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Basic perfect graphs and their extensions

by

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ABSTRACT

We characterize basic graphs introduced by Chudnovsky, Robertson, Seymour, and Thomas [2] in terms of forbidden induced subgraphs. Then we apply Reducing Pseudopath Method of Zverovich [12] to find forbidden induced subgraphs the substitutional closure of the class of basic graphs, thus obtaining a characterization of a hereditary subclass of perfect graphs.

1 Perfect graphs

A *clique* in a graph is a vertex subset that induces a complete subgraph [not necessarily maximal]. The *clique number* of a graph G, $\omega(G)$, is cardinality of a largest clique in G. A *(proper) k-coloring* of a graph G is a partition $V_1 \cup V_2 \cup \cdots \cup V_k$ of $V(G)$ into k stable sets [possibly, some V_i are empty]. If G has a k-coloring then it is a *k-colorable graph*. Usually, 2-colorable graphs are called *bipartite*. The *chromatic number* of G, $\chi(G)$, is the smallest k such that G is a k-colorable graph.

Definition 1. *A graph G is a* perfect graph *if $\omega(H) = \chi(H)$ for every induced subgraph H of G.*

A *hole* in a graph is and induced cycle C_n, $n \geq 4$. A hole is *odd* if it has an odd number of vertices. The complement of a hole (respectively, an odd hole) is called an *antihole* (respectively, an *odd antihole*).

Definition 2. *A graph is called a* Berge graph *is it does not contain all odd holes and all odd antiholes as induced subgraphs.*

In 1960 Berge proposed the following Strong Perfect Graph Conjecture.

Conjecture 1 (Berge [1]). *A graph is perfect if and only if it is a Berge graph.*

Necessity is straightforward. Indeed, any odd hole H has $\omega(H) = 2$ and $\chi(H) = 3$. For an antihole $A = \overline{C}_{2n+1}$, we have $\omega(A) = n$ and $\chi(A) = n + 1$.

A weaker conjecture of Berge that the class of perfect graphs is closed under complementation was proved by Lovász (Corollary 1). The *stability number* of a graph G, $\alpha(G)$, equals the clique number of $\overline{G}$.

Theorem 1 (Lovász [7]). *A graph G is perfect if and only if*

$$|V(H)| \leq \alpha(H)\omega(H) \tag{1}$$

for every induced subgraph H of G.

This result was obtained with using so-called Lovász Replication Lemma (Lemma 1). Let u be a vertex of a graph G. We add a new vertex u' that is adjacent to all vertices in the closed neighborhood $N[u]$ of u. The resulting graph $\mathrm{Rep}(G, u)$ is obtained by *replication* of the vertex u.

Lemma 1 (Lovász [7]). *If G is a perfect graph and $u \in V(G)$, then* $\mathrm{Rep}(G, u)$ *is also a perfect graph.*

Corollary 1 (Lovász [7]). *A graph is perfect if and only if its complement is perfect.*

2 Basic graphs

A subclass of perfect graphs called basic graphs plays an essential role in the announced proof of Strong Perfect Graph Conjecture (Chudnovsky, Robertson, Seymour, and Thomas [2]). Recall that the *complement* of a graph G is a graph $\overline{G}$ such that $V(\overline{G}) = V(G)$ and two distinct vertices u and v are adjacent in $\overline{G}$ if and only if they are non-adjacent in G.

The *line graph* $L(G)$ of a graph G is the intersection graph of edges of G, that is $V(L(G)) = E(G)$, and two distinct vertices e and e' are adjacent in $L(G)$ if and only if the edges e and e' of G have a common vertex. A graph H is called a *line graph* if $H = L(G)$ for some graph G. We consider the following classes of graphs:

- $\mathcal{BIP}$, the class of all bipartite graphs,
- $\mathcal{COBIP}$, the class of all complements of bipartite graphs,
- $\mathcal{LINEBIP}$, the class of all line graphs of bipartite graphs, and
- $\mathcal{COLINEBIP}$, the class of all complements of line graphs of bipartite graphs.

Definition 3 (Chudnovsky, Robertson, Seymour, and Thomas [2]). *The class of all* basic graphs*,* $\mathcal{BASIC}$*, is defined as*

$$\mathcal{BIP} \cup \mathcal{COBIP} \cup \mathcal{LINEBIP} \cup \mathcal{COLINEBIP}.$$

A graph is *non-basic* if it is not basic. It is easy to show that all basic graphs are perfect. A *(proper) edge k-coloring* of a graph G is a partition $W_1 \cup W_2 \cup \cdots \cup W_k$ of $E(G)$ into k matchings [possibly, some sets of the partition are empty]. The chromatic index of G is the minimum k such that G has an edge k-coloring. As usual, $\Delta(G)$ denotes the maximum vertex degree of G.

Theorem 2 (König [6]). *For every bipartite graph G, $\chi'(G) = \Delta(G)$.*

Corollary 2. *All basic graphs are perfect.*

Proof. By Corollary 1, it is sufficient to show that all bipartite graphs and all line graphs of bipartite graphs are perfect.

For a bipartite graph G, we have $\omega(G) = \chi(G) = 2$, hence all bipartite graphs are perfect. It is easy to see that $\chi(L(G)) = \chi'(G)$ and $\omega(L(G)) = \Delta(G)$. Theorem 2 implies that $\omega(L(G)) = \chi(L(G))$, therefore $L(G)$ is a perfect graph. □

3 Forbidden induced subgraphs for basic graphs

The graphs $C_3 = K_3$, Claw $= K_{1,3}$, and Diamond along with their complements O_3, co-Claw, and co-Diamond are shown in Figure 1.

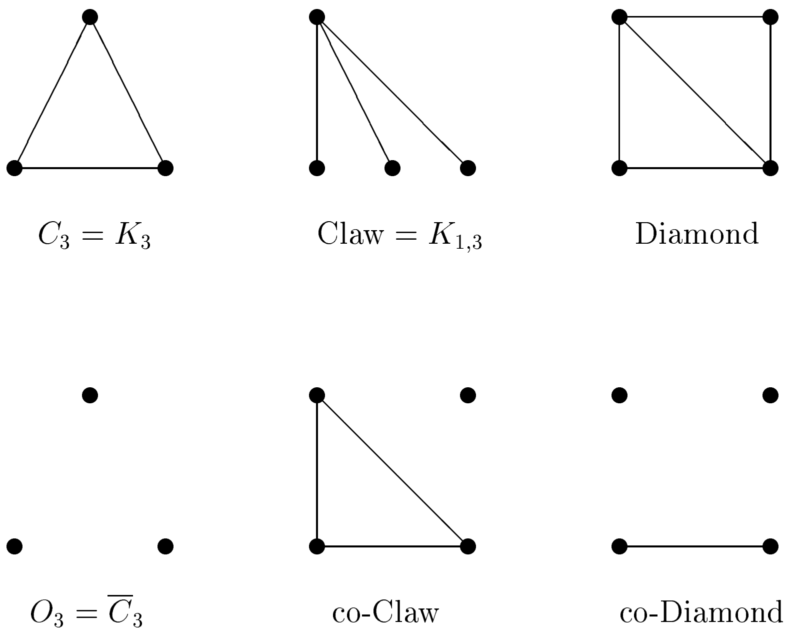


Figure 1.

For a set of graphs Z, a graph G is Z*-free* if it does not contain any graph of Z as an induced subgraph.

Theorem 3 (König [5]). *A graph is bipartite if and only if it does not contain all odd cycles as induced subgraphs.*

It follows that a Berge graph is bipartite if and only if it is C_3-free. Accordingly, a Berge graph is co-bipartite if and only if it is O_3-free.

Zverovich [13] and Gagarin and Zverovich [3] developed a general method to characterize hereditary classes of line graphs in terms of forbidden induced subgraphs. In particular, it is easy to characterize line graphs of bipartite graphs.

Corollary 3 (Hemminger and Beineke [4], Staton and Wingard [10]). *The class*
class
$\mathcal{LINEBIP}$ *coincides with the class of all* ($K_{1,3}$, Diamond, odd holes)*-free graphs.*

Corollary 4. *A Berge graph is the line graph of a bipartite graph if and only if it is a* (Claw, Diamond)*-free graph.*

It follows that $\mathcal{COLINEBIP}$ is exactly the class of all (co-Claw, co-Diamond)-free Berge graphs. A *monster* is a Berge graph which is a minimal forbidden induced subgraph for the class of all perfect graphs. Since any monster is not a basic graph, the results above imply that every monster must contain

- C_3, and
- O_3, and
- either Claw or Diamond, and
- either co-Claw or co-Diamond

as an induced subgraph. Table 1 shows four possible variants for an arbitrary non-basic Berge graph G, where "yes" (respectively, "no") means that G contains (respectively, does not contain) the corresponding induced subgraph.

Table 1. Possible variants for non-basic Berge graphs.

Class	C_3	O_3	Claw	Diamond	co-Claw	co-Diamond
Class 1			yes		yes	
Class 2	yes		yes		no	yes
Class 3		yes	no	yes	yes	
Class 4			no	yes	no	yes

For example, Class 1 consists of all Berge graphs that contain Claw and co-Claw as induced subgraphs. By Corollary 3, Class 1 is disjoint from $\mathcal{LINEBIP} \cup \mathcal{COLINEBIP}$. Since O_3 (respectively, C_3) is an induced subgraph of Claw (respectively, co-Claw), Class 1 is also disjoint from $\mathcal{BIP} \cup \mathcal{COBIP}$. Thus, Class 1 consists of non-basic Berge graphs only.

We used our computer software GraphLab (Zverovich [14]) to obtain an explicit forbidden induced subgraph characterization of the class $\mathcal{BASIC}$. For that, it is sufficient to generate all minimal [with respect to the partial order "to be an induced subgraph"] graphs in the four classes of Table 1. Orders of minimal graphs in Class 1 and Class 4 do not exceed eight. Table 1 gives a straightforward upper bound 11 on the maximum order of a minimal graph in Class 2 or in Class 3. In our computer database we have all graphs with at most 10 vertices. To reduce the upper bound, we can easily check that each minimal co-Claw-free Berge graph that contains C_3 and an induced Claw has at most five vertices [all such graphs are shown in Figure

2]. It follows that the maximum order of a minimal graph in Class 2 is at most nine. The same result holds for the complementary Class 3.

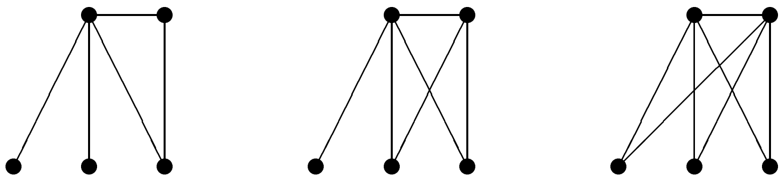


Figure 2.

We formulate our first result.

Theorem 4. *Each non-basic graph contains at least one of the graphs* $G_1, G_2, \ldots, G_{16}$ *(see Figure 3) or an* odd hole *or an* odd antihole *as an induced subgraph.*

4 Substitutional closure

Definition 4. *Let* G *and* H *be graphs. A* substitution of H in G instead of a vertex $v \in V(G)$ *is the graph* $G(v \to H)$ *consisting of disjoint union of* H *and* $G - v$ *with the additional edge-set* $\{xy : x \in V(H), y \in N_G(v)\}$.

For a class $\mathcal{P}$ of graphs, its *substitutional closure* $\mathcal{P}^*$ consists of all graphs that can be obtained from $\mathcal{P}$ by repeated substitutions, i.e., $\mathcal{P}^*$ is generated by the following rules:

(S1) $\mathcal{P} \subseteq \mathcal{P}^*$, and

(S2) if $G, H \in \mathcal{P}^*$ and $v \in V(G)$, then $G(v \to H) \in \mathcal{P}^*$.

Definition 5. *A set* $W \subseteq V(G)$ *is called* homogeneous *in a graph* G *if*

(H1) $2 \leq |W| \leq |V(G)| - 1$, *and*

(H2) $N(x) \backslash W = N(y) \backslash W$ *for each* $x, y \in W$.

According to (H2), a homogeneous set W defines a partition $W \cup W^+ \cup W^-$ of $V(G)$ such that

- every vertex of W is adjacent to every vertex of W^+ [notation $W \sim W^+$], and
- every vertex of W is non-adjacent to every vertex of W^- [notation $W \not\sim W^-$].

By (H1), $W^+ \cup W^- \neq \emptyset$ for every homogeneous set W. A simple observation is that if $|V(G)| > 1$, $|V(H)| > 1$ and $v \in V(G)$ then $V(H)$ is a homogeneous set in $G(v \to H)$.

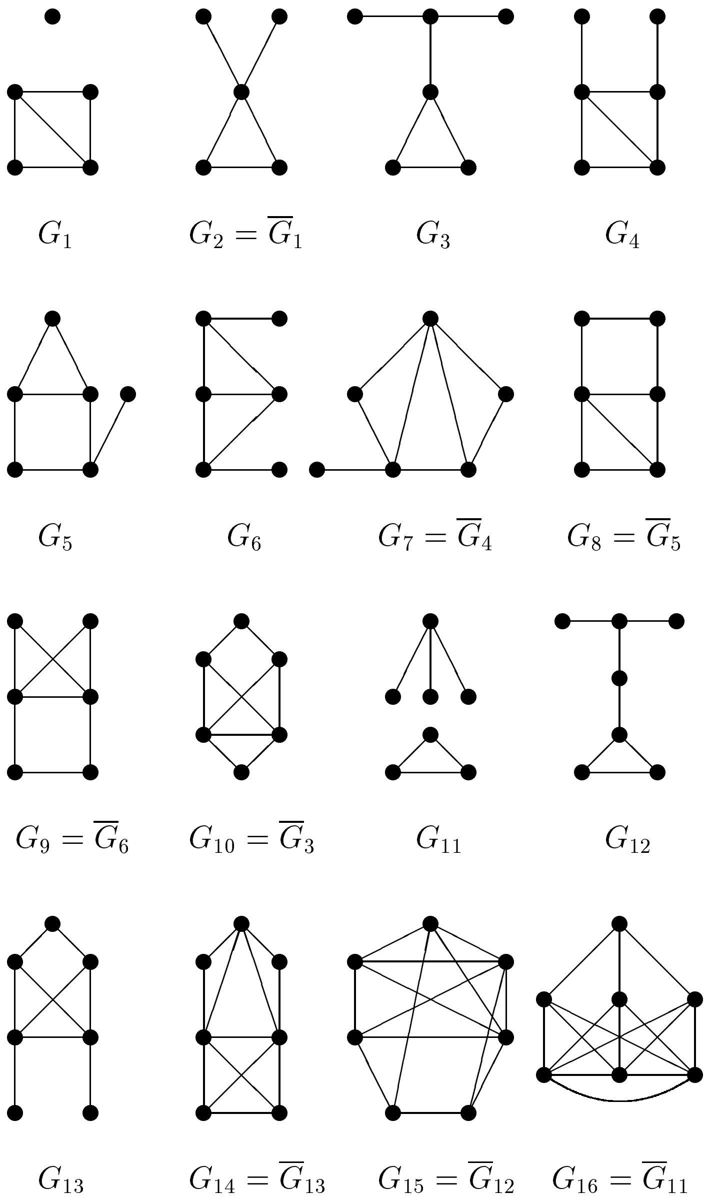


Figure 3. Minimal forbidden induced subgraphs for basic graphs within Berge graphs.

Definition 6. *A graph without homogeneous sets is called* prime. *A graph H is called a* (primal) extension *of a graph G if*

(E1) *G is an induced subgraph of H, and*

(E2) *H is a prime graph.*

Reducing Pseudopath Method for characterizing the substitutional closure of hereditary classes (Zverovich [12]) is based on the following definition.

Definition 7. *Let G be an induced subgraph of a graph H, and let W be a homogeneous set of G. We define a* reducing W-pseudopath (with respect to G) *in H as a sequence*

$$R = (u_1, u_2, \ldots, u_t),\ \ t \geq 1, \tag{2}$$

of pairwise distinct vertices of $V(H) \backslash V(G)$ satisfying the following conditions:

(R1) *there exist vertices $w_1, w_2 \in W$ such that*

- **(R1a)** *$u_1 \sim w_1$, and*
- **(R1b)** *$u_1 \not\sim w_2$;*

(R2) *for each $i = 2, 3, \ldots, t$ either*

- **(R2a)** *$u_i \sim u_{i-1}$ and $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$, or*
- **(R2b)** *$u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$*
  [if $i = 2$, $\{u_1, u_2, \ldots, u_{i-2}\} = \emptyset$];

(R3) *for every $i = 1, 2, \ldots, t-1$*

- **(R3a)** *$u_i \sim W^+$, and*
- **(R3b)** *$u_i \not\sim W^-$;*

(R4) *either*

- **(R4a)** *$u_t \not\sim x$ for a vertex $x \in W^+$, or*
- **(R4b)** *$u_t \sim y$ for a vertex $y \in W^-$.*

The *length* of a reducing pseudopath (2) is t.

Theorem 5 (Zverovich [12]). *Let H be an extension of its induced subgraph G, and let W be a homogeneous set of G. Then there exists a reducing W-pseudopath with respect to any induced copy of G in H.*

Lovász Replication Lemma (Lemma 1) implies that the class $\mathcal{PERFECT}$ of all perfect graphs is closed under substitutions of complete graphs, that is $G \in \mathcal{PERFECT}$ and $v \in V(G)$ implies that the graph $G(v \to K_n)$ is perfect for every $n \geq 1$. In fact, the class $\mathcal{PERFECT}$ is closed under substitutions of perfect graphs, i.e., $\mathcal{PERFECT} = \mathcal{PERFECT}^*$.

Theorem 6. *The class of all perfect graphs is closed under substitutions.*

Proof. Let $G, H \in \mathcal{PERFECT}$ and $v \in V(G)$. We show that the graph $F = G(v \to H)$ is perfect. We choose a maximum clique K in H, and denote $L = V(H)\backslash K$. By Lemma 1, the induced subgraph $F - L$ is perfect. In particular, we can color $F - L$ with $\omega(F - L) = \omega(F)$ colors. As a result, we color the clique K with $|K| = \omega(H)$ colors. Since H is a perfect graph, we can extend the $|K|$-coloring of K to a $|K|$-coloring of H, thus obtaining an $\omega(F)$-coloring of F.

For each induced subgraph F' of F, we also have $\chi(F') = \omega(F')$. Indeed, F' is either

- an induced subgraph of G [which is a perfect graph], or
- F' is an induced subgraph of H [which is also a perfect graph], or
- $F' = G'(v \to H')$, where G' is an induced subgraph of G and H' is an induced subgraph of H.

In the latter case, both G' and H' are perfect graphs. Therefore we may use the same proof as for G and H. □

We shall apply Reducing Pseudopath Method to the class of all basic graphs to characterize a wider class of perfect graphs.

5 Bounding a reducing pseudopaths

Zverovich [11] found some conditions on a homogeneous set W under which there exists a reducing W-pseudopath of bounded length. In particular, if W induces $P_2, P_3, \overline{P}_2$ or $\overline{P}_3$ there exists a reducing W-pseudopath of length $t = 1$.

Proposition 1. *Let W be a homogeneous set W in a graph G, and let H be a primal extension of G. If W induces $P_2, P_3, \overline{P}_2$ or $\overline{P}_3$, then there exists a set $Y \subseteq V(H)$ that induces G, and H contains a reducing W-pseudopath (2) with respect to $H(Y)$ having $t = 1$.*

Proof. Let $X \subseteq V(H)$ be a set that induces G in H. According to Theorem 5, there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ with respect to $G = H(X)$ in H. We may assume that t has the minimum value taken over all induced copies of G in H and all corresponding reducing pseudopaths.

Suppose that $t \geq 2$. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$ for some $w_1, w_2 \in W$. It is easy to see that there exists a vertex $w \in W$ such that the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G. Recall that according to (R3), $u_1 \sim W^+$ and $u_1 \not\sim W^-$, since $t \geq 2$. For example, if the vertices w_1 and w_2 are adjacent, then the set $Y = (X \backslash \{w_2\}) \cup \{u_1\}$ induces G [with u_1 instead of w_2].

The condition (R2) implies that $R' = (u_2, u_3, \ldots, u_t)$ is a reducing W-pseudopath with respect to $G = H(Y)$ in H. Since R' is shorter that R, we obtain a contradiction to minimality of R. $\square$

6 Extension of basic graphs

Recall that $\mathcal{BASIC}^*$ denotes the substitutional closure of the class $\mathcal{BASIC}$. All graphs in $\mathcal{BASIC}^*$ are called *superbasic graphs.*

Now we characterize superbasic graphs in terms of forbidden induced subgraphs.

Theorem 7. *The set of all minimal forbidden induced subgraphs for the class $\mathcal{BASIC}^*$ within Berge graphs is*

$$\mathbf{F} = \{F_1, F_2, \ldots, F_6, F_7, F_8, F_9 = \overline{F}_7, F_{10} = \overline{F}_8, F_{11}, F_{12} = \overline{F}_{11}\}$$

$$\cup\{\mathrm{Crocodile}(n), \overline{\mathrm{Crocodile}}(n) : n \geq 1\}$$

shown in Figure 4.

Proof. The fact that all graphs in $\mathbf{F}$ are minimal forbidden induced subgraphs for $\mathcal{BASIC}^*$ can be checked directly.

Now let H be an arbitrary minimal forbidden induced subgraphs for $\mathcal{BASIC}^*$. Note that H is a prime graph. We may assume that H is a Berge graph, and $H \notin \mathbf{F}$. By minimality of H, no graph in $\mathbf{F}$ is an induced subgraph of H.

Clearly, H is not a basic graph. Then Theorem 4 implies that at least one of the graphs $G_1, G_2, \ldots, G_{22}$ (Figure 3) is an induced subgraph of H. Note that $\mathbf{F}$ is a self-complementary set, and so is the class of all Berge graphs. In particular, if a graph G can not be an induced subgraph of H, then the complement of G also can not be an induced subgraph of H.

Claim 1. *H does not contain both G_4 and G_5 (Figure 3) as induced subgraphs.*

Proof. Indeed, the graphs G_4 and G_5 are included into $\mathbf{F}$ as F_1 and F_2, respectively. □

It follows that H also does not contain the complements of G_4 and G_5 [the graphs F_4 and F_5 in $\mathbf{F}$] as induced subgraphs.

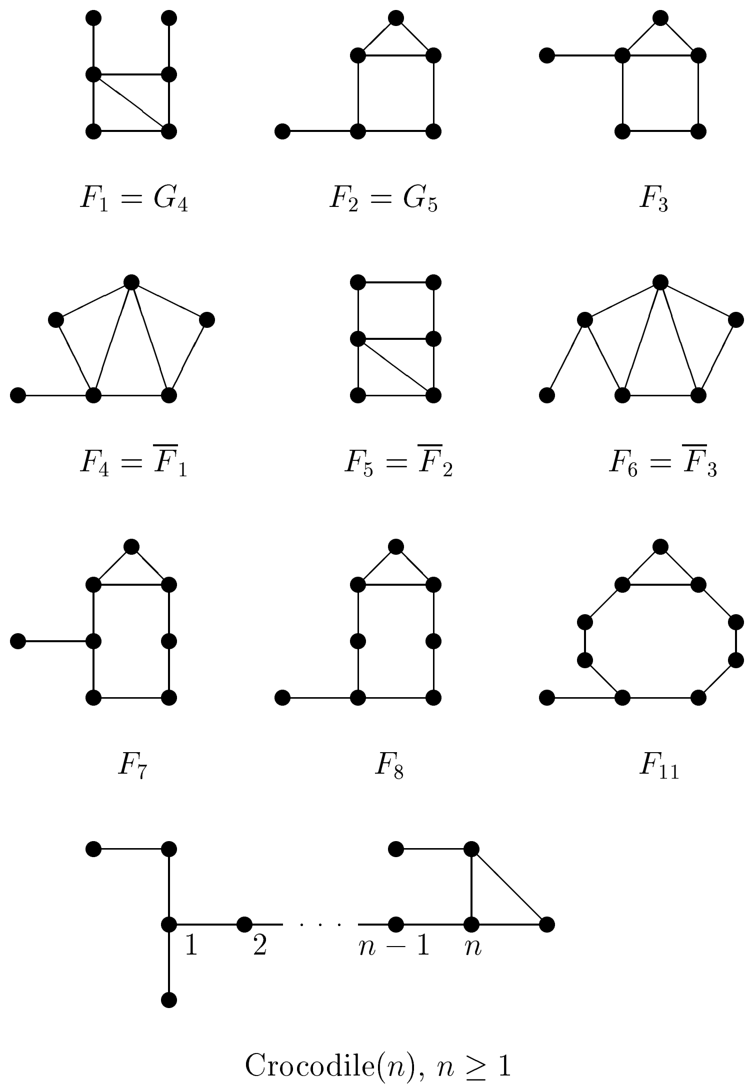


Figure 4. The set $\mathbf{F}$ [the graphs $F_9 = \overline{F}_7$, $F_{10} = \overline{F}_8$, $F_{12} = \overline{F}_{11}$ and $\overline{\text{Crocodile}(n)}$, $n \geq 1$, are not shown].

Now we consider the graph G_6 that has a unique homogeneous set.

Claim 2. *H does not contain both G_6 and G_9 (Figure 3) as induced subgraphs.*

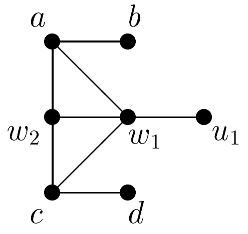


Figure 5. The graph G_6 and a reducing W-pseudopath (u_1).

Proof. Suppose that G_6 is an induced subgraph of H. The unique homogeneous set $W = \{w_1, w_2\}$ of G_6 is shown in Figure 5. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. We have the following possibilities:

- if u_1 is non-adjacent to both a and c, then we delete the vertex d and obtain either F_1 or F_5, a contradiction,
- if u_1 is adjacent to a and u is non-adjacent to c, then u_1 is non-adjacent to d, otherwise the set $\{u_1, d, c, w_2, a\}$ induces C_5, a contradiction [recall that H is a Berge graph]; then we delete b and obtain F_6, a contradiction,
- if u_1 is adjacent to both a and c, then $t = 1$ and (R4) imply that u_1 is adjacent to a vertex of $W^- = \{b, d\}$, say to b, and deleting a produces either F_1 or F_4, a contradiction.

Thus, G_6 is not an induced subgraph of H. The result for $G_9 = \overline{G_6}$ follows immediately. $\square$

The graph G_3 is more complicated, since it has two homogeneous sets.

Claim 3. *H does not contain both G_3 and G_{10} (Figure 3) as induced subgraphs.*

Proof. Suppose that G_3 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of G_3 are shown in Figure 6. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u_1')$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u_1' \sim x_1$, and $u_1' \not\sim x_2$.

Case 1. $u_1 = u_1'$.

We need only specify edges between u_1 and $\{a, b\}$: if $u \not\sim \{a, b\}$, then $H(u_1, x_1, b, a, w_1)$ is isomorphic to C_5, a contradiction. Otherwise we delete x_2 and obtain one of F_2, F_3 or F_4, a contradiction.

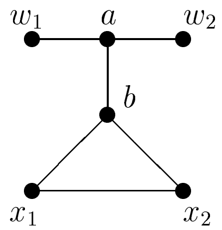


Figure 6. The graph G_3.

Case 2. $u_1 \neq u'_1$.

We consider separately the variants for the induced subgraph $H(V(G_3) \cup \{u_1\})$ [Subcase 2A] and for $H(V(G_3) \cup \{u'_1\})$ [Subcase 2B]. Then we shall compile the results together.

Subcase 2A.

The condition $t = 1$ and (R4) imply that either u_1 is non-adjacent to a vertex of $W^+ = \{a\}$, or u_1 is adjacent to a vertex of $W^- = \{a, x_1, x_2\}$. Note that u_1 can be adjacent to both x_1 and x_2 or to none of them, since $u_1 \neq u'_1$. We delete the vertex x_2. As a result we either obtain one of the forbidden induced subgraphs [F_1, F_2, F_3, F_4 or a graph containing C_5], or one of the graphs A_1, A_2 of Figure 7.

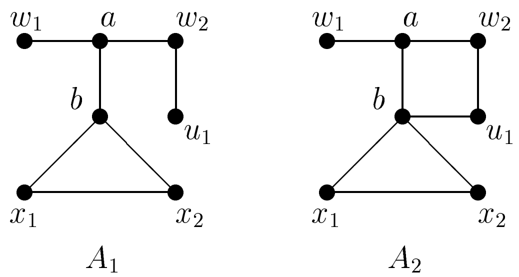


Figure 7. The variants A_1 and A_2.

Subcase 2B.

The condition $t = 1$ and (R4) imply that either u'_1 is non-adjacent to a vertex of $X^+ = \{b\}$, or u'_1 is adjacent to a vertex of $X^- = \{a, w_1, w_2\}$. Note that u'_1 can be adjacent to both w_1 and w_2 or to none of them, since $u_1 \neq u'_1$. We delete the vertex

w_2. As a result, we either obtain one of the forbidden induced subgraphs [F_2, F_5, F_6 or a graph containing C_5], or one of the graphs B_1, B_2, B_3 of Figure 8.

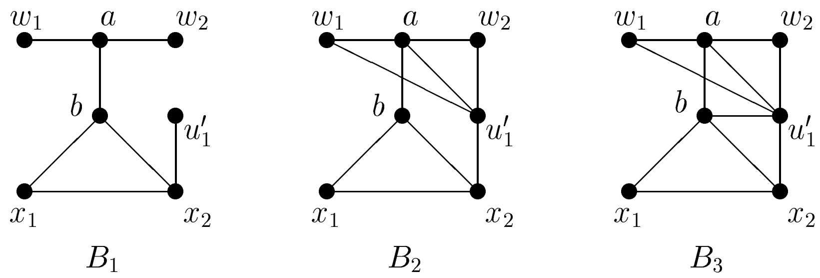


Figure 8. The variants B_1, B_2 and B_3.

Now we consider all combinations (A_i, B_j), $i = 1, 2$ and $j = 1, 2, 3$ (Table 2). Note that each of the six combinations has two variants depending on adjacency of u_1 and u_1'.

Table 2. Forbidden induced subgraphs in combinations (A_i, B_j).

Combination	Forbidden induced subgraphs
(A_1, B_1)	is isomorphic to F_7 or Crocodile(2)
(A_1, B_2)	contains F_1 or F_4
(A_1, B_3)	contains F_1 or F_4
(A_2, B_1)	contains F_2 or Crocodile(1)
(A_2, B_2)	contains F_2
(A_2, B_3)	contains F_4

Thus, all the combinations produce contradiction. The result for $G_{10} = \overline{G}_3$ is straightforward. $\square$

The graph G_1 has several homogeneous sets, the largest being $W = \{w_1, w_2, w_3, w_4\}$ (Figure 9). First we reduce G_1 to a set graphs that have simpler structures of homogeneous sets.

Claim 4. *Suppose that H contains G_1 or $G_2 = \overline{G}_1$ as an induced subgraph. Then at least one of $H_1, H_2, \ldots, H_8$ (Figure 10) or their complements is an induced subgraph of H.*

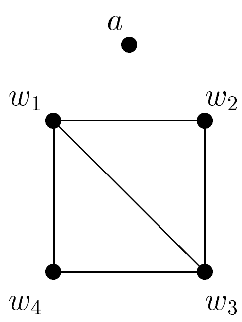


Figure 9. The graph G_1.

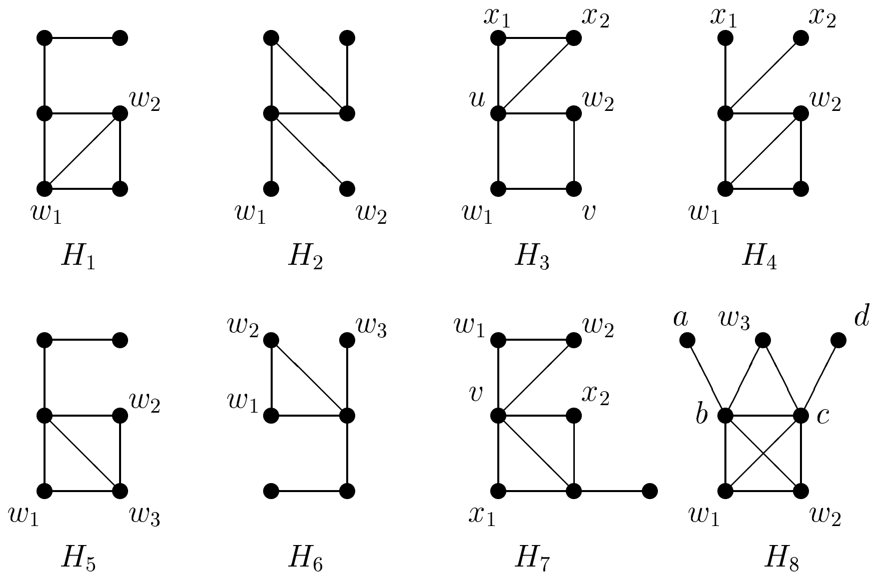


Figure 10. The graphs $H_1, H_2, \ldots, H_8$.

Proof. Suppose that G_1 is an induced subgraph of H. We consider the homogeneous set $W = \{w_1, w_2, w_3, w_4\}$ of G_1 (see Figure 9). By Theorem 5, there exists a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$ in H. We may assume that R is the shortest reducing W-pseudopath over all induced copies of G_1 in H. According to (R1), u_1 is adjacent to a vertex of W, and u_1 is non-adjacent to a vertex of W. Due to symmetry, there are seven possibilities for adjacencies u_1 and the vertices of W, namely

Possibility 1 $N(u_1) \cap W = \{w_1\}$,

Possibility 2 $N(u_1) \cap W = \{w_2\}$,

Possibility 3 $N(u_1) \cap W = \{w_2, w_4\}$,

Possibility 4 $N(u_1) \cap W = \{w_1, w_2\}$,

Possibility 5 $N(u_1) \cap W = \{w_1, w_3\}$,

Possibility 6 $N(u_1) \cap W = \{w_1, w_2, w_3\}$, and

Possibility 7 $N(u_1) \cap W = \{w_1, w_2, w_4\}$.

Case 1. $t = 1$.

Since $t = 1$, the vertex u_1 must be adjacent to a [by (R4)]. It can be easily checked that Possibilities 1-7 produce the graphs H_5, H_1, $\overline{H}_3$, F_6, $\overline{H}_4$, $\overline{H}_2$, and $\overline{H}_6$, respectively, and the result follows.

Case 2. $t \geq 2$ in Possibilities 4, 5, 6, 7.

By (R3b), u_1 is non-adjacent to the vertex $a \in W^-$. We show that each of Possibilities 4, 5, 6, 7 produce a contradiction to minimality of R. Indeed, we can change an appropriate vertex $w \in W$ with u_1 and obtain a new induced copy of G_1, with $R' = (u_2, u_3, \ldots, u_t)$ being a shorter reducing W'-pseudopath, where $W' = (W \backslash \{w\}) \cup \{u_1\}$.

In Case 3 and Case 4, we use the fact that the condition (R2) determines exactly two variants for adjacencies of each vertex u_i, $i \geq 2$, with the set $W \cup \{u_1, u_2, \ldots, u_{i-1}\}$.

Case 3. $t = 2$ in Possibilities 1, 2, 3.

We can easily construct all graphs corresponding to Possibilities 1, 2, 3 with $t = 2$.

- Possibility 1 produces H_5 and $\overline{H}_8$,
- Possibility 2 produces H_1 and F_1, and
- Possibility 3 produces $\overline{H}_3$ and $\overline{H}_7$.

Case 4. $t \geq 3$ in Possibilities 1, 2, 3.

If the vertex u_2 satisfies (R2a) [u_2 is adjacent to u_1, but u_2 is not adjacent to all the vertices in W], then Possibilities 1, 2, 3 produce H_5, H_1 and $\overline{H}_3$, respectively.

Suppose that u_2 satisfies (R2b) [u_2 is non-adjacent to u_1, but u_2 is adjacent to all the vertices in W], then we may change the vertex $w_1 \in W$ with u_2 and obtain a new copy of G_1 in H. With respect to the new copy of G_1, we have a shorter reducing W'-pseudopath $R' = (u_3, u_4, \ldots, u_t)$ where $W' = (W \backslash \{w_1\}) \cup \{u_2\}$, a contradiction to minimality of R. $\square$

By Claim 4, the variant of an induced G_1 or G_2 is reducible to the graphs $H_1, H_2, \dots, H_8$ of Figure 10. We shall show that all of them are impossible.

Claim 5. *H does not contain H_1, H_2 or their complements as induced subgraphs.*

Proof. Suppose that H_1 is an induced subgraph of H. The unique homogeneous set $W = \{w_1, w_2\}$ of H_1 is shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. It is easy to check that we obtain at least one of the following graphs: F_1, F_4, F_5, F_6 or C_5, a contradiction.

Let H contain H_2 as an induced subgraph. Again, H_2 has a unique homogeneous set, namely $W = \{w_1, w_2\}$ (Figure 10). By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$. It is easy to check that we obtain at least one of the following graphs: F_1, F_2, F_3, F_4, Crocodile(1) or C_5, and the result follows. □

Claim 6. *H does not contain H_3 or its complement as an induced subgraph.*

Proof. Suppose that H_3 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of H_3 are shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u'_1)$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u'_1 \sim x_1$, and $u'_1 \not\sim x_2$.

Case 1. $u_1 = u'_1$.

Deleting one of the vertices v, w_1 or x_1 produces F_3 or F_5, a contradiction.

Case 2. $u_1 \neq u'_1$.

Since (u_1) is not a reducing X-pseudopath, either (a1) $u_1 \not\sim X$ or (a2) $u_1 \sim X$.

(a1) The condition (R4) shows that u_1 is non-adjacent to a vertex of $W^+ = \{u, v\}$. Then deleting w_2 or x_2 produces G_3, F_2 or F_3, a contradiction [recall that H does not contain G_3 by Claim 3].

(a2) The case $N(u_1) = \{u, v, w_1, x_1, x_2\}$ will be considered later. In the other three cases, we delete one or both of the vertices w_1 and x_2, and obtain H_1, F_5 or C_5, a contradiction [H does not contain H_1 by Claim 5].

Since (u'_1) is not a reducing W-pseudopath, either (b1) $u'_1 \not\sim W$ or (b2) $u_1 \sim W$.

(b1) The condition (R4) shows that either u'_1 is non-adjacent to $u \in X^+$, or u'_1 is adjacent to $v \in X^-$. Then deleting v or w_1 produces H_2 or F_5, a contradiction [H does not contain H_2 by Claim 5].

(b2) The case $N(u'_1) = \{u, v, w_1, w_2, x_1\}$ will be considered later. In the other three cases, we delete v or w_1, and obtain $\overline{H}_1$ or F_6, a contradiction [H does not contain $\overline{H}_1$ by Claim 5].

It remains to consider the situation where $N(u_1) = \{u, v, w_1, x_1, x_2\}$ and $N(u_1') = \{u, v, w_1, w_2, x_1\}$. The vertices u_1 and u_1' may or may not be adjacent. Deleting $\{u, x_1\}$ produces either F_6 [if u_1 and u_1' are non-adjacent], or $\overline{H}_2$ [if u_1 and u_1' are adjacent], a contradiction, since H does not contain $\overline{H}_2$ by Claim 5. $\square$

Claim 7. *H does not contain H_4 or its complement as an induced subgraph.*

Proof. Suppose that H_4 is an induced subgraph of H. It is sufficient to consider one of the two homogeneous sets in H_4, namely $X = \{x_1, x_2\}$ (see Figure 10). By Proposition 1, there exists a reducing X-pseudopath $R = (u_1)$ in H. By (R1), $u_1 \sim x_1$ and $u_1 \not\sim x_2$.

If u_1 is adjacent to exactly one vertex of W, we obtain a graph containing F_1 or F_3 as an induced subgraph, a contradiction.

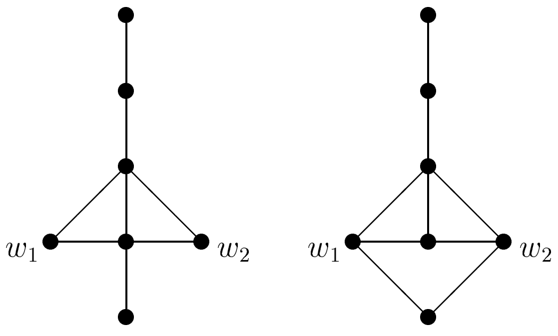
If $u_1 \not\sim W$, then either u_1 is non-adjacent to the unique vertex of X^+ or u_1 is adjacent to the unique vertex of $X^- \setminus W$ [the condition (R4) for $t = 1$]. Up to symmetry, we have three variants producing F_3 or H_1 or C_5, a contradiction.

If $u_1 \sim W$, then we have four variants containing at least one of the graphs F_1, F_2, F_4 or G_{10} as an induced subgraph, a contradiction. $\square$

Claim 8. *H does not contain H_5 or its complement as an induced subgraph.*

Proof. Let H_5 be an induced subgraph of H. We consider the maximal homogeneous set of H_5, namely $W = \{w_1, w_2, w_3\}$ (Figure 10). By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ with respect to H_5 in H.

It can be easily checked that $V(H_5) \cup \{u_1\}$ contains one of the forbidden induced subgraphs F_1, F_2, F_4, F_5, F_6 or C_5, or an induced G_3, or an induced H_3, unless we have the variants H_5^1 and H_5^2 shown in Figure 11. Both G_3 and H_3 are impossible by Claim 3 and Claim 6. Therefore it remains to consider the graphs H_5^1 and H_5^2, each of them having $W = \{w_1, w_2\}$ as a unique homogeneous set. A straightforward application of Proposition 1 produces graphs containing at least one of $F_1, F_2, \ldots, F_6$ as an induced subgraph, a contradiction. $\square$



H_5^1 H_5^2

Figure 11. The variants H_5^1 and H_5^2.

Claim 9. *H does not contain H_6 or its complement as an induced subgraph.*

Proof. Let H_6 be an induced subgraph of H. It will be sufficient to consider a homogeneous set $W = \{w_1, w_2\}$ (Figure 10) which is not maximal. We apply Proposition 1 and obtain graphs containing at least one of $F_1, F_2, \ldots, F_6$ or H_2, H_5 as an induced subgraph, a contradiction. The result follows from Claim 2 and Claim 9. $\square$

Claim 10. *H does not contain H_7 or its complement as an induced subgraph.*

Proof. Suppose that H_7 is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of H_7 are shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u_1')$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u_1' \sim x_1$, and $u_1' \not\sim x_2$.

Case 1. $u_1 = u_1'$.

Deleting two appropriate vertices produces F_1 or F_4, a contradiction.

Case 2. $u_1 \neq u_1'$.

Since (u_1) is not a reducing X-pseudopath, either (a1) $u_1 \not\sim X$ or (a2) $u_1 \sim X$.

(a1) The condition (R4) implies that we have seven variants for the subgraph induced by $V(H_7) \cup \{u_1\}$. All of them are impossible, since at least one of F_3, F_4, F_5, H_2 or H_5 appears as an induced subgraph.

(a2) We obtain induced subgraphs F_5, F_6 or $\overline{H}_1$ unless $N(u_1) = V(H_7) \backslash \{w_2\}$. The latter case will be considered later.

Since (u_1') is not a reducing W-pseudopath, either (b1) $u_1' \not\sim W$ or (b2) $u_1 \sim W$.

(b1) The condition (R4) implies that we have seven variants for $H(V(H_7) \cup \{u_1'\})$. It is easy to check that forbidden induced subgraphs F_1, F_3, F_4 appear, a contradiction.

(b2) In this case we have forbidden induced subgraphs F_4, F_5, F_6 or H_3, unless either $N(u_1') = V(H_7)\backslash\{x_2\}$ or $N(u_1') = V(H_7)\backslash\{v, x_2\}$ (see Figure 10).

It remains to consider the situation where $N(u_1) = V(H_7)\backslash\{w_2\}$ and either $N(u_1') = V(H_7)\backslash\{x_2\}$ or $N(u_1') = V(H_7)\backslash\{v, x_2\}$. The set $\{w_1, w_2, x_1, x_2, u_1, u_1'\}$ induces either F_2 [if u_1 and u_1' are non-adjacent], or F_4 [if u_1 and u_1' are adjacent], a contradiction. □

Claim 11. *H does not contain H_8 or its complement as an induced subgraph.*

Proof. Suppose that H_8 is an induced subgraph of H. The unique maximal homogeneous set $W = \{w_1, w_2, w_3\}$ H_8 in shown in Figure 10. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. According to (R1), u_1 is adjacent to a vertex of W, and u_1 is non-adjacent to a vertex of W. Due to symmetry, we have four possibilities for adjacency u_1 an W. Recall that u_1 must be either adjacent to a vertex of W^-, or non-adjacent to a vertex of W^+.

Case 1. $N_H(u_1) \cap W = \{w_1\}$.

Deleting two appropriate vertices produces F_1, F_3, F_4 or F_5, a contradiction.

Case 2. $N_H(u_1) \cap W = \{w_3\}$.

Either we can delete two vertices and obtain one of forbidden induced subgraphs F_1, F_3, F_4, F_5, or we delete a vertex of W^+ and obtain an induced H_7, a contradiction to Claim 9.

Case 3. $N_H(u_1) \cap W = \{w_1, w_3\}$.

In this case we have one of forbidden induced subgraphs F_3, F_4, or $\overline{H}_1$. By Claim 5, $\overline{H}_1$ can not be an induced subgraph of H.

Case 4. $N_H(u_1) \cap W = \{w_1, w_2\}$.

In this case we either have one of forbidden induced subgraphs F_1, F_3, F_4, F_5, G_6, or an induced H_8' shown in Figure 12.

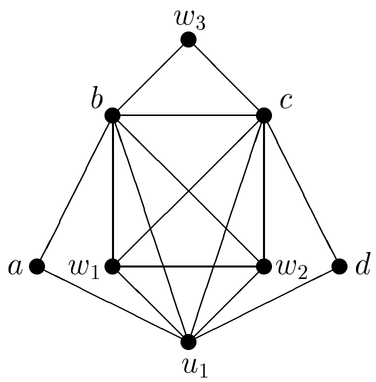


Figure 12. Graph H_8'.

Now we consider the graph H_8' having $W' = \{w_1, w_2\}$ as a homogeneous set. By Proposition 1, there exists a reducing W'-pseudopath $R' = (u_1')$ in H. According to (R1), we may assume that u_1' is adjacent to w_1, and u_1' is non-adjacent to w_2. If we delete u_1, we obtain essentially Case 1 or Case 4 above [with u_1' instead of u_1]. Therefore $u_1' \sim W^+ = \{b, c\}$ and $u_1' \not\sim W^- = \{a, d\}$. The edges $u_1'w_3$ and $u_1'u_1$ are not specified now.

If u_1' and u_1 are non-adjacent, then the set $\{a, b, d, w_1, u_1, u_1'\}$ induces F_4, a contradiction. If u_1' and u_1 are adjacent, then u_1' must be adjacent to w_3 [according to (R4)], and the set $\{b, d, w_1, w_3, u_1, u_1'\}$ induces F_4, a contradiction. □

At the moment, we have shown that H does not contain $G_1, G_2, \ldots, G_{10}$ (Figure 3) as induced subgraphs. Since $G_{14} = \overline{G}_{13}$, $G_{15} = \overline{G}_{12}$, and $G_{16} = \overline{G}_{11}$ (Figure 3), it remains to consider the graphs G_{11}, G_{12} and G_{13}.

Claim 12. *H does not contain G_{12} or its complement as an induced subgraph.*

Proof. Suppose that G_{12} is an induced subgraph of H. The two homogeneous sets $W = \{w_1, w_2\}$ and $X = \{x_1, x_2\}$ of G_{12} are shown in Figure 13. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H and there exists a reducing X-pseudopath $R' = (u_1')$ in H. According to (R1), $u_1 \sim w_1$, $u_1 \not\sim w_2$, $u_1' \sim x_1$, and $u_1' \not\sim x_2$.

Case 1. $u_1 = u_1'$.

We obtain an induced G_4, G_5, G_7 or C_5 or F_8 (Figure 13), a contradiction.

Case 2. $u_1 \neq u_1'$.

It is easy to check that $N_H(u_1) \cap V(G_{12}) = \{w_1\}$ and $N_H(u_1') \cap V(G_{12}) = \{x_1\}$, otherwise some of $G_1, G_2, G_4, G_5, G_7, G_8$ or C_5 appears as an induced subgraph. If

$u_1 \sim u_1'$, then we have an induced C_7, a contradiction. Thus, u_1 and u_1' are non-adjacent, the set $V(G_{12}) \cup \{u_1, u_1'\}$ induces Crocodile(3) (Figure 13). ☐

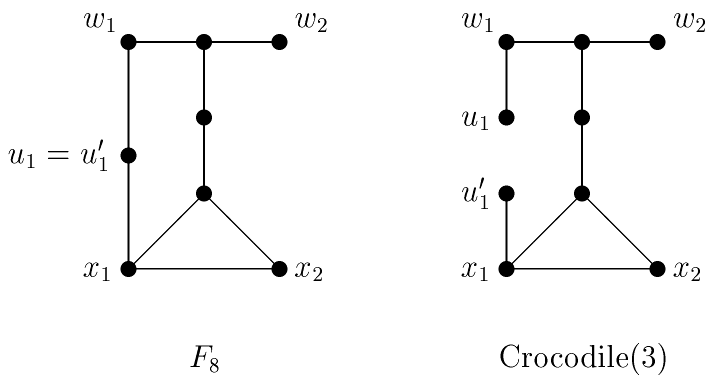


Figure 13. Variants for G_{12}.

Claim 13. *H does not contain G_{13} or its complement as an induced subgraph.*

Proof. A straightforward application of Proposition 1 to the unique homogeneous set of G_{13} produces graphs containing at least one of G_1, G_2, G_5 or G_8 as an induced subgraph, a contradiction. ☐

Now we reduce the most complicated graph G_{11} to a series of graphs having simpler structures of homogeneous sets.

Claim 14. *If H contains G_{11} as an induced subgraph, then at least one of the graphs T_n, $n \geq 4$, shown in Figure 14 as an induced subgraph of H.*

Proof. Let $W \cup X$ induce G_{12} in H, where $W = \{w_1, w_2, w_3\}$ induces a triangle, and $X = \{a, b, c, d\}$ induces a Claw centered at a. By Theorem 5, H contains a reducing W-pseudopath $R = (u_1, u_2, \ldots, u_t)$. We may assume that R is the shortest pseudopath over all induced copies of G_{11} in H.

If $t = 1$, the vertex u_1 is adjacent to a vertex of W, and u_1 is non-adjacent to a vertex of W [by (R1)], and u_1 is adjacent to a vertex of $X = W^-$ [by (R4), since $W^+ = \emptyset$]. It is easy to check that at least of G_2, G_3, G_{12} or T_4 appears as an induced subgraph, and the result follows. Thus, we may assume that $t \geq 2$.

If the vertex u_1 is adjacent to two vertices of W, say w_1 and w_2, then we can delete w_3 and obtain a copy of G_{11} induced by $\{w_1, w_2, u_1\} \cup X$. We have a shorter reducing $\{w_1, w_2, u_1\}$-pseudopath, namely $R' = (u_2, u_3, \ldots, u_t)$, a contradiction to the choice of R. Thus, u_1 is adjacent to exactly one vertex of W.

Now we show that each vertex u_i, $i = 2, 3, \ldots, t-1$, satisfies (R2a), that is $u_i \sim u_{i-1}$ and $u_i \not\sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$. If it does not hold, the condition (R2) implies that there exists $i \in \{2, 3, \ldots, t-1\}$ such that u_i satisfies (R2b). In other words, $u_i \not\sim u_{i-1}$ and $u_i \sim W \cup \{u_1, u_2, \ldots, u_{i-2}\}$. Then we can change w_3 with u_i and obtain a copy of G_{11} induced by $\{w_1, w_2, u_i\} \cup X$. As before, we have a shorter reducing $\{w_1, w_2, u_i\}$-pseudopath, namely $R_i = (u_{i+1}, u_{i+2}, \ldots, u_t)$, a contradiction to the choice of R. Thus, $u_1, u_2, \ldots, u_{t-1}$ is a path.

According to (R4), u_t is adjacent to at least one vertex of X. It is easy to see that if u_t satisfies (R2b), then the set $W \cup X \cup \{u_t\}$ induces a subgraph containing G_2, G_3 or G_{12} as an induced subgraph, a contradiction. It follows that u_t satisfies (R2a). In other words, R is an induced path with u_1 being adjacent to exactly one vertex of W.

It $N_H(u_t) \cap X$ induced a complete subgraph, then we have an induced subgraph T_n, $n \geq 4$, and the result follows. Otherwise u_t is adjacent to distinct non-adjacent vertices $x, x' \in X$. We can delete $X \backslash \{x, x'\}$ and obtain either an induced G_{12} [if $t = 2$] or an induced T_n, $n \geq 4$ [if $t \geq 3$]. ☐

Accordingly, if H contains $\overline{G}_{11}$ as an induced subgraph, then at least one of the graphs $\overline{T}_n$, $n \geq 4$, is an induced subgraph of H.

Claim 15. *H does not contain T_n, $n \geq 4$, shown in Figure 14 as an induced subgraph.*

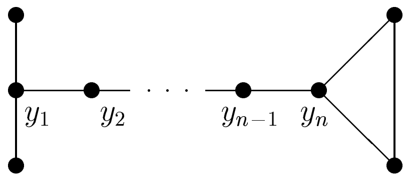


Figure 14. Graphs T_n.

Proof. Suppose that H contains an induced subgraph $T = T_n$, $n \geq 4$. We shall assume that T has the smallest possible value of $n \geq 4$. First we consider the homogeneous set $W = \{w_1, w_2\}$ in T, see Figure 14 for notation of vertices. By Proposition 1, there exists a reducing W-pseudopath $R = (u_1)$ in H. By (R1), $u_1 \sim w_1$ and $u_1 \not\sim w_2$.

Fact 1. $N_H(u_1) \cap V(T) = \{w_1\}$.

Proof. We show that u_1 is non-adjacent to y_1. Indeed, otherwise either $H(w_1, w_2, y_1, y_2, u_1) = G_2$ [if $u_1 \not\sim y_2$] or $H(w_1, w_2, y_1, y_2, y_3, u_1) \in \{G_4, G_7\}$ [if $u_1 \sim y_2$], a contradiction.

Now we prove that $u_1 \not\sim \{x_1, x_2, y_n\}$. Note that the vertices y_2 and y_n are non-adjacent, since $n \geq 4$. Assuming that u_1 is adjacent to $r \geq 1$ vertices of the triangle $C = \{x_1, x_2, y_n\}$, we have the following possibilities:

- if $u_1 \sim y_2$ and $r = 1$, then the set $C \cup \{u_1, w_1, y_2\}$ induces G_3,
- if $u_1 \sim y_2$ and $r \geq 2$, say $u_1 \sim \{x_1, x_2\}$, then $\{u_1, w_1, y_2, x_1, x_2\}$ induces G_2,
- if $u_1 \not\sim y_2$ and $r = 1$, then deleting $y_3, y_4, \dots, y_{n-1}$ produces an induced T_4, therefore minimality of T implies that $n = 4$, and we obtain either an induced G_2, C_5 or C_7,
- if $u_1 \not\sim y_2$ and $r \geq 2$, say $u_1 \sim \{x_1, x_2\}$, then the set $\{u_1, w_1, w_2, y_1, y_2, x_1, x_2\}$ induces G_{12}.

Since all the cases produces contradictions, we have $u_1 \not\sim C$.

Suppose that there exists a maximum $i \in \{3, 4, \dots, n-1\}$ such that u_1 is adjacent to y_i. Note that absence of C_5 and G_5 implies that $i \geq 4$. Hence the set $\{u_1, y_{i-1}, y_i, \dots, y_n, x_1, x_2\}$ induces either G_3 or G_{12} or T_k with $k < n$, a contradiction. Then either $N_H(u_1) \cap V(T) = \{w_1\}$ [and the proof is complete], or $N_H(u_1) \cap V(T) = \{w_1, y_2\}$. In the latter case we can delete $\{w_2, y_1\}$ and obtain a contradiction to minimality of n. □

Now we consider the homogeneous set $X = \{x_1, x_2\}$ in T, (see Figure 14). By Proposition 1, there exists a reducing X-pseudopath $R' = (u_1')$ in H. By (R1), $u_1' \sim x_1$ and $u_1' \not\sim x_2$.

Fact 2. $N_H(u_1') \cap V(T) = \{x_1\}$.

Proof. We show that u_1' is non-adjacent to y_n. Indeed, if it does not hold then either $H(y_i, u_1, y_n, x_1, x_2) = G_1$ [if u_1' is non-adjacent to some y_i, $i \leq n-2$] or we have an induced G_8 [if $u_1 \not\sim y_{n-1}$], or the set $\{u_1', y_{n-3}, y_{n-2}, y_{n-1}, x_2\}$ induces G_1, a contradiction.

Now we prove that $u_1' \not\sim C'$, where $C' = \{w_1, w_2, y_1, y_2\}$. Note that the vertices y_2 and y_n are non-adjacent, since $n \geq 4$. Assuming that u_1' is adjacent to $r \geq 1$ vertices of the C', we have either obtain induced G_3, G_{12} which is impossible, or u_1' is adjacent to exactly one vertex c of C' and $c \neq y_1$. Since an induced T_4 appears, we must have $n = 4$. Then it is easy to see that H has at least on of the following forbidden induced subgraphs C_5, C_7 or G_3, a contradiction. Thus, $u_1' \not\sim C'$.

Suppose that there exists a minimum $i \in \{2, 3, \dots, n-2\}$ such that u_1' is adjacent to y_i. Then we can easily construct an induced subgraph T_k with $k < n$, a contradiction to minimality of n.

It follows that either $N_H(u_1') \cap V(T) = \{x_1\}$ [and the proof is complete], or $N_H(u_1') \cap V(T) = \{x_1, y_{n-1}\}$. In the latter case we have an induced G_5, a contradiction. □

Fact 1 and Fact 2 imply that the set $V(T) \cup \{u_1, u_1'\}$ induces one of the following subgraphs:

- Crocodile(n), $n \geq 4$ [if u_1 and u_1' are non-adjacent], or
- F_{11} [if u_1 and u_1' are adjacent and $n = 4$], or
- a graph containing an induced C_9 [if u_1 and u_1' are adjacent and $n = 5$], or
- a graph containing an induced Crocodile(5) [if u_1 and u_1' are adjacent and $n \geq 6$]; in this case we delete the vertices $y_4, y_5, \ldots, y_{n-2}$.

□

Since all graphs G_i in Figure 3 are already considered, Theorem 7 is proved. □

Corollary 5. *Each monster contains at least one graph of the family* **F** *an induced subgraph.*

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