DIMACS Technical Report 2003-24
July 2003

A new kind of graph colorings

by

I. E. Zverovich [1]
RUTCOR–Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Rd.
Piscataway, NJ 08854-8003, U.S.A.

[1]Supported by DIMACS Summer 2003 Research Program.

ABSTRACT

A proper k-coloring $C^1, C^2, \ldots, C^k$ of a graph G is called *strong* if for every vertex $u \in V(G)$ there exists an index $i \in \{1, 2, \ldots, k\}$ such that u is adjacent to every vertex of C^i. We consider classes $\mathcal{SCOLOR}(k)$ of strongly k-colorable graphs and show that the recognition problem of $\mathcal{SCOLOR}(k)$ is NP-complete for every $k \geq 4$, but it is polynomial-time solvable for $k = 3$. We give a characterization of $\mathcal{SCOLOR}(3)$ in terms of forbidden induced subgraphs. Finally, we solve the problem of uniqueness of a strong 3-coloring.
AMS Subject Classification: 05C15.
Keywords: Graph coloring, forbidden induced subgraphs, NP-complete problem.

1 Introduction

It is well-known that recognizing of classes of graphs having a k-coloring with a given property is often a hard problem. A *proper k-coloring* of a graph is a partition of its vertex-set into stable subsets. A graph is *k-colorable* if it has a proper k-coloring. The recognition of k-colorable graphs is NP-complete for each $k \geq 3$, see Garey and Johnson [2]. For classes of graphs having (k,l)-*cocoloring* [i.e., a partition into k stable sets and l cliques] the recognition problem is NP-complete when $\max\{k,l\} \geq 3$ and it is polynomial-time solvable otherwise (Brandstädt [1]).

However, imposing of some additional conditions on subgraphs between color classes can give a possibility to obtain polynomial-time recognizible classes. For example, the classes of graphs having an l-bounded k-coloring have finite forbidden induced subgraph characterizations (Zverovich [4]) and hence they are polynomial-time recognizible.

The notation $u \sim v$ (respectively, $u \not\sim v$) means that vertices u and v are adjacent (respectively, non-adjacent). If $X, Y \subseteq V(G)$ and $X \cap Y = \emptyset$, then $X \sim Y$ (respectively, $X \not\sim Y$) means that $x \sim y$ (respectively, $x \not\sim y$) for all $x \in X$ and $y \in Y$. In case of $X = \{x\}$ we simply write $x \sim Y$ and $x \not\sim Y$ instead of $\{x\} \sim Y$ and $\{x\} \not\sim Y$. A subgraph of a graph G induced by a set $X = \{x_1, x_2, \ldots, x_n\} \subseteq V(G)$ is denoted by either $G(X)$ or $G(x_1, x_2, \ldots, x_n)$. The *neighborhood* $N(x) = N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices in G that are adjacent to x.

Definition 1. *A proper k-coloring $C^1, C^2, \ldots, C^k$ of a graph G is called* strong *if for every vertex $u \in V(G)$ there exists an index $i \in \{1, 2, \ldots, k\}$ such that u is adjacent to every vertex of C^i. A graph is* strongly k-colorable *if it has a strong k-coloring. We denote by $\mathcal{SCOLOR}(k)$ the class of all strongly k-colorable graphs.*

Clearly, $\mathcal{SCOLOR}(1) = \emptyset$ and $\mathcal{SCOLOR}(2)$ is the set of all complete bipartite graphs, a hereditary class which is defined by two forbidden induced subgraphs $K_1 \cup K_2$ and K_3. It is obvious that the class $\mathcal{SCOLOR}(k)$ is hereditary for each $k \geq 3$ and hence it admits a forbidden induced subgraph characterization. We give such a characterization for $\mathcal{SCOLOR}(3)$ and construct a polynomial-time algorithm for recognizing $\mathcal{SCOLOR}(3)$.

Proposition 1. *The recognition problem $G \in \mathcal{SCOLOR}(k)$ is NP-complete for every $k \geq 4$.*

Proof. Let G be a graph and $k \geq 3$. We construct a graph G' by adding a new isolated vertex u to G. Note that G has a proper k-coloring if and only if G' has a strong $(k+1)$-coloring.

Let $C^1, C^2, \ldots, C^k$ be a proper k-coloring of G. We define a strong $(k+1)$-coloring of G' as follows: $D^1 = C^1 \cup \{u\}, D^i = C^i, i = 2, 3, \ldots, k,$ and $D^{k+1} = \emptyset$. Now let

$C^1, C^2, \ldots, C^{k+1}$ be a strong $(k+1)$-coloring of G' with $u \in C^1$. By definition, there exists $i \in \{2, 3, \ldots, k+1\}$ such that $u \sim C^i$. Since u is an isolated vertex, $C^i = \emptyset$. Hence G is a k-colorable graph. □

2 A characterization of $\mathcal{SCOLOR}(3)$

Theorem 1. *A graph G belongs to $\mathcal{SCOLOR}(3)$ if and only if it does not contain any of $G_1, G_2, \ldots, G_{10}$ (Figure 1) and odd cycles C_{2n+1}, for all $n \geq 2$, as induced subgraphs.*

Proof. Necessity. We show that $H = C_{2n+1} \notin \mathcal{SCOLOR}(3)$ for any $n \geq 2$. Let $H = (v_1, v_2, \ldots, v_{2n+1})$, where the vertices are cyclically ordered. Suppose that H has a strong 3-coloring C^1, C^2, C^3 We may assume that $v_1 \in C^1$ and $v_2 \in C^2$. Every vertex v_i, $4 \leq i \leq 2n$, is non-adjacent to both v_1 and v_2, therefore $v_i \notin C^3$. Since H is not a bipartite graph, $C^3 \neq \emptyset$. It follows that either $v_3 \in C^3$ or $v_{2n+1} \in C^3$. By symmetry, we may assume that $v_3 \in C^3$.

Since v_3 is non-adjacent to both $v_1 \in C^1$ and v_{2n+1}, we have $v_{2n+1} \notin C^2$. Hence $v_{2n+1} \in C^3$. Since v_1 is non-adjacent to both $v_3 \in C^3$ and v_4, we have $v_4 \notin C^2$. Therefore $v_4 \in C^1$. Thus, $v_2 \in C^2$ is non-adjacent to both $v_4 \in C^1$ and $v_{2n+1} \in C^3$, a contradiction. We have proved that $C_{2n+1} \notin \mathcal{SCOLOR}(3)$ for all $n \geq 2$.

It can checked directly that graphs $G_1, G_2, \ldots, G_{10}$ shown in Figure 1 do not have strong 3-colorings. Thus, all graphs $G_1, G_2, \ldots, G_{10}$ and C_{2n+1}, $n \geq 2$, are minimal forbidden induced subgraphs for $\mathcal{SCOLOR}(3)$.

Sufficiency. Let G be a graph without induced subgraphs $G_1, G_2, \ldots, G_{10}$ and induced odd cycles C_{2n+1}, $n \geq 2$. A vertex $u \in V(G)$ is called *dominating* if u is adjacent to each vertex of $V(G) \setminus \{u\}$.

Claim 1. *Let u be a dominating vertex of G. Then $G \in \mathcal{SCOLOR}(3)$ if and only if $G - u$ is a bipartite graph.*

Proof. Necessity. Let C^1, C^2, C^3 be a 3-coloring of G with $C^1 = \{u\}$. Then $G - u$ has a 2-coloring C^1, C^2.

Sufficiency. Let C^1, C^2 be a bipartition of $G - u$. Then G has a strong 3-coloring $C^1, C^2, \{u\}$. □

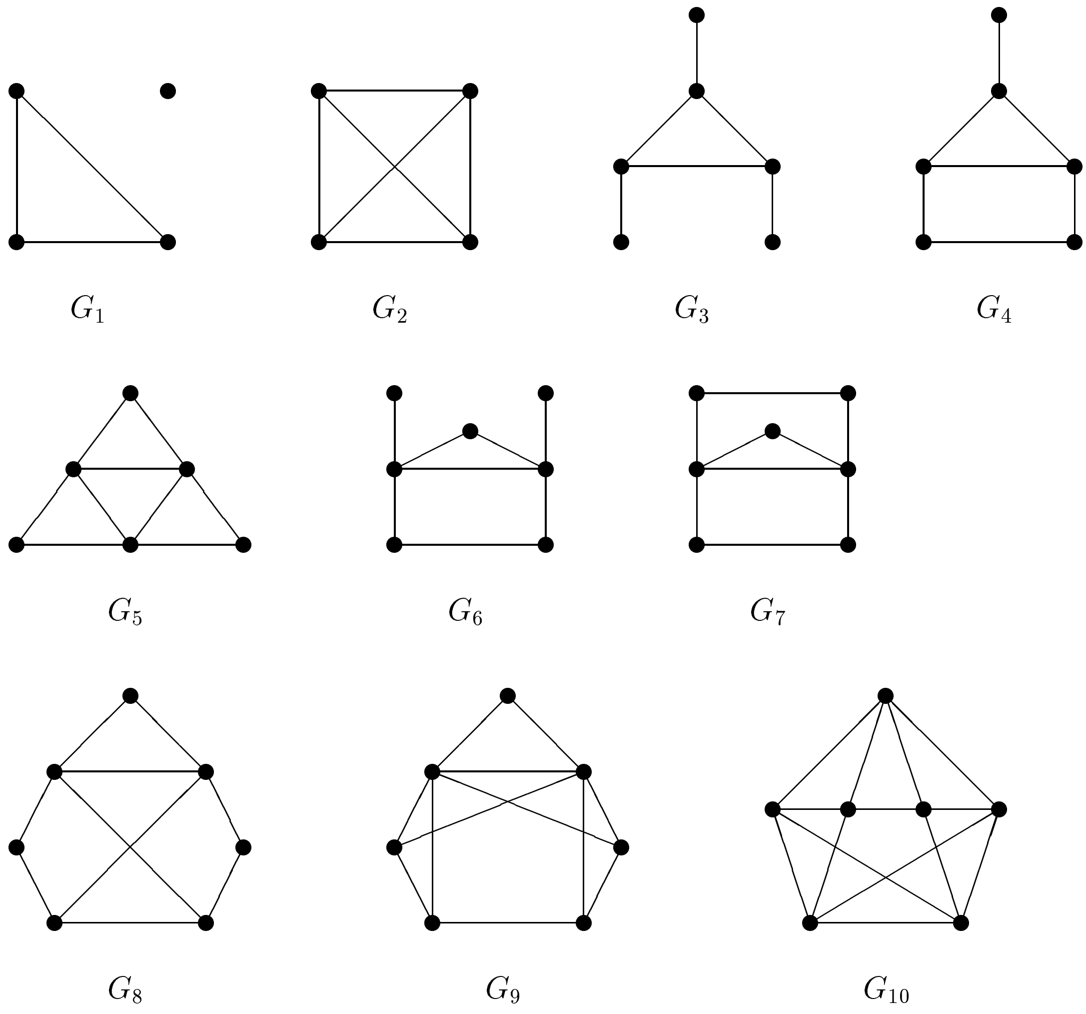

Figure 1. Some minimal forbidden induced subgraphs for $\mathcal{SCOLOR}(3)$.

Claim 2. *A bipartite graph has a strong* 3*-coloring.*

Proof. We add the empty set as the third color class to a bipartition. $\square$

Suppose that G has a dominating vertex u. Then $G - u$ has no triangles, since $G_2 = K_4$ is not an induced subgraph of G. Besides, all cycles C_{2n+1}, $n \geq 2$, are not induced subgraphs of $G-u$. By König's theorem, $G-u$ is a bipartite graph. Claim 2 implies that $G - u \in \mathcal{SCOLOR}(3)$. By Claim 1, $G \in \mathcal{SCOLOR}(3)$. Therefore we shall assume that G has no dominating vertices.

Two vertices u and v of a graph are called *duplicated* if $N(u) = N(v)$. In particular, duplicated vertices are not adjacent.

Claim 3. *Let u and v be distinct duplicated vertices in G. Then $G \in \mathcal{SCOLOR}(3)$ if and only if $G - u \in \mathcal{SCOLOR}(3)$.*

Proof. Necessity follows from the fact that $\mathcal{SCOLOR}(3)$ is a hereditary class.

Sufficiency. Let C^1, C^2, C^3 be a strong 3-coloring of $G - u$ with $v \in C^1$. It is easy to see that $C^1 \cup \{u\}, C^2, C^3$ is a strong 3-coloring of G. □

By Claim 3, we may assume that G does not have duplicated vertices.

Claim 4. *If G does not contain dominating and duplicated vertices then either G is bipartite or at least one of the graphs F or H (Figure 2) is an induced subgraph of G.*

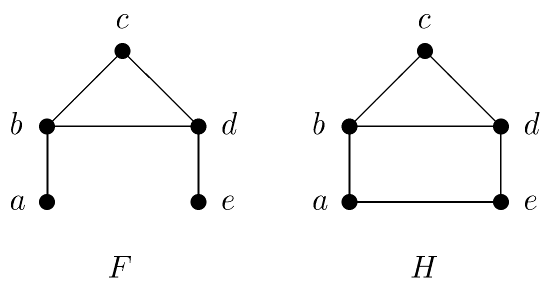

Figure 2. Graphs F and H.

Proof. Suppose that G is not a bipartite graph. Then G has an odd induced cycle $C = C_{2n+1}$. By the condition, $n = 1$, i.e., $C = C_3$. Let $V(C) = \{u, v, w\}$. First we prove that G contains Q (Figure 3) as an induced subgraph. We consider a maximal stable set I that contains u. Since u is not a dominating vertex, we have $|I| \geq 2$. Let $x \in I \setminus \{u\}$. If $x \not\sim \{v, w\}$ then $\{u, v, w, x\}$ induces G_1, a contradiction. By symmetry, we may assume that $x \sim v$. If $x \not\sim w$ then $\{u, v, w, x\}$ induces Q. Let $x \sim w$.

The vertices u and x are not duplicated, therefore there exists a vertex y which is adjacent to exactly one of them. By symmetry, we may assume that $y \sim u$ and $y \not\sim x$. If $y \not\sim \{v, w\}$ then $\{v, w, x, y\}$ induces G_1, a contradiction. By symmetry, we may assume that $y \sim v$. The set $\{u, v, w, y\}$ does not induce G_2. Hence $y \not\sim w$. Then the set $\{u, v, x, y\}$ induces a subgraph isomorphic to Q. Thus, G contains an induced subgraph isomorphic to Q. Let us fix the notation of $V(Q)$ shown in Figure 3.

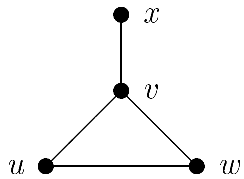

Figure 3. The graph Q.

Now we consider a maximal stable set J that contains $v \in V(Q)$. Since v is not a dominating vertex, we have $|J| \geq 2$. We choose a vertex $z \in J \setminus \{v\}$. If $z \not\sim \{u, w\}$ then $\{u, v, w, z\}$ induces G_1, a contradiction. If $|N(z) \cap \{u, w\}| = 1$ then $V(Q) \cup \{z\}$ induces either F or H, and we the proof is complete.

It remains to consider the case $z \sim \{u, w\}$. Since $\{u, w, x, z\}$ does not induces G_1, $z \sim x$. Further, v and z are not duplicated vertices, i.e., there exists a vertex t that is adjacent to exactly one of them. By symmetry, we may assume that $t \sim v$ and $t \not\sim z$. If $t \not\sim \{u, w\}$ then $\{u, w, z, t\}$ induces G_1, a contradiction. If $t \sim \{u, w\}$ then $\{u, v, w, t\}$ induces G_2, a contradiction. Thus, we may assume that $t \not\sim u$ and $t \sim w$. We obtain that the set $\{u, w, x, z, t\}$ induces either F or H. $\square$

To finish the proof of Theorem 1, we consider two cases.

Case 1. F is an induced subgraph of G.

We use the notation of $V(F)$ shown in Figure 2. It can be directly checked that F has a unique strong 3-coloring, namely, $C^1 = \{b\}, C^2 = \{a, c, e\}, C^3 = \{d\}$. If $G = F$ then $G \in \mathcal{SCOLOR}(3)$ and the proof is complete. Otherwise there is a vertex $u \in V(G) \setminus V(F)$. We show that C^1, C^2, C^3 induces a unique strong 3-coloring of G [which will be also denoted by C^1, C^2, C^3].

All possible variants for the subgraph induced by $V(F) \cup \{u\}$ are listed in Table 1, where 0 (respectively, 1) means that u is non-adjacent (respectively, adjacent) to the corresponding vertex of F.

As it can be seen, we either obtain a forbidden induced subgraph [a contradiction], or the color class of u is uniquely defined.

Claim 5. *Both C^1 and C^3 are stable sets in G.*

Proof. By symmetry we may consider C^1 only. Suppose that there are adjacent vertices $u, v \in C^1$. According to Table 1, $\{a, d\} \sim \{v, w\}$ and $b \not\sim \{v, w\}$.

The set $\{a, e, v, w\}$ does not induce G_1. So we may assume that $e \sim w$. The set $\{d, e, v, w\}$ does not induce G_2. Therefore $e \not\sim v$.

Table 1. Variants for the subgraph induced by $V(F) \cup \{u\}$.

Adjacency of u to					A forbidden induced subgraph,
a	b	c	d	e	or a color class of u
0	0	0	0	0	$G(b,c,d,u) \cong G_1$
0	0	0	0	1	$G(b,c,d,u) \cong G_1$
0	0	0	1	0	$u \in C^2$
0	0	0	1	1	$G(a,d,e,u) \cong G_1$
0	0	1	0	0	$G(a,b,c,d,e,u) \cong G_3$
0	0	1	0	1	$G(a,b,c,d,e,u) \cong G_4$
0	0	1	1	0	$G(a,c,d,u) \cong G_1$
0	0	1	1	1	$G(a,c,d,u) \cong G_1$
0	1	0	0	0	$u \in C^2$
0	1	0	0	1	$u \in C^3$
0	1	0	1	0	$u \in C^2$
0	1	0	1	1	$G(a,d,e,u) \cong G_1$
0	1	1	0	0	$G(b,c,e,u) \cong G_1$
0	1	1	0	1	$u \in C^3$
0	1	1	1	0	$G(b,c,d,u) \cong G_2$
0	1	1	1	1	$G(b,c,d,u) \cong G_2$
1	0	0	0	0	$G(b,c,d,u) \cong G_1$
1	0	0	0	1	$G(b,c,d,u) \cong G_1$
1	0	0	1	0	$u \in C^1$
1	0	0	1	1	$u \in C^1$
1	0	1	0	0	$G(a,b,c,d,e,u) \cong G_4$
1	0	1	0	1	$G(a,b,d,e,u) \cong C_5$
1	0	1	1	0	$u \in C^1$
1	0	1	1	1	$u \in C^1$
1	1	0	0	0	$G(a,b,e,u) \cong G_1$
1	1	0	0	1	$u \in C^3$
1	1	0	1	0	$G(a,b,e,u) \cong G_1$
1	1	0	1	1	$G(a,b,c,d,e,u) \cong G_5$
1	1	1	0	0	$G(b,c,e,u) \cong G_1$
1	1	1	0	1	$u \in C^3$
1	1	1	1	0	$G(b,c,d,u) \cong G_2$
1	1	1	1	1	$G(b,c,d,u) \cong G_2$

It remains to consider possible edges between c and $\{v, w\}$:

- if $c \not\sim v$ and $c \not\sim w$ then $\{a, c, v, w\}$ induces G_1, a contradiction,

- if $c \not\sim v$ and $c \sim w$ then $V(F) \cup \{v, w\}$ induces G_9, a contradiction,
- if $c \sim v$ and $c \not\sim w$ then $\{a, b, c, e, v, w\}$ induces G_4, a contradiction,
- if $c \sim v$ and $c \sim w$ then $\{c, d, v, w\}$ induces G_2, a contradiction.

□

Claim 6. *C^2 is a stable sets in G.*

Proof. Suppose that there exist adjacent vertices $v, w \in C^2$. Clearly, $v, w \notin V(F)$. According to Table 1, $\{v, w\} \sim \{a, c, e\}$, $N(v) \cap \{b, d\} \neq \emptyset$, and $N(w) \cap \{b, d\} \neq \emptyset$. Let $v \sim b$. The set $\{b, e, v, w\}$ does not induce G_1. So $w \not\sim b$. Then $w \sim d$ and $v \not\sim d$. It follows that $V(F) \cup \{v, w\}$ induces G_6, a contradiction. □

Claim 5 and Claim 6 imply that C^1, C^2, C^3 is a proper vertex coloring of G.

Claim 7. *Let $u \in C^1$ and $v \in C^3$ are non-adjacent vertices. Then $u, v \notin V(F)$ and the set $V(F) \cup \{u, v\}$ induces F_1 (Figure 4).*

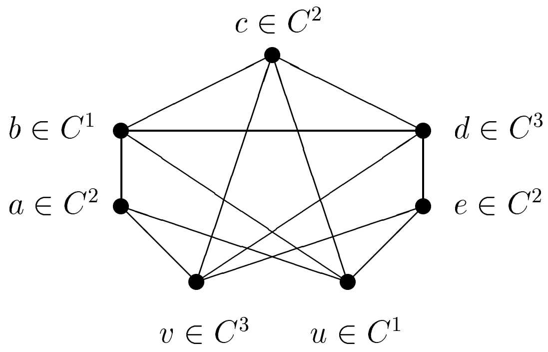

Figure 4. Graph F_1.

Proof. According to Table 1, $b \sim C^3$ and $d \sim C^1$. Therefore $u, v \notin V(F)$. It is clear from Table 1 that $u \sim \{a, d\}$, $u \not\sim b$, $v \sim \{b, e\}$, and $v \not\sim d$. We have undefined edges av, eu, cu, and cv only.

We show that either $c \not\sim \{u, v\}$ or $c \sim \{u, v\}$. Indeed, otherwise we may assume that $c \not\sim u$ and $c \sim v$. The set $\{b, c, u, v\}$ induces G_1, a contradiction. Let us show that either $av, eu \in E(G)$ or $av, eu \notin E(G)$. Indeed, otherwise we may assume that $av \notin E(G)$ and $eu \in E(G)$. The set $\{a, b, e, u, v\}$ induces C_5, a contradiction. Thus, there are four variants only:

- if $c \not\sim \{u, v\}$, and $av, eu \notin E(G)$ then $V(F) \cup \{u, v\}$ induces G_7, a contradiction,
- if $c \not\sim \{u, v\}$, and $av, eu \in E(G)$ then $\{a, b, c, e, u, v\}$ induces G_4, a contradiction,
- if $c \sim \{u, v\}$, and $av, eu \notin E(G)$ then $\{a, b, c, e, u, v\}$ induces G_4, a contradiction, and
- if $c \sim \{u, v\}$, and $av, eu \in E(G)$ then $V(F) \cup \{u, v\}$ induces F_1, as required.

□

Claim 8. (i) *For each vertex $u \in C^1$, either $u \sim C^2$ or $u \sim C^3$.*

(ii) *For each vertex $u \in C^3$, either $u \sim C^2$ or $u \sim C^1$.*

Proof. By symmetry, it is sufficient to prove the statement (i) only. Suppose that there exist vertices $u \in C^1$, $v \in C^3$ and $w \in C^2$ such that $u \not\sim v$ and $u \not\sim w$. By Claim 7, $u, v \notin V(F)$ and $V(F) \cup \{u, v\}$ induces F_1 (Figure 4).

Since $u \sim \{a, c, e\} = V(F) \cap C^2$, we have $w \notin V(F)$. Also, $w \not\sim \{a, c, e\}$. The set $\{c, d, u, w\}$ does not induce G_1. Hence $w \sim d$. If $w \not\sim v$ then $\{a, d, e, u, v, w\}$ induces G_4, a contradiction. Thus, $w \sim v$ and $\{a, d, u, v, w\}$ induces C_5, a contradiction. □

Claim 9. *For each vertex $u \in C^2$ either $u \sim C^1$ or $u \sim C^3$.*

Proof. Suppose that there exist vertices $u \in C^2$, $v \in C^1$ and $w \in C^3$ such that $u \not\sim v$ and $u \not\sim w$. We apply Claim 8(i) to v: since $v \in C^1$, $u \in C^2$, and $v \not\sim u$, $v \sim C^3$. In particular, $v \sim w$. Since $v \sim a$ and $w \sim e$, either $u = c$ or $u \notin V(F)$.

First, let $u = c$. Then $v, w \in V(G) \setminus V(F)$. The set $\{c, e, v, w\}$ does not induce G_1, so $v \not\sim e$. Similarly, $w \not\sim a$. Then $V(F) \cup \{v, w\}$ induces G_8, a contradiction.

Now let $u \notin V(F)$. According to Table 1, either $u \sim b$ or $u \sim d$. Let $u \sim d$. Hence $w \notin V(F)$. We show that $u \sim b$. Suppose that $u \not\sim b$. The set $\{b, c, u, w\}$ does not induce G_1, so $w \not\sim c$. The set $\{a, b, u, w\}$ does not induce G_1, therefore $w \not\sim a$. Then $V(F) \cup \{u, w\}$ indexes G_6, a contradiction. Thus, $u \sim b$ and $v \notin V(F)$.

The set $\{a, u, v, w\}$ does not induce G_1, therefore $a \not\sim w$. Similarly, $e \not\sim v$. Then $\{a, b, d, e, u, v, w\}$ induces G_8, a contradiction. □

Claim 8 and Claim 9 imply that the 3-coloring C^1, C^2, C^3 of G is strong. Thus, $G \in \mathcal{SCOLOR}(3)$.

Case 2. F is not an induced subgraph of G.

By Claim 4, either G is a bipartite graph [then $G \in \mathcal{SCOLOR}(3)$ and the proof is complete] or G contains H (Figure 2) as an induced subgraph. We use the notation of $V(H)$ shown in Figure 2.

Table 2. Variants for the subgraph induced by $V(F) \cup \{u\}$.

Adjacency of u to					A forbidden induced subgraph,
a	b	c	d	e	or subgraph induced by $V(H) \cup \{u\}$
0	0	0	0	0	$G(b,c,d,u) \cong G_1$
0	0	0	0	1	$G(b,c,d,u) \cong G_1$
0	0	0	1	0	$G(a,b,c,d,u) \cong F$
0	0	0	1	1	$G(a,b,c,d,u) \cong F$
0	0	1	0	0	$G(a,b,c,d,u) \cong F$
0	0	1	0	1	$G(a,b,c,d,u) \cong F$
0	0	1	1	0	$G(a,c,d,u) \cong G_1$
0	0	1	1	1	$G(a,c,d,u) \cong G_1$
0	1	0	0	0	$G(b,c,d,e,u) \cong F$
0	1	0	0	1	H_2 (Figure 5)
0	1	0	1	0	H_1 (Figure 5)
0	1	0	1	1	$G(a,c,d,e,u) \cong F$
0	1	1	0	0	$G(b,c,e,u) \cong G_1$
0	1	1	0	1	H_5 (Figure 5)
0	1	1	1	0	$G(a,c,d,u) \cong G_1$
0	1	1	1	1	$G(a,c,d,u) \cong G_1$
1	0	0	0	0	$G(b,c,d,u) \cong G_1$
1	0	0	0	1	$G(b,c,d,u) \cong G_1$
1	0	0	1	0	H_3 (Figure 5)
1	0	0	1	1	$G(a,c,e,u) \cong G_1$
1	0	1	0	0	$G(b,c,d,e,u) \cong F$
1	0	1	0	1	H_6 (Figure 5)
1	0	1	1	0	H_4 (Figure 5)
1	0	1	1	1	H_8 (Figure 5)
1	1	0	0	0	$G(b,c,d,e,u) \cong F$
1	1	0	0	1	$G(a,c,e,u) \cong G_1$
1	1	0	1	0	$G(a,b,c,e,u) \cong F$
1	1	0	1	1	$G(a,c,e,u) \cong G_1$
1	1	1	0	0	$G(b,c,e,u) \cong G_1$
1	1	1	0	1	H_7 (Figure 5)
1	1	1	1	0	$G(b,c,e,u) \cong G_1$
1	1	1	1	1	$G(b,c,d,u) \cong G_2$

If $G = H$ then $G \in \mathcal{SCOLOR}(3)$ and the proof is complete. Otherwise there exists a vertex $u \in V(G) \setminus V(H)$. As in Case 1, we list all possible variants for the subgraph induced by $V(H) \cup \{u\}$ (Table 2). Table 2 implies that there are exactly eight possible variants $H_1, H_2, \ldots, H_8$ (Figure 5) for the subgraph induced by $V(H) \cup \{u\}$.

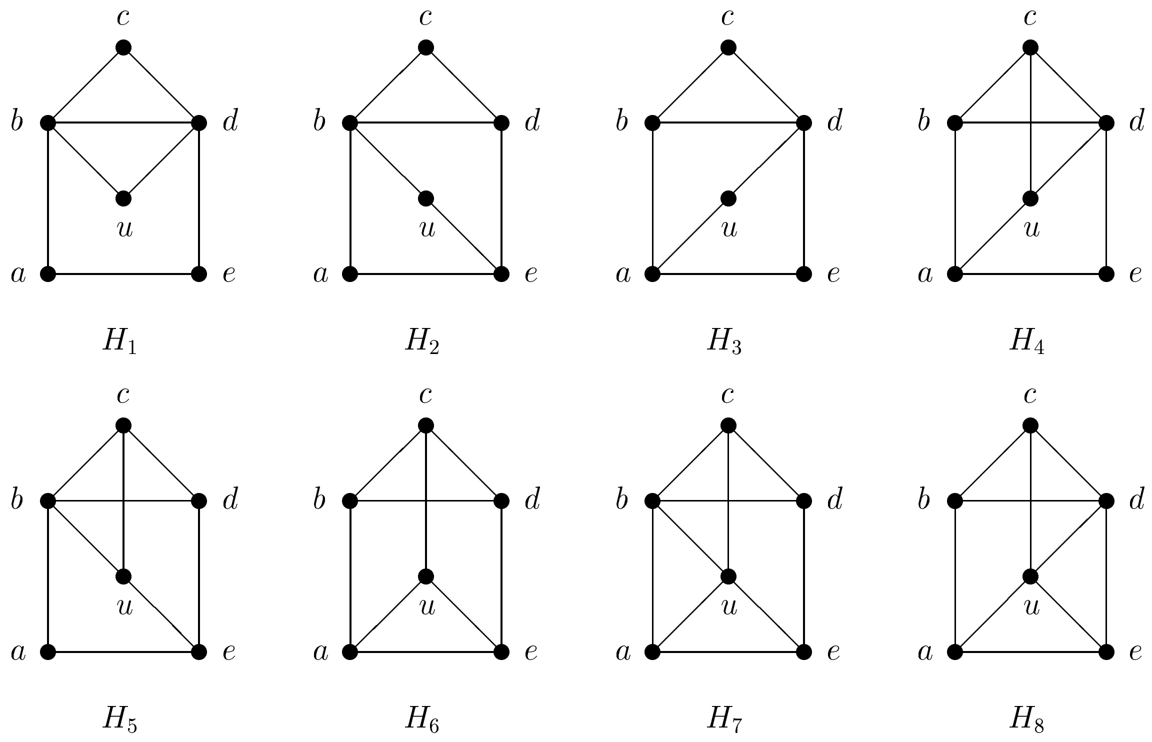

Figure 5. Graphs $H_1, H_2, \ldots, H_8$.

Claim 10. *The variant H_1 is impossible.*

Proof. The vertices u and c are not duplicated in G. So there is a vertex v such that $v \sim u$ and $v \not\sim c$ or vice versa. According to Table 2, there are three possibilities:

- $v \sim b$ and $v \sim d$ [then $\{b, d, u, v\}$ induces G_2, a contradiction],
- $v \sim b$ and $v \sim e$ [then $\{b, c, e, u, v\}$ induces F, a contradiction], and
- $x \sim a$ and $v \sim d$ [then $\{a, c, d, u, v\}$ induces F, a contradiction].

$\square$

Claim 11. *Variants H_2 and H_3 are impossible.*

Proof. We consider H_2 only. The vertices u and a are duplicated in H_2. Hence there is a vertex v which is adjacent to u and non-adjacent to a. As it follows from Table 2 and Claim 10 that there are two possibilities only:

- $v \sim e$ and $v \not\sim c$ [then $\{c, e, u, v\}$ induces G_1, a contradiction], and
- $v \sim e$ and $v \sim c$ [then $\{a, c, e, u, v\}$ induces F, a contradiction].

$\square$

Claim 12. *The variants H_4 and H_5 are impossible.*

Proof. We consider H_4 only. Since vertices u and b are duplicated in H_4, there is a vertex v such that $v \sim b$ and $v \not\sim u$. It follows from Table 2, Claim 10 and Claim 11 that $v \sim \{c, e\}$. Then $\{c, b, e, u, v\}$ induces F, a contradiction. $\square$

Claim 13. *$V(G) \setminus V(H)$ is a stable set.*

Proof. Claim 10, Claim 11 and Claim 12 imply that $V(H) \cup \{u\}$ induces either H_6 or H_7 or H_8 for each vertex $u \not\sim V(H)$. If $u, v \in V(G) \setminus V(H)$ are adjacent vertices then $\{a, e, u, v\}$ induces G_2, a contradiction. $\square$

Claim 14. *$G \in \{H, H_6, H_7, H'\}$, where H' is shown in Figure 6.*

Note that symmetric variants with H_8 instead of H_7 are omitted.

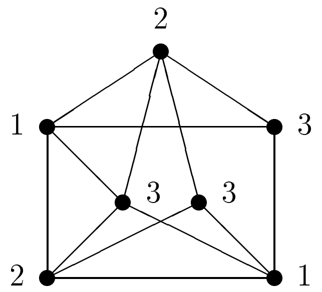

Figure 6. The graph H'.

Proof. It is sufficient to consider the case $|V(G) \setminus V(H)| \geq 2$. Let $u, v \in V(G) \setminus V(H)$ and $u \neq v$. By Claim 10, Claim 11 and Claim 12, both $V(H) \cup \{u\}$ and $V(H) \cup \{v\}$ induce either H_6 or H_7 or H_8. By Claim 13, the set $V(G) \setminus V(H)$ is stable. In particular, $u \not\sim v$. Since G does not contain duplicated vertices, $N(u) \cap V(H) \neq N(v) \cap V(H)$. If $V(H) \cup \{v\}$ induces H_8 then $V(H) \cup \{u, v\}$ induces G_{10}, a contradiction. Hence $|V(G) \setminus V(H)| = 2$ and $G \cong H'$. $\square$

The graphs H, H_6, H_7, and H' have strong 3-coloring [see Figure 6, where a strong 3-coloring of H' induces strong 3-colorings of H, H_6 and H_7]. By Claim 14, $G \in \mathcal{SCOLOR}(3)$. □

3 Recognizing of strongly 3-colorable graphs

The following algorithm recognizes strongly 3-colorable graphs in time $O(mn)$, where m is the number of edges and n is the number of vertices.

Algorithm 1 (Recognizing strongly 3-colorable graphs).

Input: *a graph* G.

Output (1): $G \notin \mathcal{SCOLOR}(3)$.

Output (2): $G \in \mathcal{SCOLOR}(3)$ *and a strong* 3*-coloring of* G.

Step 1. *If* G *is a bipartite graph with bipartition* C^1, C^2 *then return* Output (2) *with* $C^1, C^2, C^3 = \emptyset$. *Otherwise go to* Step 2.

Step 2. *Construct the equivalence classes* $R_1, R_2, \ldots, R_k$ *of duplicated vertices of* G. *Fix a vertex* $r_i \in R_i$ *for each* $i = 1, 2, \ldots, k$. *Let* G' *be a subgraph of* G *induced by* $\{r_1, r_2, \ldots, r_k\}$. *For* $X \subseteq V(G')$, *we denote*

$$R(X) = \bigcup_{X \cap R_i \neq \emptyset} R_i \subseteq V(G).$$

Go to Step 3.

Step 3. *If* G' *contains a dominating vertex* d *and* $G' - d$ *is a bipartite graph* [with parts C'_1 and C'_2], *then return* Output (2) *with* $C^1 = R(C'_1)$, $C^2 = R(C'_2)$, $C^3 = R(\{d\})$.

If $G' - d$ *is not a bipartite graph then return* Output (1).

If $G' - d$ *does not contain a dominating vertex then go to* Step 4.

Step 4. *If* G' *does not have a triangle then return* Output (1). *Otherwise find a triangle* C *in* G' *and go to* Step 5.

Step 5. *As in the proof of* Claim 4, *it is possible to find either an induced subgraph* Q (Figure 3), *or one of forbidden induced subgraphs* G_1, G_2. *In the latter case, return* Output (1). *Go to* Step 6.

Step 6. *Find a vertex* y *which is adjacent to exactly one of* u, w, *and non-adjacent to* v (see Figure 3) *and go to* Step 7. *If it is impossible to find* y *then return* Output (1).

Step 7. *Let the set* $U = \{u, v, w, x, y\}$ *induces* F (Figure 2). *Using the proofs in* Case 1, *either find a forbidden induced subgraph and then return* Output (1), *or construct the unique* 3*-coloring* C^1, C^2, C^3 *of* G' *and then return* Output (2) *with* $R(C^1), R(C^2), R(C^3)$. *Go to* Step 8.

Step 8. *Let the set* U *induces* H (Figure 2). *Using the proofs in* Case 2, *either*

- *find a a forbidden induced subgraph and then return* Output (1), *or*
- *find an induced subgraph* F *and then go to* Step 7*], or*
- *find out that* $G' \in \{H, H_6, H_7, H'\}$ *and then return* Output (2) *with* $R(C^1)$, $R(C^2)$, $R(C^3)$, *where* C^1, C^2, C^3 *is a strong* 3*-coloring of* G'.

4 Uniqueness of a strong 3-coloring

Here we solve the problem of the uniqueness of a strong 3-coloring.

Definition 2. *Two* 3*-colorings* C^1, C^2, C^3 *and* D^1, D^2, D^3 *of a graph are* distinct *if* $\{C^1, C^2, C^3\} \neq \{D^1, D^2, D^3\}$.

For example, each of graphs P_4, $\overline{P}_5$, and $\overline{C}_6$ has two different 3-colorings, see Figure 7.

Recall that the *join* $G + H$ of vertex-disjoint graphs G and H is a graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

Theorem 2. *A graph* $G \in \mathcal{SCOLOR}(3)$ *has at least two distinct strong* 3*-colorings if and only if either it is a disconnected bipartite graph or it can be obtained by vertex duplications from the following graphs:*

(UN1) $K_1 + B$*, where* B *is a disconnected bipartite graph, and*

(UN2) P_4*,* $\overline{P}_5$*, and* $\overline{C}_6$ *(Figure 7).*

Proof. Necessity. Let $C = \{C^1, C^2, C^3\}$ and $D = \{D^1, D^2, D^3\}$ be different strong 3-colorings of G. We put $A_{ij} = C^i \cap D^j$, where $i, j \in \{1, 2, 3\}$.

Claim 15. (i) *For each* $i \in \{1, 2, 3\}$ *there exists* $j \in \{1, 2, 3\}$ *such that* $A_{ij} = \emptyset$.

(ii) *For each* $j \in \{1, 2, 3\}$ *there exists* $i \in \{1, 2, 3\}$ *such that* $A_{ij} = \emptyset$.

Proof. (i) Suppose that $A_{11} \neq \emptyset$, $A_{12} \neq \emptyset$, and $A_{13} \neq \emptyset$. We choose vertices $u_j \in A_{1j}$, $j = 1, 2, 3$. Since $u_1, u_2, u_3 \in C^1$, these vertices are pairwise non-adjacent. We have $u_1 \in D^1$, $u_2 \in D^2$, $u_3 \in D^3$, and $u_1 \not\sim \{u_2, u_3\}$. Therefore D is not a strong 3-coloring, a contradiction.

(ii) It is similar to (i). $\square$

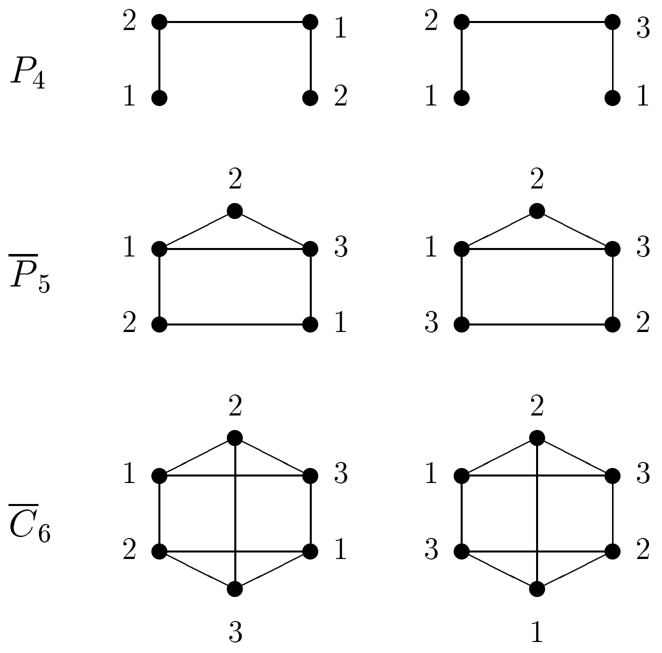

Figure 7. Different strong 3-colorings of P_4, $\overline{P}_5$ and $\overline{C}_6$.

Claim 16. *There exist $i, j, k \in \{1, 2, 3\}$ and $u, v \in V(G)$ such that $j \neq k$ and either*

(i) *$u, v \in C^i$, $u \in D^j$ and $v \in D^k$, or*

(ii) *$u, v \in D^i$, $u \in C^j$ and $v \in C^k$.*

Proof. Suppose that the statement does not hold. We choose a vertex $u \in V(G)$. We may assume that $u \in C^1$ and $u \in D^1$. If there is a vertex $v \in C^1 \setminus D^1$ then (i) is satisfied for $i = j = 1$ and some $k \in \{2, 3\}$. So $C^1 \setminus D^1 = \emptyset$. Similarly, $D^1 \setminus C^1 = \emptyset$. It follows that $C^1 = D^1$. If $C^2 = C^3 = \emptyset$ then $D^2 = D^3 = \emptyset$ and $C = D$, a contradiction. Therefore we may assume that there is a vertex $u' \in C^2$. Let $u' \in D^2$. As before, $C^2 = D^2$. Then $C^3 = D^3$ and $\mathcal{C} = \mathcal{D}$, a contradiction. $\square$

By symmetry, we may assume that Claim 2(i) takes place with $i = 1$, $j = 1$ and $k = 2$, i.e., $u \in A_{11}$ and $v \in A_{12}$. By Claim 15, $A_{13} = \emptyset$.

Claim 17. (i) $A_{ij} \not\sim A_{ik}$ *for each* $i, j, k \in \{1, 2, 3\}$, $j \neq k$, *and*

(ii) $A_{ij} \not\sim A_{kj}$ *for each* $i, j, k \in \{1, 2, 3\}$, $i \neq k$.

Proof. (i) $A_{ij}, A_{ik} \subseteq C^i$ and C^i is a stable set.

(ii) $A_{ij}, A_{kj} \subseteq D^j$ and D^j is a stable set. □

Claim 18. (i) $A_{11} \sim A_{23} \cup A_{33}$, *and*

(ii) $A_{12} \sim A_{23} \cup A_{33}$.

Proof. (i) Let $A_{11} \not\sim A_{12}$. In particular, $x \sim v$, where $v \in A_{12}$. Since $\mathcal{D}$ is a strong coloring, either $x \sim D^2$ or $x \sim D^3$. But $x \not\sim v \in D^2$, therefore $x \sim D^3$. In particular, $x \sim A_{23} \cup A_{33}$. Thus, $A_{11} \sim A_{23} \cup A_{33}$.

(ii) It is similar to (i). □

To enumerate possible variants we define a bipartite graph F of order six with parts $\{C^1, C^2, C^3\}$ and $\{D^1, D^2, D^3\}$ in which vertices C^i and D^j are linked if and only if $A_{ij} \neq \emptyset$. By Claim 15, vertex degrees of F do not exceed two. By Claim 16, there is a vertex of degree two in F.

Claim 19. *There are* 10 *pairwise non-isomorphic variants for* F *shown in Figure 7.*

Proof. Straightforward. □

□

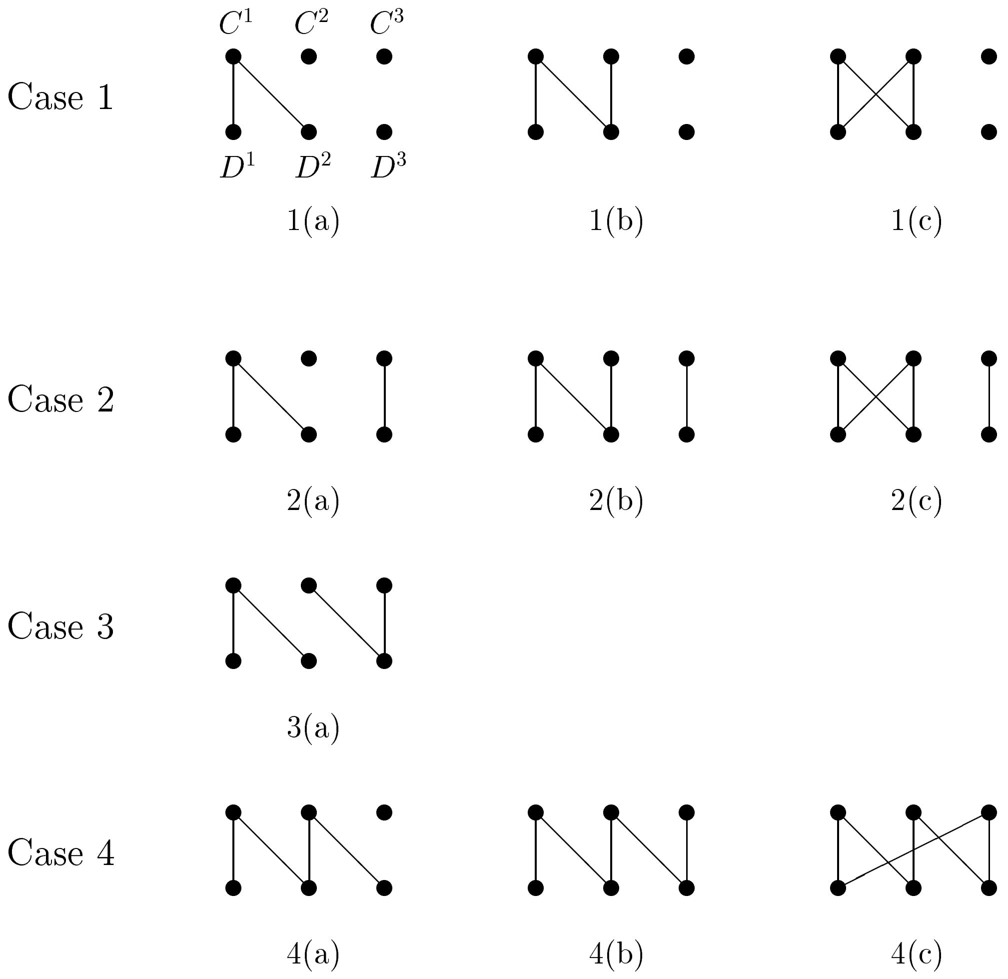

Figure 8. Case1, Case2, Case3, and Case 4.

Case 1. There are at most four non-empty sets A_{ij}, namely, A_{11}, A_{12}, A_{21}, A_{22}. According to Claim 17, $A_{11} \not\sim A12$, $A_{12} \not\sim A22$, $A_{22} \not\sim A21$, and $A_{21} \sim A11$. It follows that G is a disconnected graph. Also, G is a bipartite graph, since $C^3 = \emptyset$.

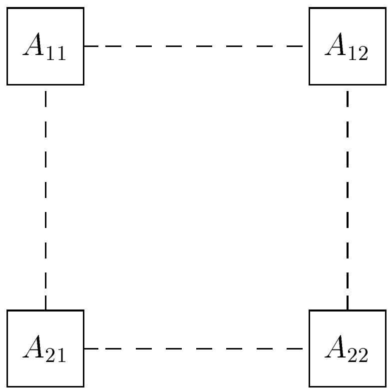

Figure 9. An illustration to Case 1.

Case 2. There are at most five non-empty sets A_{ij}, namely, A_{11}, A_{12}, A_{21}, A_{22} and A_{33}. By Claim 18, $A_{11} \cup A_{12} \not\sim A_{33}$. Similarly, $A_{21} \cup A_{22} \not\sim A_{33}$. As in Case 1, $A_{11} \cup A_{12} \cup A_{21} \cup A_{22}$ induces a disconnected bipartite graph B. It is easy to see that G can be obtained from $K_1 + B$ by vertex duplications.

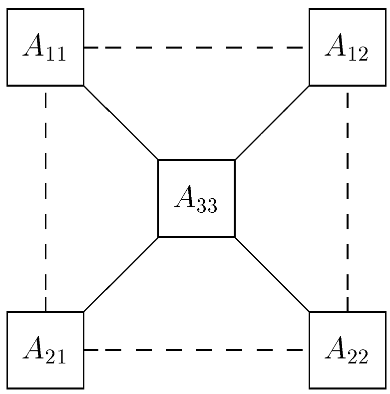

Figure 10. An illustration to Case 2.

Case 3. By Claim 17, $A_{11} \not\sim A_{12}$ and $A_{23} \not\sim A_{33}$. By Claim 18 $A_{11} \cup A_{12} \sim A_{23} \cup A_{33}$.

Thus, G can be obtained from the path P_3 by vertex duplications.

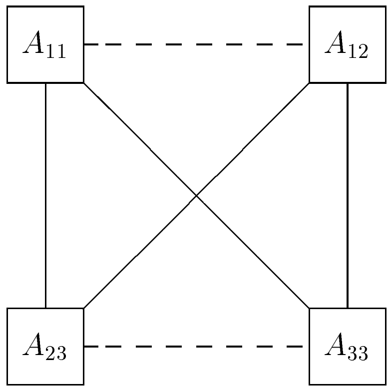

Figure 11. An illustration to Case 3.

Case 4. We have $A_{13} = A_{21} = A_{32} = \emptyset$, and $A_{11} \neq \emptyset$, $A_{12} \neq \emptyset$, $A_{23} \neq \emptyset$. By Claim 17, $A_{11} \not\sim A_{12}$, $A_{12} \not\sim A_{22}$, $A_{22} \not\sim A_{23}$, $A_{23} \not\sim A_{33}$, $A_2 \not\sim A_{31}$, and $A_{31} \not\sim A_{11}$. By Claim 18, $A_{11} \sim A_{12} \cup A_{33}$ and $A_{12} \sim A_{23} \cup A_{33}$.

Since $A_{22} \not\sim A_{23}$, $A_{22} \subseteq C^2$, $A_{12} \subseteq C^1$, and $\mathcal{C}$ is a strong coloring, $A_{22} \sim C^3$. In particular, $A_{22} \sim A_{33}$. Similarly, $A_{23} \sim C^1$ and $A_{23} \sim D^3$. In particular, $A_{23} \sim A_{31}$ and $A_{12} \sim A_{31}$. Thus, G can be obtained either from P_4 [Subcase 4(a)], or from $\overline{P}_5$ [Subcase 4(b)], or from $\overline{C}_6$ [Subcase 4(c)] by vertex duplications.

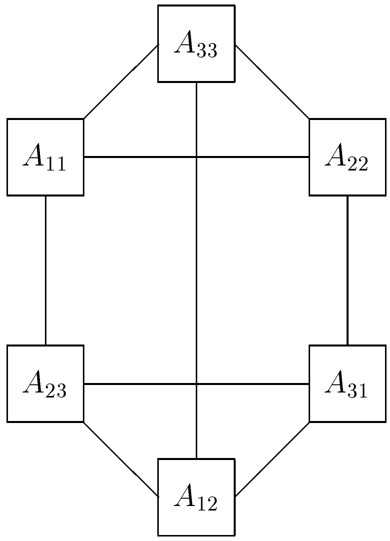

Figure 12. An illustration to Case 4.

References

[1] A. Brandstädt, Partitions of graphs into one or two stable sets and cliques, Discrete Math. **152** (1-3) (1998) 47–54; Corrigendum Discrete Math. **186** (1-3) (1998) 295

[2] M. R. Garey and D. S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness* (W. H. Freeman and Co., San Francisco, Calif., 1979) x+338 pp.

[3] T. R. Jensen and B. Toft, *Graph coloring problems* (John Wiley and Sons, New York, 1995) xxii+295 pp.

[4] I. E. Zverovich, Locally bounded hereditary subclasses of k-colorable graphs, Discrete Appl. Math. **114** (1-3) (2001) 301–311