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Algorithmic aspects of Reducing Pseudopath Method

by

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ABSTRACT

Let G and H be graphs. A *substitution* of H in G instead of a vertex $v \in V(G)$ is the graph $G(v \to H)$, which consists of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$. For a hereditary class of graphs $\mathcal{P}$, the *substitutional closure* of $\mathcal{P}$ is defined as the class $\mathcal{P}^*$ consisting of all graphs which can be obtained from graphs in P by repeated substitutions.
Let $\mathcal{P}$ be an arbitrary hereditary class for which a characterization in terms of forbidden induced subgraphs is known. Zverovich [20] proposed Reducing Pseudopath Method for constructing a forbidden induced subgraph characterization of $\mathcal{P}^*$. However, an implementation of the method is not straightforward.
We find conditions that guarantee the existence of a very simple reducing pseudopath [for example, of length 1]. As a result, a number of known and new characterizations may be obtained immediately by applying the improved method. In particular, we essentially simplify proofs given in Brandstädt, Hoàng, and Zverovich [3]. As examples, we consider some interesting hereditary classes.

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1 Introduction

A *substitution* of H in G for of a vertex $v \in V(G)$ is the graph $G(v \to H)$, which consists of disjoint union of H and $G - v$ with the additional edge-set $\{xy : x \in V(H), y \in N_G(v)\}$. For a hereditary class of graphs $\mathcal{P}$, the *substitutional closure* of $\mathcal{P}$ is defined as the class $\mathcal{P}^*$ consisting of all graphs which can be obtained from graphs in P by repeated substitutions. Let $\mathcal{P}$ be an arbitrary hereditary class for which a characterization in terms of forbidden induced subgraphs is known. Zverovich [20] proposed Reducing Pseudopath Method for constructing a forbidden induced subgraph characterization of $\mathcal{P}^*$.

Implementation of the method is not always straightforward. Indeed, it is known (Zverovich [20]) that a graph may have infinitely many primal extensions Also, in general the total number of reducing W-pseudopaths of a fixed length t is exponential in t and in order of the underlying graph. Finally, sometimes we apply the method to hereditary classes with infinitely many minimal forbidden induced subgraphs, in which case additional problems may arise.

We find conditions that guarantee the existence of a very simple reducing pseudopath [for example, of length one]. As a result, a number of known and new characterizations may be obtained immediately by applying the improved method. As examples, we consider some interesting hereditary classes, for which proofs become either redundant or straightforward. It concerns several results obtained by Brandstädt, Hoàng, and Zverovich [3].

2 Bounding reducing pseudopaths

We use main definitions and results of Reducing Pseudopath Method from Zverovich [20]. A graph G is *supersimple* if it is O_2, K_2, P_3 or $\overline{P}_3$. For isomorphic graphs G and G', we say that a set $X' \subseteq V(G')$ *corresponds* to a set $X \subseteq V(G)$ if there exists an isomorphism $\phi : V(G) \to V(G')$ such that $\phi(X) = X'$. The *mid-vertices* of $P_4 = (a, b, c, d)$ are b and c.

Theorem 1. *Let W be a homogeneous set in a graph G, and let H be an extension of G.*

(i) *If W induces a supersimple graph, then there exist an induced copy G' of G in H and a reducing W'-pseudopath $R = (u_1)$ with respect to G' in H, where the set $W' \subseteq V(G')$ corresponds to $W \subseteq V(G)$.*

(ii) *If W induces P_4, then there exist an induced copy G' of G in H and a reducing W'-pseudopath either $R = (u_1)$ or $R = (u_1, u_2)$ with respect to G' in H, where the set $W' \subseteq V(G')$ corresponds to $W \subseteq V(G)$. Moreover, if $R = (u_1, u_2)$ then $N_H(u_1) \cap W'$ consists of the two mid-vertices of the induced $P_4 = G'(W')$.*

Proof. Let $X \subseteq V(H)$ be a set that induces G in H. Then there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ with respect to $G = H(X)$ in H. We may assume that t has the minimum value taken over all induced copies of G in H and all corresponding reducing pseudopaths.

(i) Suppose that $t \geq 2$. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$ for some $w_1, w_2 \in W$. It is easy to see that there exists a vertex $w \in W$ such that the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G Recall that $G \in \{O_2, K_2, P_3, \overline{P}_3\}$, and, according to (R3), $u_1 \sim W^+$ and $u_1 \not\sim W^-$, since $t \geq 2$.

Let $W' = (W \backslash \{w_2\}) \cup \{u_1\}$. The condition (R2) implies that $R' = (u_2, u_3, \ldots, u_t)$ is a reducing W'-pseudopath with respect to $G = H(Y)$ in H. Since R' is shorter that R, we obtain a contradiction to minimality of R.

(ii) Suppose that $t \geq 2$. Let W induce $P_4 = (w_1, w_2, w_3, w_4)$. By (R1), u_1 is adjacent to a vertex of W and u_1 is non-adjacent to a vertex of W. Up to symmetry we have eight possibilities for adjacencies between u_1 and W. Unless $N(u_1) = \{w_2, w_3\}$, the induced subgraph $H(\{u_1\} \cup W)$ contains an induced P_4 with u_1. Therefore we have essentially the same situation as in (i).

Suppose that $N(u_1) = \{w_2, w_3\}$ and $t \geq 3$. If u_2 satisfies (R2a), then we delete $S = \{w_1, w_2\}$ and obtain an induced copy of P_4. If u_2 satisfies (R2b), then we delete $S = \{w_1, w_3\}$ and obtain an induced copy of P_4. Recall that according to (R3), $u_2 \sim W^+$ and $u_2 \not\sim W^-$, since $t \geq 3$.

Let $W' = (W \backslash S) \cup \{u_1, u_2\}$. The condition (R2) implies that $R' = (u_3, u_4, \ldots, u_t)$ is a reducing W'-pseudopath with respect to an induced copy $H(W' \cup W^+ \cup W^-)$ of G in H. Since R' is shorter that R, we obtain a contradiction to minimality of R. □

Here we apply Reducing Pseudopath Method to some hereditary classes. We shall use a simple observation that if $\mathcal{P}^* = \mathrm{FIS}(Z)$ then $(\overline{\mathcal{P}})^* = \mathrm{FIS}(\overline{Z})$, where $\overline{S} = \{\overline{G} : G \in S\}$ is the *complementary class* for a class of graphs S.

A straightforward application of Theorem 1 immediately gives characterizations of the substitutional closure of Chair-free graphs (see Figure 5), P-free graphs (Figure 5), $2K_2$-free graphs, $K_1 \cup P_3$-free graphs, and Double Bull (Figure 8), as well as their complements. A simple observation is that if $\mathcal{P}^* = \mathrm{FIS}(Z)$ then $(\overline{\mathcal{P}})^* = \mathrm{FIS}(\overline{Z})$, where $\overline{S} = \{\overline{G} : G \in S\}$ is the *complementary class* for a class of graphs S.

In all these examples, each homogeneous set W is supersimple, so we only need to add a new vertex u_1 such that

- distinguishes W [$2^{|W|} - 2$ variants], and
- either u_1 is non-adjacent to a vertex of W^+ or u_1 is adjacent to a vertex of W^- [$2^{|W^+|+|W^-|} - 1$ variants].

The number of variants may be reduced due to symmetry.

Corollary 1. *(*Chair*-free graphs)* is the class of all $(G_1, G_2, \ldots, G_7)$-free graphs (see Figure 1).*

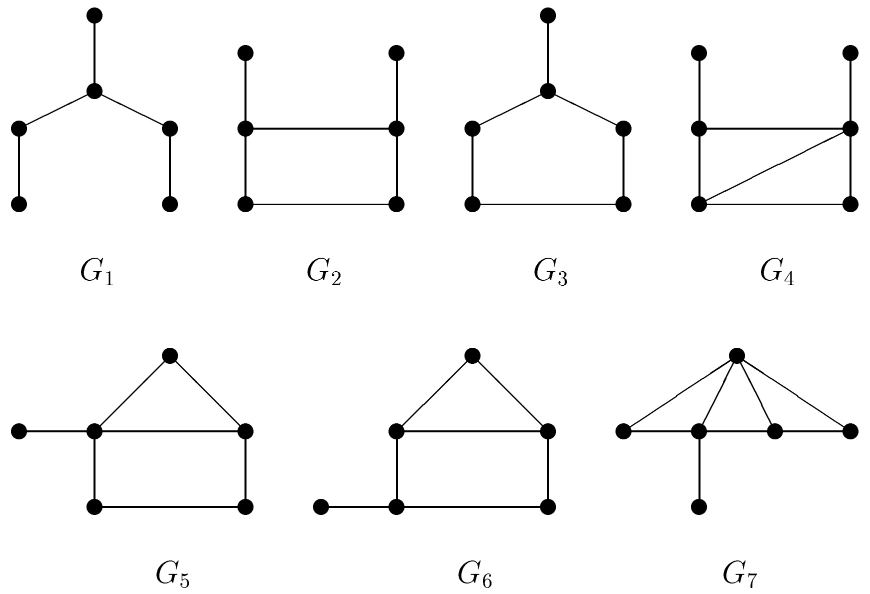

Figure 1. Minimal forbidden induced subgraphs for (Chair-free graphs)*.

The graph P is shown in Figure 5.

Corollary 2. *The minimal forbidden induced subgraphs for the substitutional closure of P-free graphs are $H_1, H_2, \ldots, H_7$ (see Figure 2).*

Corollary 3. *The minimal forbidden induced subgraphs for the substitutional closure of $\overline{P}$-free graphs are $\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_7$ (see Figure 2).*

A result of Balas and Yu [1] implies that the class of all mK_2-free graphs is α_w-polynomial. Corollary 4 below implies that Weighted Stable Set is a polynomial-time solvable problem within a wider class of all (G_1, G_2, G_3)-free graphs (see Figure 3).

Corollary 4 (Hoàng, Reed [9]). *The minimal forbidden induced subgraphs for the substitutional closure of $2K_2$-free graphs are G_1, G_2, G_3 (see Figure 3).*

Recall that a graph is *chordal* if and only if it does not contain cycles $C_n (n \geq 4)$ as induced subgraphs.

Corollary 5 (Olariu [16]). *The minimal forbidden induced subgraphs for the substitutional closure of chordal graphs are* $\overline{P}_5, A$, Domino, *and* C_n, $n \geq 5$.

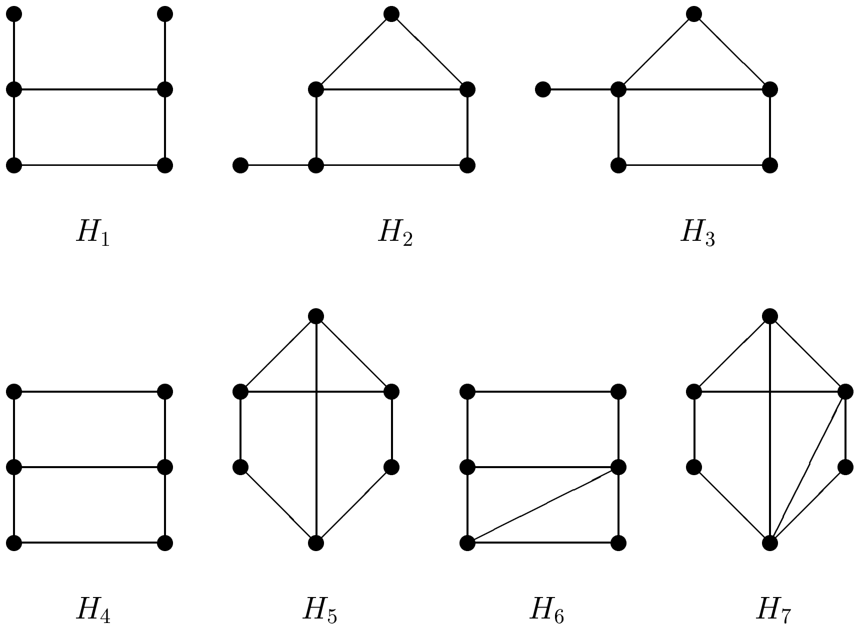

Figure 2. Minimal forbidden induced subgraphs for (P-free graphs)*.

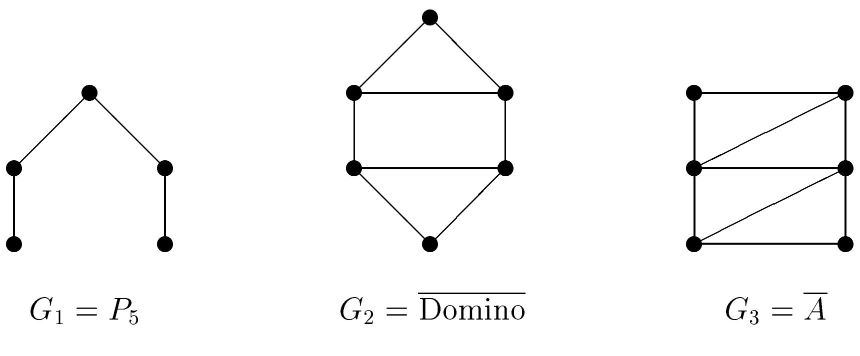

Figure 3. Minimal forbidden induced subgraphs for ($2K_2$-free graphs)*.

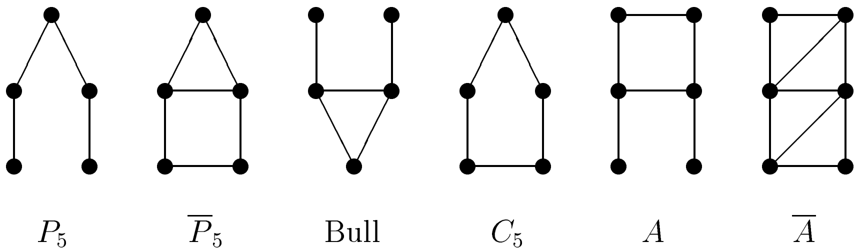

Figure 4.

Recall that the class of all *split* graphs can be characterized by three forbidden induced subgraphs, namely $2K_2, C_4$ and C_5 (Földes and Hammer [5]).

Corollary 6. *The minimal forbidden induced subgraphs for the substitutional closure of split graphs are* $P_5, \overline{P}_5, C_5, A, \overline{A}$ *(Figure 4).*

The graph $K_1 \cup P_3$ consists of two components, namely, the path P_3 and an isolated vertex.

Corollary 7. *The minimal forbidden induced subgraphs for the substitutional closure of* $(K_1 \cup P_3)$*-free graphs are* P_5, Bull, A *(see Figure 4).*

Proof. Let $G = K_1 \cup P_3$ with $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, bc\}$. We apply Theorem 1(i) to the homogeneous set $W = \{a, b, c\}$. As a result, we obtain graphs P_5, Bull, P or Chair.

By Corollary 2, $\mathrm{Ext}(P) = \{H_1, H_2, \ldots, H_7\}$ (Figure 2). All graphs in this set, except $G_2 = A$ contain either an induced P_5 or an induced Bull. By Corollary 1, $\mathrm{Ext}(P) = \{G_1, G_2, \ldots, G_7\}$ (Figure 1). All graphs in this set, except $H_1 = A$ contain either an induced P_5 or an induced Bull. $\square$

Since Weighted Stable Set Problem can be solved in polynomial time for P_3-free graphs, it can be effectively solved on $(K_1 \cup P_3)$-free graphs. Then Corollary 5 implies that Weighted Stable Set Problem is polynomial-time solvable within (P_5, Bull, A)-free graphs.

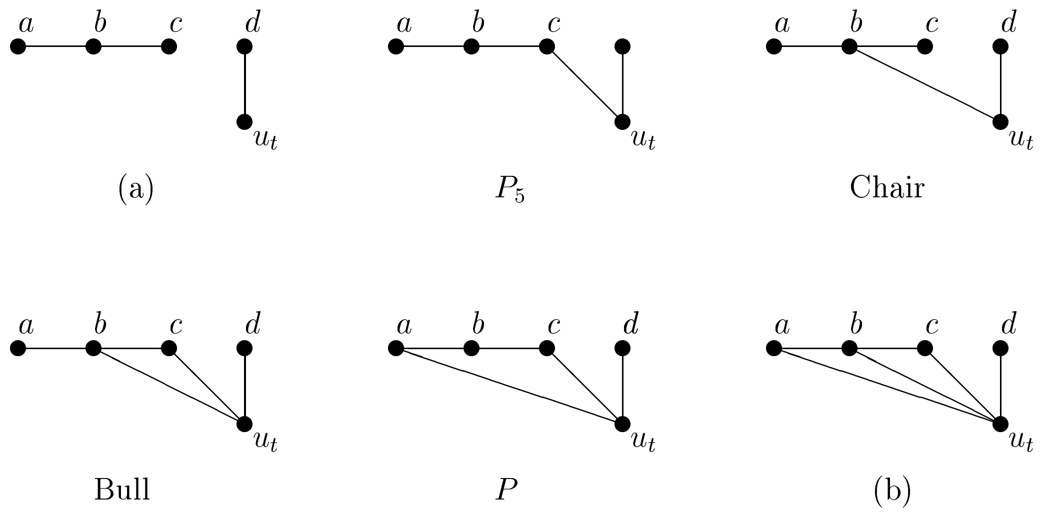

Figure 5.

Paw is the complement of $K_1 \cup P_3$.

Corollary 8 (Olariu [15]). *The minimal forbidden induced subgraphs for the substitutional closure of triangle-free graphs are* $\overline{P}_5, \text{Bull}, \overline{A}$ *(see Figure 4).*

Proof. Let $G = K_3$ with $V(G) = \{a, b, c\}$. We have a homogeneous set $W = \{a, b\}$ with $W^+ = \{c\}$ and $W^- = \emptyset$. There exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$. It can be directly checked that either $\{a, b, c, u_t\}$ or $\{a, b, c, u_t, u_{t-1}\}$ contains a set that induces Paw. Then we can apply the result complementary to Corollary 7. $\square$

Recall that $O_3 = \overline{K}_3$.

Corollary 9. *The substitutional closure of O_3-free graphs is the same as the substitutional closure of $(K_1 \cup P_3)$-free graphs, i.e., it is the class of all (P_5, Bull, A)-free graphs.*

Since cographs constitute an α_w-polynomial class, the class of all $(K_1 \cup P_4)$-free graphs is also α_w-polynomial. Therefore it is interesting to characterize $(K_1 \cup P_4)$-free graphs)*.

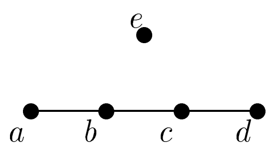

Figure 6. $K_1 \cup P_4$.

Theorem 2 (Brandstädt, Hoàng, and Zverovich [3]). *The substitutional closure of $(K_1 \cup P_4)$-free graphs is the class of all $(H_1, H_2, \ldots, H_9)$-free graphs (see Figure 7).*

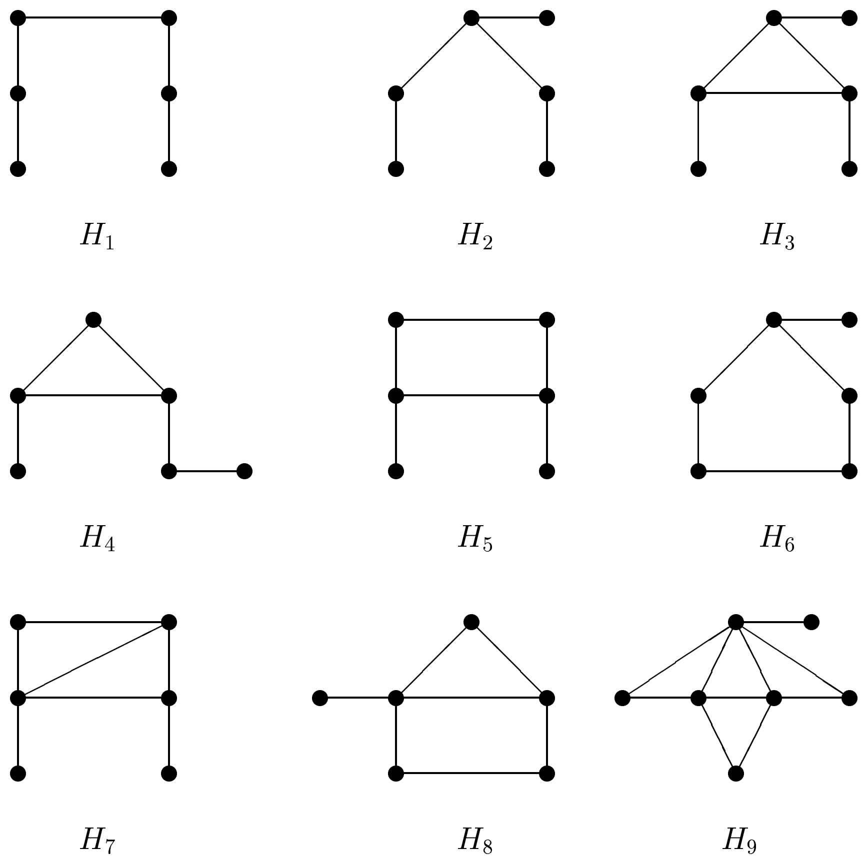

Figure 7. Minimal forbidden induced subgraphs for $((K_1 \cup P_4)\text{-free graphs})^*$.

Proof. Let $G = K_1 \cup P_4$ (Figure 6). There is a unique homogeneous set in G, namely $W = \{a, b, c, d\}$ (see Figure 6) with $W^+ = \emptyset$ and $W^- = \{e\}$. Since W induces P_4, we may apply Theorem 1(ii). If $R = (u_1)$ is a reducing W-pseudopath, we obtain the graphs $H_1, H_2, \ldots, H_8$ (Figure 7).

Let $R = (u_1, u_2)$. By Theorem 1(ii), we may assume that $u_1 \sim \{b, c\}$ and $u_1 \not\sim \{a, d\}$. If u_2 satisfies (R2a), then we have an induced H_3. If u_2 satisfies (R2b), then the set $\{a, b, c, d, e, u_1, u_2\}$ induces H_9. □

3 Claw-free graphs

We need an auxiliary result concerning Double Bull [the graph shown in Figure 8].

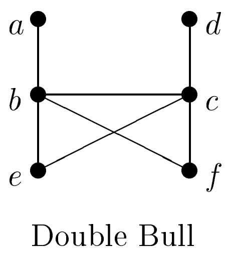

Figure 8. Double Bull.

Proposition 1 (Brandstädt, Hoàng, and Zverovich [3]). *The substitutional closure of* Double Bull*-free graphs is the class of all* $(G_1, G_2, \ldots, G_9)$*-free graphs (see Figure 9).*

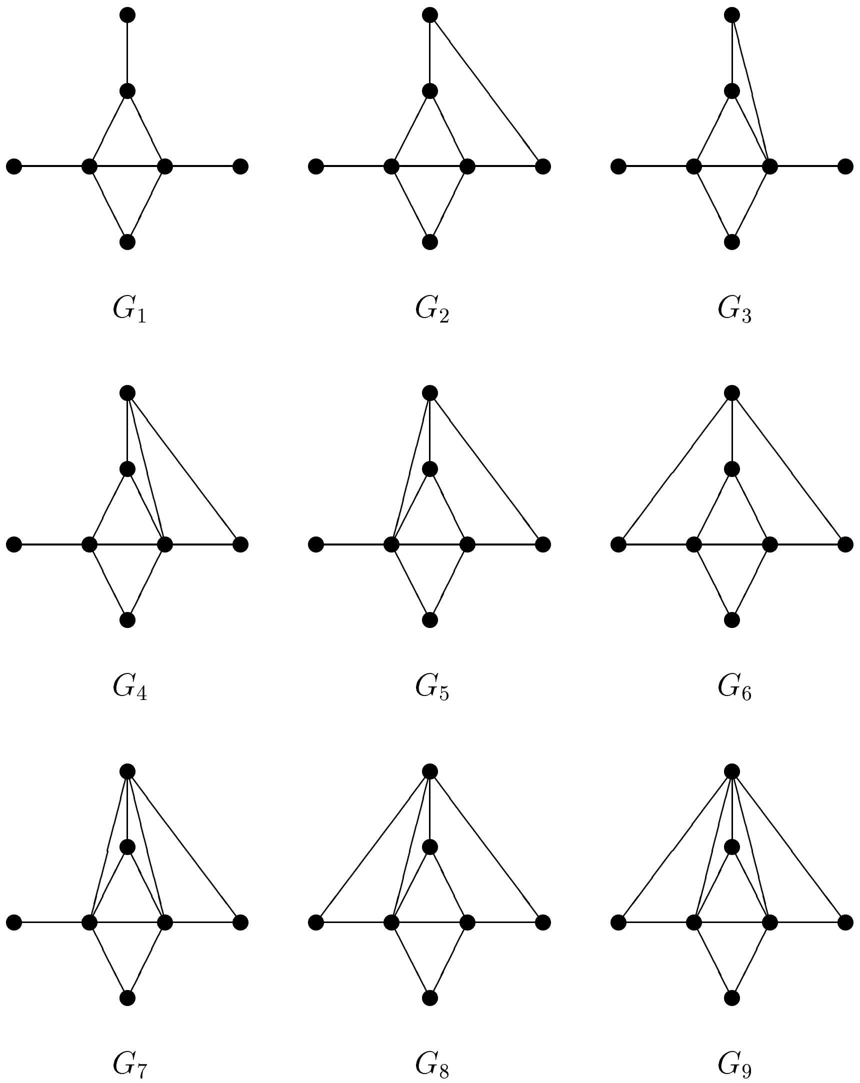

Figure 9. Minimal forbidden induced subgraphs for (Double Bull-free graphs)*.

Weighted Stable Set Problem can be solved in polynomial time within Claw-free graphs (Minty [13]). We extend this result.

Theorem 3 (Brandstädt, Hoàng, and Zverovich [3]). *The substitutional closure of Claw-free graphs coincides with the class of all* $(H_1, H_2, \ldots, H_{12})$*-free graphs (see Figure 10).*

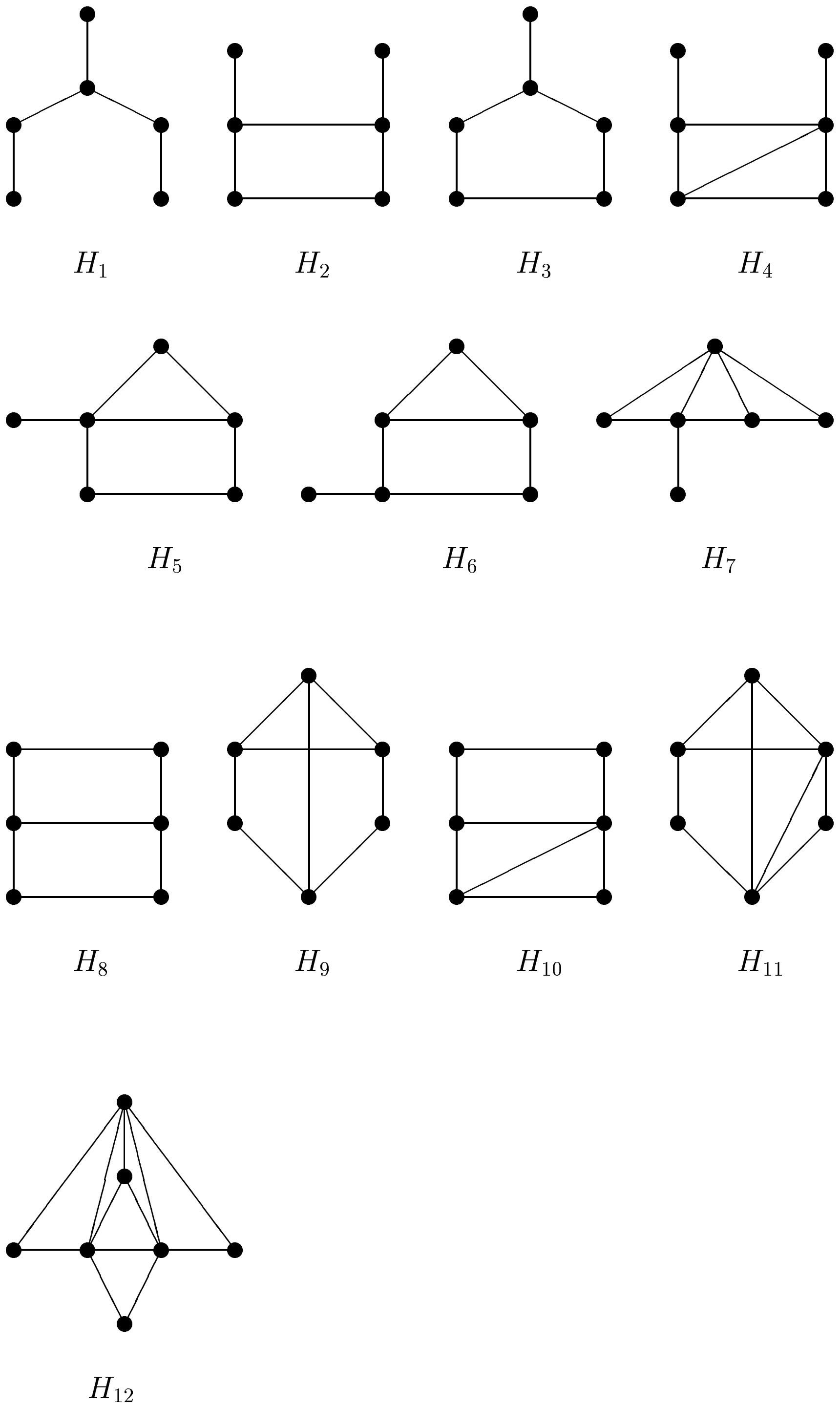

Figure 10. Minimal forbidden induced subgraphs for (Claw-free graphs)*.

Proof. Let G be Claw with the vertex-set $\{a, b, c, d\}$ and edge-set $\{ad, bd, cd\}$. The set $W = \{a, b, c\}$ is homogeneous in G; $W^+ = \{d\}$ and $W^- = \emptyset$.

Let $R = (u_1, u_2, \ldots, u_t)$ be a reducing W-pseudopath. By (R1), we may assume that $u_1 \not\sim a$ and $u_1 \not\sim c$. All possible variants [up to isomorphism] for the subgraphs induced by $\{a, b, c, d, u_1\}$ are shown in Figure 11. The first and the second produce Chair and P, respectively. The third variant is reducible by deleting the vertex c, since $t \geq 2$. We have another copy of a Claw with a shorter reducing W'-pseudopath $R = (u_2, u_3, \ldots, u_t)$, where $W' = \{a, b, u_1\}$.

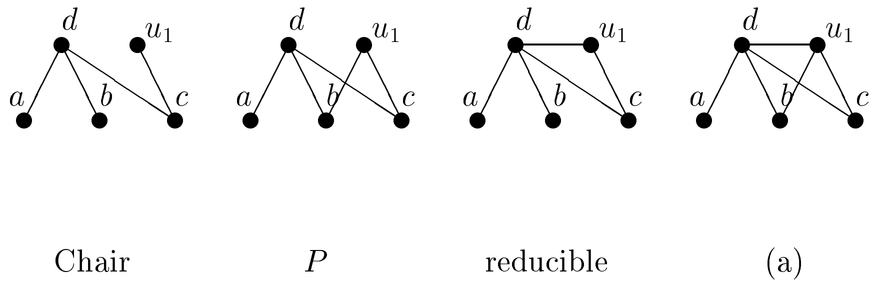

Figure 11.

If we have an induced Chair then, by Corollary 1, we obtain one of induced $G_1, G_2, \ldots, G_7$ of Figure 1. These graphs are minimal forbidden induced subgraphs $H_1, H_2, \ldots, H_9$ (Figure 10), respectively.

If we have an induced P then, by Corollary 2, we obtain one of induced $H_1, H_2, \ldots, H_7$ of Figure 2. These graphs are minimal forbidden induced subgraphs $H_2, H_6, H_5, H_8, H_9, H_{10}$, and H_{11} (Figure 10), respectively. Thus, we may assume that there are no induced Chair or P.

It remains to consider the variant (a) of Figure 11. Since $u_1 \sim d$, we have $t \geq 2$. If u_2 satisfies (R2b), we delete d and obtain an induced P. Therefore we may assume that u_2 satisfies (R2a). If $u_2 \sim d$ then $t \geq 3$. We delete $\{a, u_1\}$, and obtain a new induced Claw with a shorter reducing W'-pseudopath $R = (u_3, u_4, \ldots, u_t)$, where $W' = \{b, c, u_2\}$. If $u_2 \not\sim d$ then we have an induced Double Bull.

Also, if we have an induced Double Bull then, by Proposition 1, we obtain one of induced $G_1, G_2, \ldots, G_9$ of Figure 9. The graphs G_1, G_2, G_3, G_4, G_5 and G_7 of Figure 9 contain an induced Chair. The graphs G_6 and G_8 of Figure 9 contain an induced P. The graph G_9 of Figure 9 is the minimal forbidden induced subgraph H_{12} (Figure 10). $\square$

4 Another effective implementation of Reducing Pseudopath Method

Let W be a homogeneous set of a graph G. We denote by $\mathcal{H}_t(G,W)$ the set of all graphs that can be obtained from G by adding a reducing W-pseudopath of length t. We investigate implementation of Reducing Pseudopath Method in the case where W induces a graph having dominating or isolated vertex.

Definition 1. *We say that the set $\mathcal{H}(G,W)$ is* reducible *to its subset $\mathcal{H}'$ if every graph $H \in \mathcal{H}(G,W)$ contains some graph $H' \in \mathcal{H}'$ as an induced subgraph.*

Theorem 4. *Let W be a homogeneous set of a graph G. If the induced subgraph $F = G(W)$ has a dominating vertex then $\mathcal{H}(G,W)$ is reducible to*

$$\mathcal{H}_1(G,W) \cup \mathcal{H}_2(G,W) \cup \mathcal{H}^a(G,W),$$

$\mathcal{H}^a$ where $\mathcal{H}^a(G,W)$ consists of all reducing W-pseudopaths $R = (u_1,u_2,\ldots,u_t)$ such that $t \geq 3$ and each vertex u_i $(i = 2,3,\ldots,t-1)$ satisfies (R2a).

Proof. We consider an arbitrary graph $H \in \mathcal{H}(G,W)$. By the definition, H is obtained from G and W by adding a reducing W-pseudopath $R = (u_1,u_2,\ldots,u_t)$. If $H \in \mathcal{H}_1(G,W) \cup \mathcal{H}_2(G,W)$ then the proof is complete. Otherwise $t \geq 3$.

If $H \in \mathcal{H}^a(G,W)$ then the proof is complete. Otherwise there exists the maximum $i \in \{2,3,\ldots,t-1\}$ such that the condition (R2b) holds for u_i. By (R2b), $u_i \sim W$. Let d be a dominating vertex of F. We denote $U = (W\backslash\{d\}) \cup \{u_i\}$. The graphs F and $G(U)$ are isomorphic. Moreover, according to (R3), the set $(V(G)\backslash\{d\}) \cup \{u_i\}$ induces a graph G' isomorphic to G.

Further, the sequence $R' = (u_{i+1},u_{i+2},\ldots,u_t)$ is a reducing U-pseudopath (with respect to G'). Indeed, $|W| \geq 2$. Let $e \in W\backslash\{d\}$. The vertex u_{i+1} satisfies (R1), since it is adjacent to exactly one of $e, u_i \in U$. The conditions (R2), (R3) and (R4) are trivially holds for R'. By maximality of i, $R' \in \mathcal{H}^a(G,W)$, as required. $\square$

Corollary 10. *Let W be a homogeneous set of a graph G. If the induced subgraph $F = G(W)$ has an isolated vertex then $\mathcal{H}(G,W)$ is reducible to*

$$\mathcal{H}_1(G,W) \cup \mathcal{H}_2(G,W) \cup \mathcal{H}^b,$$

where $\mathcal{H}^b$ consists of all reducing W-pseudopaths $R = (u_1,u_2,\ldots,u_t)$ such that $t \geq 3$ and each vertex u_i, $i = 2,3,\ldots,t-1$, satisfies (R2b).

Corollary 11. *Let W be a homogeneous set of a graph G. If $|W| = 2$ then $\mathcal{H}(G,W)$ is reducible to $\mathcal{H}_1(G,W)$.*

Proof. By Theorem 4 and Corollary 10 it is sufficient to show that

- $\mathcal{H}_2(G,W) \cup \mathcal{H}^a(G,W)$ is reducible to $\mathcal{H}_1(G,W)$ if W consists of two adjacent vertices, and
- $\mathcal{H}_2(G,W) \cup \mathcal{H}^b(G,W)$ is reducible to $\mathcal{H}_1(G,W)$ if W consists of two non-adjacent vertices.

First, let $H \in \mathcal{H}_2(G,W)$, i.e., H is obtained from G by adding a reduced W-pseudopath $R = (u_1, u_2)$. By (R1), u_1 is adjacent to some vertex of $w_1 \in W$ and u_1 is non-adjacent to some vertex of $w_2 \in W$. In any case, there exists a vertex $w \in W$ such that $\{u_1, w\}$ induces a subgraph isomorphic to $F = G(W)$. Then we delete the unique vertex of $W \backslash \{w\}$ from H and obtain a graph isomorphic to a graph from $\mathcal{H}_1(G,W)$.

Suppose that W consists of two adjacent vertices. Now let $H \in \mathcal{H}^a(G,W)$, i.e., H is obtained from G by adding a reduced W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ such that $t \geq 3$ and each vertex u_i $(i = 2, 3, \ldots, t-1)$ satisfies (R2a). By (R2a), the vertices u_{t-1} and u_{t-2} are adjacent and u_t is adjacent to exactly one of them. Thus, we can delete $W \cup \{u_1, u_2, \ldots, u_{t-3}\}$ from H and obtain a graph isomorphic to a graph from $\mathcal{H}_1(G,W)$. Indeed, the set $(V(G)\backslash W) \cup \{u_{t-2}, u_{t-1}\}$ induces in H a subgraph G' isomorphic to G (with $U = \{u_{t-2}, u_{t-1}\}$ instead of W), and (u_t) is a reduced U pseudopath in G'.

The case where W consists of two non-adjacent vertices is similar, since u_{t-1} and u_{t-2} are non-adjacent according to (R2b). □

Corollary 12. *Let W be a homogeneous set of a graph G. If W induces P_3 or $\overline{P}_3$ then $\mathcal{H}(G,W)$ is reducible to $\mathcal{H}_1(G,W)$.*

Proof. By Theorem 4 it is sufficient to show that

- $\mathcal{H}_2(G,W) \cup \mathcal{H}^a(G,W)$ is reducible to $\mathcal{H}_1(G,W)$ if W induces P_3, and
- $\mathcal{H}_2(G,W) \cup \mathcal{H}^b(G,W)$ is reducible to $\mathcal{H}_1(G,W)$ if W induces $\overline{P}_3$.

First, let $H \in \mathcal{H}_2(G,W)$, i.e., H is obtained from G by adding a reduced W-pseudopath $R = (u_1, u_2)$. By (R1), u_1 is adjacent to some vertex of $w_1 \in W$ and u_1 is non-adjacent to some vertex of $w_2 \in W$. It is easy to see that if W induces P_3 then we may assume that w_1 and w_2 are adjacent. Similarly, if W induces $\overline{P}_3$ then we may assume that w_1 and w_2 are non-adjacent. In any case, $\{u_1, w_1, w_2\}$ induces a subgraph isomorphic to $F = G(W)$. Then we delete the unique vertex of $W \backslash \{w_1, w_2\}$ from H and obtain a graph isomorphic to a graph from $\mathcal{H}_1(G,W)$

Suppose that W induces P_3. Now let $H \in \mathcal{H}^a(G,W)$, i.e., H is obtained from G by adding a reduced W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ such that $t \geq 3$ and each vertex u_i $(i = 2, 3, \ldots, t-1)$ satisfies (R2a). By (R2a), the vertices u_{t-1} and u_{t-2} are adjacent and u_t is adjacent to exactly one of them. The conditions (R3) and (R1) [if $t = 3$] imply that there exists a vertex $x \in W \cup \{u_{t-3}\}$ which is adjacent to u_{t-2}. The set $\{x, u_{t-2}, u_{t-1}\}$ induces P_3. Thus, we can delete $(W \cup \{u_1, u_2, \dots, u_{t-3}\}) \backslash \{x\}$ from H and obtain a graph isomorphic to a graph from $\mathcal{H}_1(G, W)$.

The case where W induces $\overline{P}_3$ is similar, since u_{t-1} and u_{t-2} are non-adjacent according to (R2b). $\square$

Now we consider some examples. First we construct the set $\mathrm{Ext}(K_1 \cup \mathrm{Paw})$, where $K_1 \cup \mathrm{Paw}$ is the graph shown in Figure 12.

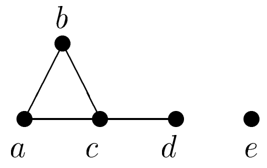

Figure 12. $K_1 \cup \mathrm{Paw}$.

Theorem 5. $\mathrm{Ext}(K_1 \cup \mathrm{Paw})$ *consists of graphs* $F_1, F_2, \dots, F_6$ *(Figure 13).*

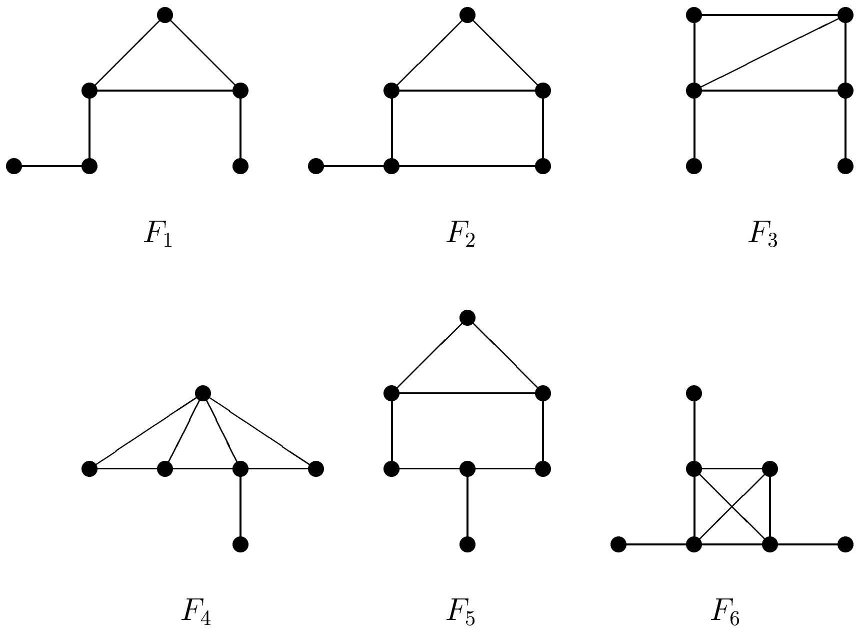

Figure 13. The set $\mathrm{Ext}(K_1 \cup \mathrm{Paw})$.

Proof. We consider the homogeneous set $W = \{a, b, c, d\}$ of $G = K_1 \cup \text{Paw}$ (Figure 12). The graph $F = G(W)$ contains a dominating vertex c. By Theorem 4 it is sufficient to construct the set

$$\mathcal{H}_1(G, W) \cup \mathcal{H}_2(G, W) \cup \mathcal{H}^a(G, W).$$

The set $\mathcal{H}_1(G, W)$ is constructed in Figure 14.

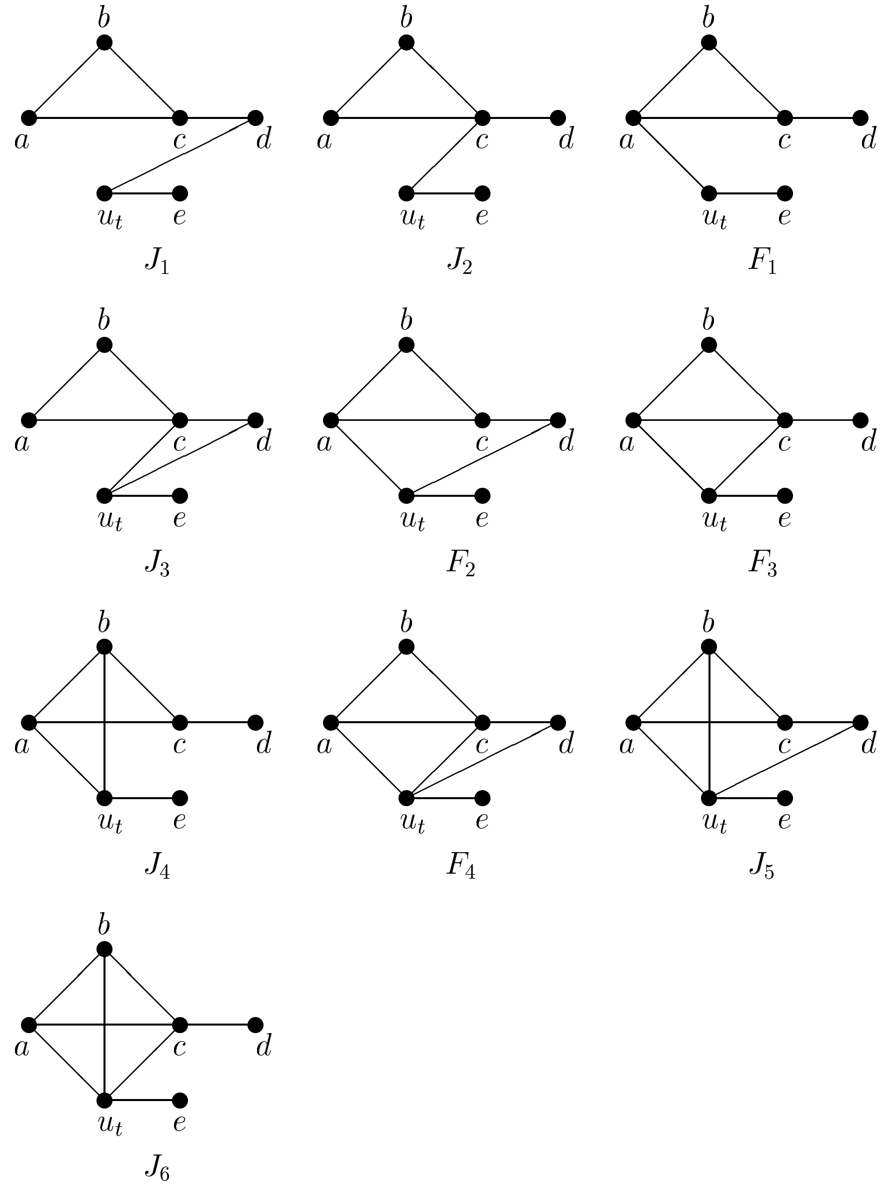

Figure 14. The set $\mathcal{H}_1(K_1 \cup \text{Paw})$.

Let $t = 2$. If $u_t = u_2$ satisfies (R2a) then we obtain essentially the same situation as before [we can use u_{t-1} and u_t instead u_t and e, respectively]. So we may assume that u_2 satisfies (R2b). The direct construction shows that either

- there is a subset of $\{a, b, c, d, e, u_1, u_2\}$ inducing one of F_3 or F_4, or
- $\{a, b, c, e, u_1, u_2\}$ induces J_6 [if $N(u_1) \cap W = \{c\}$], or
- $\{a, b, d, e, u_1, u_2\}$ induces J_5 [if $N(u_1) \cap W = \{a, b, d\}$].

Let $H \in \mathcal{H}^a(G, W)$, i.e., H is obtained from G by adding a reduced W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ such that $t \geq 3$ and each vertex u_i, $i = 2, 3, \ldots, t-1$, satisfies (R2a). In that case we obtain essentially the same situation as for $t = 1$ [we can use u_1 and u_2 instead u_t and e, respectively].

It remains to consider the graphs $J_1, J_2, \ldots, J_6$ shown in Figure 14. Each of the graphs J_1, J_3, J_4, J_6 has a unique homogeneous set W. Moreover, $|W| = 2$. By Corollary 11, we need only to construct the set $\mathcal{H}_1(J_j, W)$ for every $j = 1, 3, 4, 5, 6$. This simple and straightforward procedure gives the following results:

- every graph in $H \in \mathcal{H}_1(J_1, W)$ either contains one of F_1, F_2, F_3, F_4 as an induced subgraph or $H \cong F_5$,
- every graph in $\mathcal{H}_1(J_3, W)$ contains one of F_1, F_2, F_3, F_4 as an induced subgraph,
- every graph in $\mathcal{H}_1(J_4, W)$ contains one of F_1, F_2, F_3, F_4 as an induced subgraph, and
- every graph in $H \in \mathcal{H}_1(J_6, W)$ either contains one of F_1, F_2, F_3, F_4 as an induced subgraph or $H \cong F_6$.

Now we apply Corollary 12 to J_2. It can be directly checked that each graph $H \in \mathcal{H}_1(J_2, W)$ contains one of $F_1, F_2, F_3, F_4, J_1, J_3, J_4, J_6$ as an induced subgraph or $H \in \{L_1, L_2, L_3, L_4\}$ (Figure 15).

Similarly, we can apply Corollary 12 to J_5. It can be directly checked that each graph $H \in \mathcal{H}_1(J_5, W)$ contains one of $F_1, F_2, F_3, F_4, J_1, J_3, J_4, J_6$ as an induced subgraph or $H \in \{L_4, L_5, L_6, L_7\}$ (Figure 15).

Each of the graphs $L_1, L_2, \ldots, L_7$ has a unique homogeneous set W consisting of two adjacent vertices. By Corollary 11, we need only to construct the set $\mathcal{H}_1(L_l, W)$ for every $l = 1, 2, 3$. This straightforward procedure shows that every graph in $\mathcal{H}_1(L_l, W)$, $l = 1, 2, \ldots, 7$, contains at least one of F_1, F_2, F_3, F_4 as an induced subgraph. $\square$

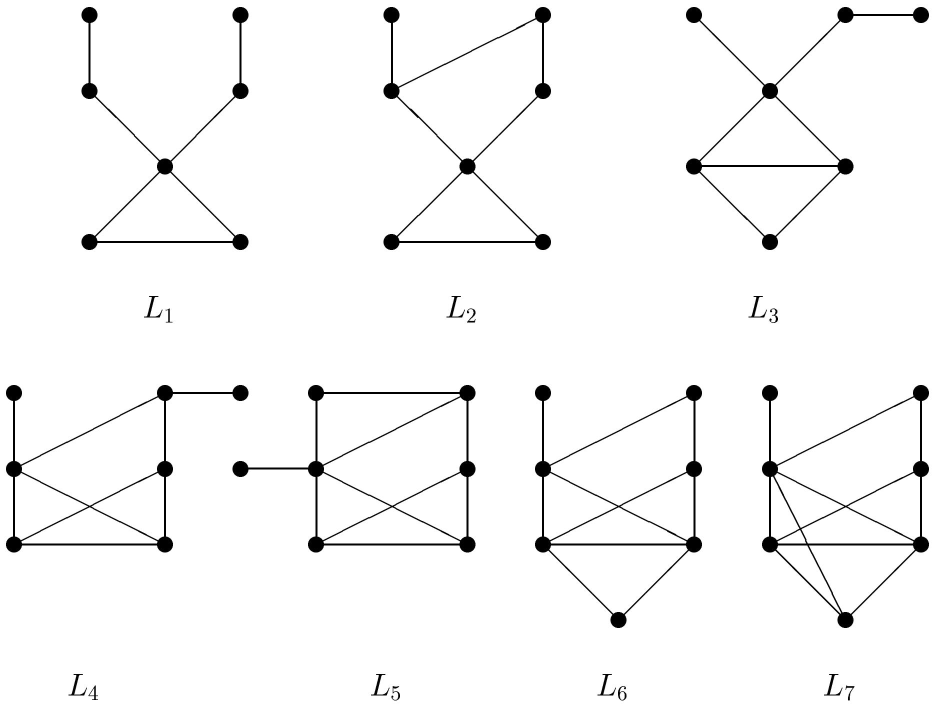

Figure 15.

Note that all homogeneous sets in the graph $O_2 \cup K_2 = \overline{\text{Diamond}}$ induce super-simple graphs. However, adding a reducing pseudopath may produce graphs with more complicated homogeneous sets.

Figure 16. $O_2 \cup K_2$.

Theorem 6. *The set of all minimal forbidden induced subgraphs for* $((O_2 \cup K_2)$*-free graphs*$)^*$ *consists of* $G_1, G_2, \ldots, G_7$ *(Figure 1) and* H_1, H_3, H_4 *(Figure 7).*

Proof. We consider the homogeneous set $W = \{b, c, d\}$. According to Theorem 1(i), every graph $H \in \mathrm{Ext}(G)$ contains an induced copy of G and a reducing W-pseudopath $R = (u_1)$. It is easy to check that we have four variants for $H(a, b, c, d, u_1)$ [up to symmetry]. They produce $K_1 \cup P_4$, Chair, $K_1 \cup \mathrm{Paw}$, and a reducible graph.

Thus, the problem is reduced to the extensions of Chair-free graphs, $(K_1 \cup P_4)$-free graphs, and $(K_1 \cup \mathrm{Paw})$-free graphs

- By Corollary 1, $\mathrm{Ext}(\mathrm{Chair}) = \{G_1, G_2, \ldots, G_7\}$ (Figure 1).
- By Theorem 2, $\mathrm{Ext}(K_1 \cup P_4) = \{H_1, H_2, \ldots, H_9\}$ (Figure 7).
- By Theorem 5, $\mathrm{Ext}(K_1 \cup \mathrm{Paw}) = \{F_1, F_2, \ldots, F_6\}$. (Figure 13).

We can exclude graphs H_2, H_5, H_6, H_7 and H_8 (Figure 7), since they are isomorphic to G_1, G_2, G_3, G_4 and G_5 (Figure 1), respectively. Also, the graph H_9 (Figure 7) contains an induced G_7 (Figure 1), and therefore it also can be excluded. The graphs F_1, F_2, F_3, F_4 and F_5 (Figure 13) are isomorphic to H_4 (Figure 7), G_6, G_4 and G_7 (Figure 1), respectively. Also, F_5 and F_6 (Figure 13) contain an induced G_3 (Figure 1) and H_3 (Figure 7), respectively. Hence all the graphs $F_1, F_2, \ldots, F_6\}$ (Figure 13) are redundant.

Thus, $\mathrm{Ext}(O_2 \cup K_2) = \{G_1, G_2, \ldots, G_7, H_1, H_3, H_4\}$. □

Diamond is the complement of $O_2 \cup K_2$.

Corollary 13. *The set of all minimal forbidden induced subgraphs for* (Diamond-*free graphs*)* *consists of* $\overline{G}_1, \overline{G}_2, \ldots, \overline{G}_7$ *(Figure 1) and* $\overline{H}_1, \overline{H}_3, \overline{H}_4$ *(Figure 7).*

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