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Independent domination on $2P_3$-free perfect graphs

by

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ABSTRACT

We show that the independent domination problem is NP-complete for $2P_3$-free perfect graphs.
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1 Introduction

Let G be a graph. A set $S \subseteq V(G)$ is called an *independent set* (or a *stable set*) if no vertices in S are adjacent. The maximum cardinality of a stable set in G is the *independence number* (or the *stability number*) of G, denoted by $\alpha(G)$. A set $D \subseteq V(G)$ is called a *dominating set* if each vertex $u \in V(G) \setminus D$ is adjacent to a vertex of D. The minimum cardinality of a dominating set in G is the *domination number* of G, denoted by $\gamma(G)$. A set $D \subseteq V(G)$ is called an *independent dominating set* if D is an independent set and D is a dominating set. Equivalently, independent dominating sets are exactly inclusion-wise maximal stable sets. The minimum cardinality of an independent dominating set in G, $\iota(G)$, is the *independent domination number* of G. As it follows from the definitions, $\alpha(G) \geq \iota(G) \geq \gamma(G)$ for every graph G.

The following three decision problems are known to be NP-complete.

Decision Problem 1 (Stability Problem).

Instance: *A graph G and an integer k.*

Question: *Is there a stable set $S \subseteq V(G)$ such that $|S| \geq k$?*

Decision Problem 2 (Domination Problem).

Instance: *A graph G and an integer k.*

Question: *Is there a dominating set $D \subseteq V(G)$ such that $|D| \leq k$?*

Decision Problem 3 (Independent Domination Problem).

Instance: *A graph G and an integer k.*

Question: *Is there an independent dominating set $I \subseteq V(G)$ such that $|I| \leq k$?*

Recall that a graph G is *perfect* if $\omega(H) = \chi(H)$ for each induced subgraph H of G, where $\omega(H)$ is the clique number of H [the size of the largest complete subgraph in H], and $\chi(H)$ is the chromatic number of H [the minimum number of colors in proper vertex colorings of H]. Stability Problem is polynomial-time solvable within perfect graphs.

Theorem 1 (Grotschel, Lovász, and Schrijver [10]). Stability Problem [even its weighted version] *can be solved in polynomial time on perfect graphs.*

Recall that a graph G is a *split graph* if there exists a partition $A \cup B = V(G)$ such that A induces a complete subgraph [possibly, $A = \emptyset$], and B is an independent set [possibly, $B = \emptyset$] (Foldes and Hammer [6]).

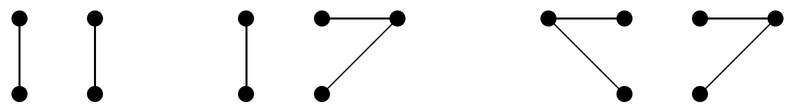
Theorem 2 (Foldes and Hammer [6]). *The class of all split graphs is characterized by the following three minimal forbidden induced subgraph:* $2K_2$ [the graph of order 4 with two disjoint edges], C_4 [the 4-cycle] *and* C_5 [the 5-cycle].

Theorem 3 (Bertossi [3]). *The domination problem is NP-complete on split graphs.*

Since each split graph is a perfect graph (see, for example, Golumbic [9]), we have the following result.

Corollary 1. *The domination problem is NP-complete within* $(2K_2, C_4)$*-free perfect graphs.*

Note that Domination problem is easy for $2K_2$-free graphs [not necessarily perfect]. As it was proved by Balas and Yu [2] (and later by Alekseev [1] and Prisner [11]), the class of all mK_2-free graphs (with a fixed m) has a polynomially bounded number of maximal stable sets. This result implies that (Weighted) Independent Domination Problem can be solved in polynomial time for mK_2-free graphs if m is fixed.



$2K_2$ $\quad$ $P_2 \cup P_3$ $\quad$ $2P_3$

Figure 1. The graphs $2K_2$, $P_2 \cup P_3$ and $2P_3$.

We consider complexity status of the independent domination problem restricted to $2P_3$-free perfect graphs.

Theorem 4. Independent Domination *is an NP-complete problem for* $2P_3$*-free perfect graphs.*

Here $2P_3$ is a graph of order 6 consisting of two disjoint copies of the path P_3. For a set of graphs Z, a graph G is called Z*-free* if no induced subgraph of G is isomorphic to a graph in Z.

Open Problem 1. *What is complexity the independent domination problem restricted to* $P_2 \cup P_3$*-free perfect graphs?*

2 Proofs

To prove Theorem 4 we consider the class of all $(1,2)$-polar graphs and show that they are perfect graphs (Corollary 2) and that the independent domination problem is NP-complete within $(1,2)$-polar graphs (Theorem 7). Since each $(1,2)$-polar graph is a $2P_3$-free graph, the result will follow.

A subgraph of a graph G induced by a set $X \subseteq V(G)$ is denoted by $G(X)$. Recall that K_n denotes a complete graph of order n.

Definition 1 (Tyshkevich and Chernyak [12]). *A graph G is called an* $(1,2)$-polar graph *if there exists a partition $A \cup B = V(G)$ (an* $(1,2)$-partition*) such that*

- *A induces a complete subgraph* [possibly, $A = \emptyset$], *and*
- *$G(B)$ is a disjoint union of complete graphs K_2 and K_1* [possibly, $B = \emptyset$].

Note that the set B induces a (P_3, K_3)-free graph. Also, each split graph is an $(1,2)$-polar graph. A result of Zverovich [13] implies that the class of all $(1,2)$-polar graphs has a finite forbidden induced subgraph characterization. Such a characterization was found by Gagarin and Metel'skii [7].

Theorem 5 (Gagarin and Metel'skii [7]). *The class of all $(1,2)$-polar graphs coincides with the class of all $(G_1, G_2, \ldots, G_{18})$-free graphs (see Figure 2).*

A *hole* in a graph is an induced cycle C_{2n+1}, where $n \geq 2$. Each $(1,2)$-polar graph is a hole-free graph. Indeed, C_5 is the forbidden induced subgraph G_1 (Figure 1), C_7 is the forbidden induced subgraph G_{18} (Figure 2), and each C_{2n+1} with $n \geq 4$ contains an induced $2P_3 = G_3$ (Figure 2).

An *antihole* in a graph is the complement of a hole C_{2n+1}, denoted by $\overline{C}_{2n+1}$, $n \geq 2$. Each $(1,2)$-polar graph is an antihole-free graph. Indeed, $\overline{C}_5 = C_5$ is the forbidden induced subgraph G_1 (Figure 1), and each $\overline{C}_{2n+1}$ with $n \geq 3$ contains an induced $\overline{P}_6 = G_{14}$ (Figure 2).

The following theorem is the former Strong Perfect Graph Conjecture (C. Berge, 1961).

Theorem 6 (Chudnovsky, Robertson, Seymour, and Thomas [4]). *A graph G is perfect if and only if it is a* (hole, antihole)*-free graph.*

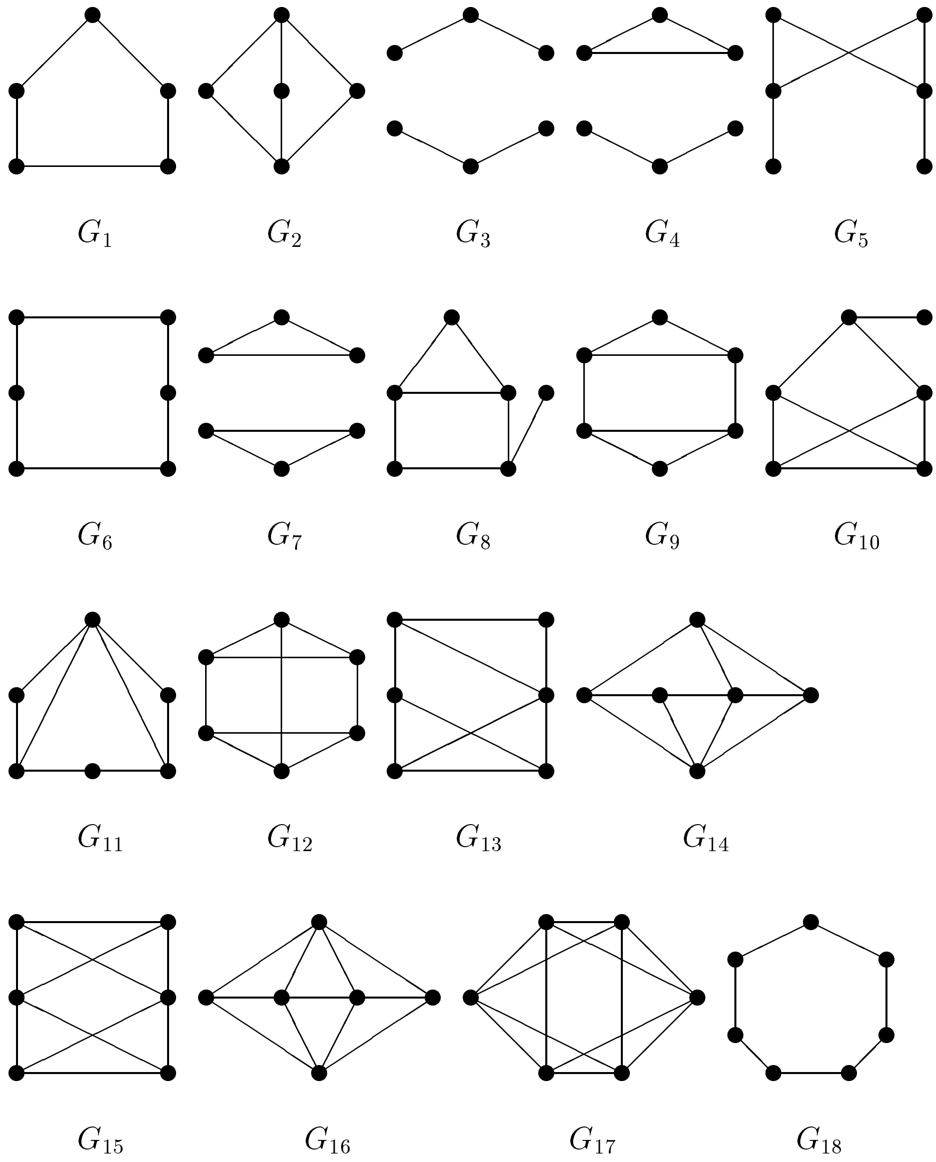


Figure 2. Minimal forbidden induced subgraphs for the class of $(1, 2)$-polar graphs.

Corollary 2. *All* $(1, 2)$*-polar graphs are perfect.*

Since Theorem 6 is still unpublished, we give an independent proof of Corollary 2 here.

Proof. Let G be an arbitrary $(1, 2)$-polar graph. It is sufficient to show that $\omega(G) = \chi(G)$, since the class of all $(1, 2)$-polar graphs is hereditary [i.e., it is closed under taking induced subgraphs]. We choose an $(1,2)$-polar partition (A,B) of G such that the set A has the largest cardinality among all $(1,2)$-polar partitions of G.

Case 1. $\omega(G) = |A|$.

We show that there exists a proper coloring of $V(G)$ with $|A| = \omega(G)$ colors. Indeed, let $A = \{a_1, a_2, \ldots, a_l\}$. We color each vertex $a_i \in A$ with the color i. Further, we consider a vertex $b \in B$. By maximality of $|A|$, $(A \cup \{b\}, B \setminus \{b\})$ is not a $(1,2)$-polar partition of G. Therefore b is non-adjacent to some vertex $a_i \in A$. If b is an isolated vertex in $G(B)$, we assign the color i to b. Otherwise b is adjacent to some vertex c in B. As before, c is non-adjacent to a vertex $a_j \in A$. If $a_j \neq a_i$, we assign the color i (respectively, j) to b (respectively, c). Suppose that $a_i = a_j$. Since $\omega(G) = |A|$, the set $(A \setminus \{a_i\}) \cup \{b, c\}$ does not induce a complete subgraph. Hence there exists a vertex $a_k \in A \setminus \{a_i\}$ that is non-adjacent to at least one b, c. By symmetry, we may assume that a_k is non-adjacent to c. Then we can assign the color i (respectively, k) to b (respectively, c). Thus, we have constructed a proper vertex coloring of G with $|A| = \omega(G)$ colors.

Case 2. $\omega(G) > |A|$.

Let C be a clique with $|C| = \omega(G)$. The condition $|A| < \omega(G)$ and the maximality of A imply that $|C \cap A| \leq |A| - 1$. Also, $|C \cap B| \leq 2$ [since $G(B)$ is a triangle-free graph]. Thus, $|C| \leq |C \cap A| + |C \cap B| \leq |A| + 1$. On the other hand, $\omega(G) \geq |A| + 1$. Therefore $|C| = |A| + 1$, i.e., C contains all but one vertices of A and exactly two vertices of B. As before, we color the set A with $\omega(G) - 1$ colors [we use i as the color of a vertex $a_i \in A$]. Then we choose a maximal stable set S in $G(B)$ and color it with an additional color. Finally, each vertex $b \in B \setminus S$ is non-adjacent to a vertex $a_i \in A$. Therefore we can color b with the color i.

Again, we have constructed a proper vertex coloring of G with $\omega(G)$ colors. $\square$

The following satisfiability problem (or SAT) is a well known NP-complete problem (Cook [5], see also Garey and Johnson [8]).

Decision Problem 4 (SAT).

Instance: *A collection* $C = \{c_1, c_2, \ldots, c_m\}$ *of clauses over a set* $X = \{x_1, x_2, \ldots, x_n\}$ *of* 0-1 *variables.*

Question: *Is there a truth assignment for* X *that satisfies all the clauses in* C*?*

A *literal* over a set X of 0-1 variables is either a variable $x_i \in X$ or a negation of a variable $x_j \in X$, denoted by $\overline{x}_j$. A *clause* over a set X of 0-1 variables is a conjunction of some literals. A truth assignment $\mathbf{x}^0$ [a 0-1 vector of length $n = |X|$] *satisfies* a clause c_i if $c_i(\mathbf{x}^0) = 1$, that is c_i involves at least one true literal l. Recall that l is a *true literal* if $l = 1$ according to the assignment $\mathbf{x}^0$.

Theorem 7. *The independent domination problem is NP-complete when restricted to* $(1,2)$*-polar graphs.*

Proof. Clearly, the problem is in NP. We shall construct a polynomial-time reduction from SAT. Given an instance $C = \{c_1, c_2, \ldots, c_m\}$ and $X = \{x_1, x_2, \ldots, x_n\}$ to SAT, we define a $(1,2)$-polar graph G as follows:

- $V(G) = A \cup B$, where
$$A = \{a_1, a_2, \ldots, a_m\}$$
  and
$$B = \{b_1, b'_1, b_2, b'_2, \ldots, b_n, b'_n\},$$
- $E(G)$ is defined by the following rules:
  - A induces a complete subgraph,
  - B induces a matching $\{b_i b'_i : i = 1, 2, \ldots, n\}$, and
  - a vertex $a_i \in A$ is adjacent to a vertex $b_j \in B$ (respectively, $b'_j \in B$) if and only if the clause c_i involves x_j (respectively, $\overline{x}_j$).

An illustration of this construction is shown in Figure 3 for $X = \{x_1, x_2, x_3, x_4\}$ and $C = \{c_1 = (x_1 \vee \overline{x}_2 \vee x_3), c_2 = (\overline{x}_1 \vee x_2 \vee x_3), c_3 = (\overline{x}_3 \vee \overline{x}_4)\}$. The graph G can be constructed in polynomial time in m and n.

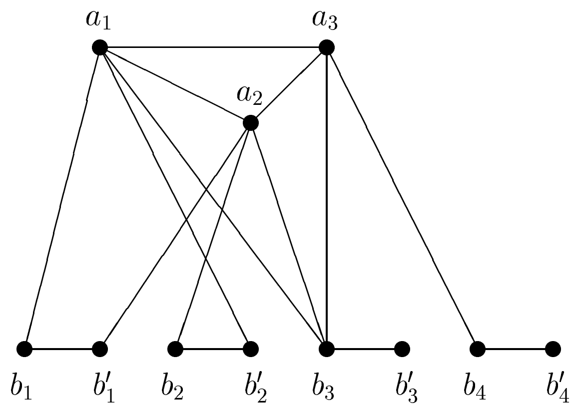


Figure 3. An example of the construction.

Without loss of generality we may assume that no clause in C contains a variable from X and its negation. Indeed, if a clause c_i involves both x_i and $\overline{x}_i$ then c_i is identically true, and hence it can be deleted from C. In terms of G, our assumption means that no vertex of A is adjacent to both b_i and b'_i, for all $i \in \{1, 2, \ldots, n\}$.

We show that there exists a satisfying truth assignment for C if and only if $\iota(G) = n = |X|$.

Each independent dominating set in G has cardinality either n or $n+1$. Indeed, let I be an independent dominating set in G. If there exists some $a_j \in I \cap A$ then $I \cap A = \{a_j\}$ and exactly one vertex from each set $\{b_i, b'_i\}$ must belong to I. It follows that $|I| = |B|/2 + 1 = n + 1$. If I is disjoint from A then $|I| = |B|/2 = n$. Suppose that there exists a satisfying truth assignment $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ for C. We define the corresponding independent subset I^0 of B:

- if $x_i^0 = 1$ then $b_i \in I^0$, and
- if $x_i^0 = 0$ then $b'_i \in I^0$.

Since $\mathbf{x}^0$ satisfies all the clauses in C, the set I^0 dominates A. Thus, I^0 is an independent dominating set in G. Since $|I^0| = n$, I^0 is a minimum independent dominating set in G, i.e., $\iota(G) = |I^0| = n$.

Conversely, suppose that there are no satisfying truth assignments for C, but $\iota(G) = n$. Let I be a minimum independent dominating set in G, $|I| = iota(G) = n$. As we have seen, $I \subseteq B$. We define a truth assignment $\mathbf{x}^0$ to C accordingly:

- if $b_i \in I$ then $x_i^0 = 1$ and
- if $b'_i \in I$ then $x_i^0 = 0$.

It is easy to see that $\mathbf{x}^0$ is a satisfying truth assignment to C, a contradiction. $\square$

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