 4 & 7 & 2 & 4 & 7 & 0 & 6 & 1 & 5 & 7 & 4 & 0 & 3 & 7 & 2 & 6 & 1 \\ 0 & 3 & 6 & 1 & 5 & 2 & 4 & 7 & 3 & 0 & 2 & 6 & 1 & 4 & 0 & 3 & 5 \\ 2 & 5 & 0 & 7 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 & 5 & 2 & 6 & 1 & 7 \\ 4 & 7 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 7 & 0 & 3 & 5 & 0 \\ 6 & 1 & 4 & 0 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 5 & 1 & 7 & 3 \end{pmatrix}$$

$$M(9,9) = \begin{pmatrix} 0 & 6 & 1 & 3 & 6 & 4 & 1 & 6 & 3 \\ 2 & 4 & 7 & 0 & 2 & 7 & 3 & 0 & 7 \\ 6 & 0 & 2 & 6 & 4 & 0 & 6 & 2 & 4 \\ 3 & 7 & 4 & 1 & 7 & 2 & 4 & 7 & 1 \\ 0 & 2 & 6 & 3 & 0 & 6 & 1 & 3 & 5 \\ 6 & 4 & 0 & 7 & 2 & 4 & 7 & 0 & 2 \\ 1 & 7 & 2 & 4 & 6 & 1 & 3 & 6 & 4 \\ 3 & 0 & 6 & 1 & 3 & 5 & 0 & 2 & 7 \\ 5 & 2 & 4 & 7 & 0 & 2 & 7 & 4 & 1 \end{pmatrix}$$

$$M(9,17) = \left(\begin{array}{ccccccccccccccccc}
0 & 3 & 7 & 2 & 0 & 4 & 1 & 3 & 6 & 1 & 3 & 6 & 0 & 3 & 6 & 2 & 5 \\
2 & 5 & 0 & 6 & 3 & 7 & 5 & 0 & 4 & 7 & 0 & 4 & 2 & 7 & 1 & 4 & 7 \\
4 & 7 & 3 & 1 & 5 & 0 & 3 & 7 & 1 & 5 & 2 & 7 & 5 & 0 & 3 & 6 & 1 \\
0 & 2 & 6 & 4 & 2 & 6 & 1 & 5 & 3 & 0 & 6 & 3 & 1 & 4 & 7 & 2 & 5 \\
3 & 5 & 1 & 7 & 0 & 3 & 7 & 0 & 6 & 2 & 4 & 0 & 7 & 2 & 5 & 0 & 7 \\
1 & 7 & 4 & 2 & 6 & 1 & 5 & 3 & 1 & 5 & 7 & 2 & 5 & 0 & 3 & 6 & 4 \\
5 & 2 & 6 & 0 & 3 & 7 & 0 & 6 & 4 & 0 & 3 & 6 & 1 & 4 & 7 & 2 & 0 \\
7 & 0 & 3 & 7 & 1 & 5 & 3 & 1 & 7 & 2 & 5 & 0 & 3 & 6 & 0 & 5 & 3 \\
4 & 6 & 1 & 4 & 6 & 2 & 7 & 5 & 0 & 4 & 7 & 2 & 5 & 1 & 4 & 7 & 1
\end{array}\right)$$

$$M(9,25) = \left(\begin{array}{ccccccccccccccccccccccccc}
0 & 3 & 6 & 1 & 4 & 7 & 2 & 0 & 5 & 2 & 6 & 0 & 3 & 7 & 0 & 6 & 4 & 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
2 & 5 & 0 & 3 & 6 & 1 & 5 & 3 & 7 & 0 & 3 & 7 & 1 & 5 & 3 & 1 & 7 & 2 & 5 & 0 & 3 & 6 & 0 & 4 & 7 \\
4 & 7 & 2 & 5 & 0 & 3 & 7 & 1 & 4 & 6 & 1 & 4 & 6 & 2 & 7 & 5 & 0 & 4 & 7 & 2 & 5 & 1 & 3 & 6 & 1 \\
0 & 3 & 6 & 1 & 4 & 6 & 2 & 5 & 0 & 3 & 7 & 2 & 0 & 4 & 1 & 3 & 6 & 1 & 3 & 6 & 0 & 4 & 7 & 2 & 5 \\
2 & 5 & 0 & 3 & 7 & 1 & 4 & 7 & 2 & 5 & 0 & 6 & 3 & 7 & 5 & 0 & 4 & 7 & 0 & 4 & 2 & 6 & 1 & 4 & 7 \\
4 & 7 & 2 & 5 & 0 & 3 & 6 & 1 & 4 & 7 & 3 & 1 & 5 & 0 & 3 & 7 & 1 & 5 & 2 & 7 & 5 & 0 & 3 & 6 & 1 \\
0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 0 & 2 & 6 & 4 & 2 & 6 & 1 & 5 & 3 & 0 & 6 & 3 & 1 & 4 & 7 & 2 & 5 \\
2 & 5 & 0 & 7 & 2 & 5 & 0 & 7 & 3 & 5 & 1 & 7 & 0 & 3 & 7 & 0 & 6 & 2 & 4 & 0 & 7 & 2 & 5 & 0 & 7 \\
4 & 7 & 2 & 5 & 0 & 3 & 6 & 4 & 1 & 7 & 4 & 2 & 6 & 1 & 5 & 3 & 1 & 5 & 7 & 2 & 5 & 0 & 3 & 6 & 1
\end{array}\right)$$

$$M(10,10) = \left(\begin{array}{cccccccccc}
0 & 2 & 4 & 0 & 2 & 4 & 0 & 2 & 4 & 6 \\
3 & 5 & 7 & 3 & 5 & 7 & 3 & 5 & 7 & 1 \\
7 & 0 & 2 & 6 & 0 & 2 & 6 & 0 & 2 & 4 \\
2 & 6 & 4 & 1 & 7 & 5 & 1 & 3 & 6 & 0 \\
5 & 1 & 7 & 5 & 2 & 0 & 7 & 5 & 1 & 3 \\
0 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 4 & 6 \\
3 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 7 & 1 \\
6 & 2 & 0 & 7 & 4 & 2 & 0 & 6 & 2 & 4 \\
1 & 4 & 6 & 2 & 0 & 6 & 3 & 1 & 5 & 7 \\
5 & 7 & 1 & 5 & 7 & 1 & 5 & 7 & 0 & 2
\end{array}\right)$$

$$M(10,13) = \left(\begin{array}{ccccccccccccc}
0 & 3 & 6 & 1 & 4 & 0 & 3 & 6 & 1 & 4 & 6 & 1 & 4 \\
2 & 5 & 0 & 3 & 6 & 2 & 5 & 0 & 3 & 7 & 0 & 3 & 6 \\
4 & 7 & 2 & 5 & 1 & 4 & 7 & 2 & 6 & 1 & 4 & 7 & 1 \\
0 & 3 & 6 & 0 & 3 & 6 & 1 & 5 & 0 & 3 & 6 & 2 & 5 \\
2 & 5 & 1 & 4 & 7 & 0 & 4 & 7 & 2 & 5 & 1 & 4 & 7 \\
4 & 0 & 3 & 6 & 1 & 3 & 6 & 1 & 4 & 0 & 3 & 6 & 1 \\
6 & 2 & 5 & 0 & 4 & 7 & 0 & 3 & 6 & 2 & 5 & 0 & 3 \\
1 & 4 & 7 & 3 & 6 & 1 & 4 & 7 & 1 & 4 & 7 & 2 & 5 \\
3 & 6 & 2 & 5 & 0 & 3 & 6 & 0 & 3 & 6 & 1 & 4 & 0 \\
5 & 1 & 4 & 7 & 2 & 5 & 1 & 4 & 7 & 0 & 3 & 7 & 2
\end{array}\right)$$

$$M(10,17)=\begin{pmatrix}
0&3&6&1&4&0&3&6&0&5&1&4&7&2&6&1&4\\
2&5&0&3&6&2&5&1&4&2&7&0&3&5&0&3&7\\
4&7&2&5&1&4&7&3&6&0&5&2&6&1&4&6&1\\
0&3&6&0&3&6&1&5&2&4&1&7&0&3&7&2&5\\
2&5&1&4&7&0&4&7&0&6&3&5&2&6&1&4&7\\
4&0&3&6&1&3&6&1&4&2&7&1&4&0&3&6&1\\
6&2&5&0&4&7&2&5&7&0&5&3&6&2&5&0&3\\
1&4&7&3&6&1&4&0&3&6&2&7&1&4&7&2&5\\
3&6&2&5&0&3&6&2&5&1&4&0&3&6&1&4&0\\
5&1&4&7&2&5&1&4&7&3&6&2&5&0&3&7&2
\end{pmatrix}$$

$$M(11,11)=\begin{pmatrix}
0&2&4&0&6&1&3&5&1&3&7\\
3&5&7&2&4&7&0&2&7&5&1\\
6&0&3&5&0&2&4&6&0&2&4\\
2&4&6&1&3&5&7&1&3&6&0\\
5&7&0&4&6&0&2&5&7&1&3\\
1&3&5&7&2&4&6&0&2&4&7\\
4&6&1&3&5&7&1&4&6&0&2\\
0&2&4&6&1&3&5&7&1&3&6\\
3&5&7&2&4&6&0&3&5&7&1\\
7&0&3&5&0&2&4&6&2&0&5\\
4&6&1&7&3&5&7&0&4&6&2
\end{pmatrix}$$

$$M(11,17)=\begin{pmatrix}
0&3&6&1&4&0&3&6&2&5&0&3&6&2&5&1&4\\
2&5&0&3&6&2&5&0&4&7&2&5&0&4&7&3&6\\
4&7&2&5&0&4&7&2&6&1&4&7&2&6&1&5&0\\
6&1&4&7&2&6&1&4&0&3&6&1&4&0&3&7&2\\
0&3&6&1&4&0&3&7&2&5&0&3&7&2&6&1&4\\
2&5&0&3&7&2&6&1&4&7&2&6&1&5&0&3&6\\
4&7&2&6&1&5&0&3&6&1&5&0&4&7&2&5&1\\
6&1&5&0&4&7&2&5&0&4&7&3&6&1&4&0&3\\
0&4&7&3&6&1&4&7&3&6&2&5&0&3&6&2&5\\
3&6&2&5&0&3&6&0&5&1&4&7&2&5&1&4&7\\
5&1&4&7&2&5&1&4&7&3&6&1&4&0&3&6&2
\end{pmatrix}$$

$$M(11,19)=\begin{pmatrix}
0&3&6&2&5&1&4&0&3&6&2&5&1&4&0&3&6&1&5\\
6&1&4&0&3&7&2&6&1&4&0&3&7&2&5&1&4&7&3\\
4&7&2&6&1&5&0&4&7&2&6&1&5&0&3&6&0&5&1\\
2&5&0&4&7&3&6&2&5&0&4&7&3&6&1&4&7&3&6\\
7&3&6&1&5&0&4&7&3&6&1&5&0&4&7&2&5&0&4\\
5&0&4&7&2&6&1&5&0&4&7&2&6&1&5&0&3&6&1\\
2&6&1&5&0&3&7&2&6&1&5&0&3&7&2&6&1&4&7\\
0&3&7&2&6&1&4&0&3&7&2&6&1&4&0&3&7&2&5\\
6&1&4&0&3&7&2&6&1&4&0&3&7&2&6&1&4&0&3\\
4&7&2&6&1&5&0&4&7&2&6&1&5&0&4&7&2&5&1\\
2&5&0&4&7&3&6&2&5&0&4&7&3&6&2&5&0&3&7
\end{pmatrix}$$

$$M(13,13) = \begin{pmatrix}
0 & 2 & 4 & 0 & 2 & 4 & 0 & 2 & 4 & 6 & 0 & 3 & 5 \\
3 & 5 & 7 & 3 & 5 & 7 & 3 & 5 & 0 & 2 & 4 & 6 & 1 \\
6 & 0 & 2 & 6 & 0 & 2 & 6 & 1 & 3 & 5 & 7 & 2 & 4 \\
1 & 3 & 5 & 1 & 3 & 5 & 0 & 4 & 6 & 0 & 3 & 5 & 7 \\
4 & 6 & 0 & 4 & 6 & 1 & 3 & 7 & 1 & 4 & 6 & 0 & 2 \\
7 & 1 & 3 & 7 & 2 & 4 & 6 & 0 & 3 & 7 & 1 & 3 & 5 \\
2 & 4 & 6 & 1 & 5 & 0 & 2 & 4 & 6 & 0 & 4 & 6 & 0 \\
6 & 0 & 2 & 4 & 7 & 3 & 5 & 7 & 1 & 3 & 7 & 1 & 4 \\
1 & 3 & 5 & 0 & 2 & 6 & 0 & 2 & 4 & 6 & 0 & 3 & 7 \\
4 & 6 & 1 & 3 & 5 & 1 & 4 & 6 & 0 & 2 & 4 & 6 & 0 \\
7 & 0 & 4 & 6 & 0 & 3 & 7 & 1 & 3 & 5 & 7 & 1 & 3 \\
1 & 3 & 7 & 2 & 4 & 6 & 2 & 4 & 6 & 0 & 2 & 4 & 6 \\
4 & 6 & 1 & 5 & 7 & 1 & 5 & 7 & 1 & 3 & 5 & 7 & 2
\end{pmatrix}$$

$$M(13,17) = \begin{pmatrix}
0 & 3 & 7 & 1 & 4 & 6 & 0 & 4 & 7 & 1 & 5 & 0 & 2 & 5 & 7 & 3 & 6 \\
4 & 6 & 0 & 3 & 7 & 1 & 3 & 6 & 2 & 4 & 7 & 3 & 6 & 0 & 2 & 5 & 1 \\
7 & 1 & 4 & 6 & 2 & 5 & 7 & 1 & 5 & 0 & 2 & 5 & 1 & 3 & 6 & 0 & 3 \\
2 & 5 & 7 & 1 & 4 & 0 & 2 & 4 & 7 & 3 & 6 & 0 & 4 & 7 & 1 & 4 & 6 \\
4 & 0 & 2 & 5 & 7 & 3 & 5 & 0 & 2 & 5 & 1 & 3 & 6 & 2 & 5 & 7 & 1 \\
7 & 3 & 6 & 0 & 2 & 6 & 1 & 3 & 6 & 0 & 4 & 7 & 1 & 4 & 0 & 2 & 5 \\
2 & 5 & 1 & 3 & 5 & 0 & 4 & 7 & 1 & 3 & 6 & 2 & 5 & 7 & 3 & 6 & 0 \\
6 & 0 & 4 & 7 & 1 & 3 & 6 & 2 & 5 & 7 & 1 & 4 & 0 & 2 & 5 & 1 & 3 \\
1 & 3 & 6 & 0 & 4 & 7 & 1 & 4 & 0 & 2 & 5 & 7 & 3 & 6 & 0 & 4 & 7 \\
5 & 7 & 1 & 3 & 6 & 2 & 5 & 7 & 3 & 6 & 0 & 2 & 5 & 1 & 3 & 6 & 2 \\
0 & 2 & 5 & 7 & 1 & 4 & 0 & 2 & 5 & 1 & 3 & 6 & 0 & 4 & 7 & 1 & 4 \\
3 & 6 & 0 & 2 & 5 & 7 & 3 & 6 & 0 & 4 & 7 & 1 & 3 & 6 & 2 & 5 & 7 \\
5 & 1 & 4 & 6 & 0 & 2 & 5 & 1 & 3 & 6 & 2 & 4 & 7 & 1 & 4 & 0 & 2
\end{pmatrix}$$

$$M(17,17) = \begin{pmatrix}
0 & 3 & 6 & 1 & 3 & 6 & 0 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 5 & 1 & 4 \\
2 & 5 & 0 & 4 & 7 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 0 & 4 & 7 & 3 & 6 \\
4 & 7 & 2 & 6 & 0 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 2 & 6 & 0 & 5 & 1 \\
6 & 0 & 5 & 1 & 4 & 7 & 1 & 4 & 7 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 \\
1 & 4 & 7 & 3 & 6 & 0 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 & 6 & 0 & 5 \\
3 & 6 & 0 & 5 & 1 & 4 & 7 & 1 & 4 & 7 & 3 & 6 & 0 & 5 & 1 & 4 & 7 \\
5 & 1 & 4 & 7 & 3 & 6 & 0 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 & 6 & 0 \\
7 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 \\
0 & 5 & 1 & 4 & 7 & 3 & 6 & 0 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 3 & 6 \\
4 & 7 & 3 & 6 & 0 & 5 & 1 & 4 & 7 & 1 & 4 & 2 & 7 & 0 & 2 & 5 & 1 \\
6 & 2 & 5 & 1 & 4 & 7 & 3 & 6 & 2 & 5 & 7 & 0 & 5 & 3 & 6 & 0 & 3 \\
1 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 2 & 7 & 1 & 4 & 7 \\
3 & 6 & 1 & 5 & 0 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 6 & 0 \\
7 & 0 & 3 & 7 & 2 & 6 & 1 & 5 & 0 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 \\
1 & 4 & 6 & 1 & 4 & 0 & 3 & 7 & 2 & 6 & 1 & 5 & 0 & 4 & 7 & 3 & 6 \\
3 & 7 & 0 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 7 & 2 & 6 & 1 & 5 & 0 \\
5 & 1 & 4 & 7 & 0 & 4 & 7 & 3 & 6 & 2 & 5 & 1 & 4 & 0 & 3 & 7 & 2
\end{pmatrix}$$

$$M(17,25)=\begin{pmatrix}
0&2&4&0&3&7&0&2&4&0&3&7&0&2&4&6&1&3&5&7&0&4&6&1&5\\
3&5&7&2&6&1&3&5&7&2&6&1&3&5&7&0&4&6&1&3&5&7&2&4&7\\
6&0&3&5&0&4&6&0&3&5&0&4&6&0&2&5&7&0&4&6&1&3&5&0&2\\
1&4&6&1&3&7&1&4&6&1&3&7&1&4&6&1&3&5&7&0&4&6&1&3&5\\
3&7&2&4&6&0&3&7&2&4&6&0&3&7&0&4&6&1&3&5&7&0&4&6&0\\
6&1&5&0&2&4&6&1&5&0&2&4&6&1&3&7&0&4&6&1&3&5&7&2&4\\
0&4&7&3&5&7&0&4&7&3&5&7&0&4&6&1&3&7&0&4&6&1&3&5&7\\
5&2&0&6&1&3&5&2&0&6&1&3&5&7&0&4&6&1&3&7&0&4&6&1&3\\
1&7&5&2&4&6&1&7&5&2&4&6&1&3&5&7&0&4&6&1&3&7&2&4&6\\
4&0&3&7&0&2&4&0&3&7&0&2&4&6&1&3&5&7&0&4&6&1&5&0&2\\
7&2&6&1&3&5&7&2&6&1&3&5&0&2&4&6&1&3&5&7&0&4&7&3&5\\
3&5&0&4&6&1&3&5&0&4&6&1&3&5&7&0&4&6&1&3&5&2&0&6&1\\
6&1&3&7&2&4&6&1&3&7&2&4&6&0&2&5&7&0&4&6&1&7&5&2&4\\
2&4&6&1&5&0&2&4&6&1&5&0&2&4&6&1&3&5&7&2&4&0&3&7&0\\
5&7&0&4&7&3&5&7&0&4&7&3&5&7&0&4&6&1&3&5&7&2&6&1&3\\
1&3&5&2&0&6&1&3&5&2&0&6&1&3&5&7&0&4&6&1&3&5&0&4&6\\
4&6&1&7&5&2&4&6&1&7&5&2&4&6&1&3&5&7&0&4&6&1&3&7&2
\end{pmatrix}$$

$$M(17,29)=\begin{pmatrix}
0&4&7&0&4&7&2&4&7&2&5&1&4&0&3&5&0&2&5&0&3&7&2&6&1&5&0&3&5\\
3&6&1&3&6&1&5&0&3&6&0&3&7&2&6&1&3&6&1&4&6&1&5&0&3&7&2&6&1\\
7&0&4&7&2&4&7&2&5&1&4&6&1&5&0&4&7&0&3&7&2&4&7&2&6&1&5&0&4\\
1&3&6&1&5&0&3&6&0&3&7&2&4&7&3&6&1&4&6&1&5&0&3&5&0&3&7&2&6\\
4&7&2&4&7&2&5&1&4&6&1&5&0&2&5&0&3&7&2&4&7&2&6&1&4&6&1&5&0\\
6&1&5&0&3&6&0&3&7&0&4&7&3&6&1&4&6&1&5&0&3&5&0&3&7&2&4&7&2\\
0&3&7&2&5&1&4&6&1&3&6&2&5&0&3&7&0&4&7&2&6&1&4&6&1&5&0&3&5\\
4&6&1&4&0&3&7&0&4&7&1&4&7&2&6&1&3&6&1&4&0&3&7&2&4&7&2&6&1\\
7&2&5&7&2&6&1&3&6&2&5&0&3&5&0&4&7&2&5&7&2&6&1&5&0&3&5&0&3\\
1&4&0&3&5&0&4&7&1&4&7&2&6&1&3&6&1&4&0&3&5&0&3&7&2&6&1&4&6\\
5&7&2&6&1&3&6&2&5&0&3&5&0&4&7&2&5&7&2&6&1&4&6&1&5&0&3&7&2\\
0&3&5&0&4&7&0&4&7&2&6&1&3&6&1&4&0&3&5&0&3&7&2&4&7&2&5&1&4\\
2&6&1&3&6&1&3&6&1&5&0&4&7&0&3&7&2&6&1&4&6&1&5&0&3&6&0&3&7\\
5&0&4&7&0&4&7&2&4&7&2&6&1&4&6&1&4&0&3&7&2&4&7&2&5&1&4&6&1\\
7&2&6&1&3&6&1&5&0&3&5&0&3&7&0&3&7&2&6&1&5&0&3&6&0&3&7&2&4\\
3&5&0&4&7&0&4&7&2&6&1&4&6&1&4&6&1&5&0&4&7&2&5&1&4&6&1&5&0\\
6&1&3&6&1&3&6&1&5&0&3&7&2&5&7&2&4&7&3&6&1&4&0&3&7&2&4&7&2
\end{pmatrix}$$

$$
M(19,19) = \begin{pmatrix}
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 7 & 5 & 2 & 0 & 6 & 4 & 2 \\
3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 0 & 6 & 3 & 1 & 7 & 5 \\
6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 4 & 2 & 0 \\
2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 5 & 2 & 0 & 6 & 4 \\
5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 2 & 0 & 6 & 3 & 1 & 7 \\
1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 0 & 6 & 3 & 1 & 7 & 5 & 3 \\
4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 4 & 2 & 0 & 6 \\
0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 7 & 5 & 2 & 0 & 6 & 4 & 2 \\
3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 0 & 6 & 3 & 1 & 7 & 5 \\
6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 4 & 2 & 0 \\
2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 5 & 2 & 0 & 6 & 4 \\
5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 2 & 0 & 6 & 3 & 1 & 7 \\
1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 0 & 6 & 3 & 1 & 7 & 5 & 3 \\
4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 4 & 2 & 0 & 6 \\
0 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 2 & 0 & 6 & 4 & 2 \\
3 & 1 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 3 & 1 & 7 & 5 \\
6 & 4 & 0 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 4 & 2 & 0 \\
1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 1 & 7 & 5 & 3 & 0 & 6 & 4 & 2 & 0 & 5 & 3 \\
5 & 3 & 1 & 6 & 4 & 2 & 0 & 6 & 4 & 2 & 0 & 6 & 4 & 1 & 7 & 5 & 3 & 1 & 7
\end{pmatrix}
$$