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Perfect Graphs, Kernels, and Cores of Cooperative Games[1]

by

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[1] *Dedicated to the memory of Claude Berge*

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ABSTRACT

A kernel in a directed graph is an independent set, which is reachable from each other vertex by an arc. A directed graph in which every induced subgraph has a kernel is called kernel-perfect. The existence of a kernel in directed graphs, and in particular, kernel-perfect orientations of undirected graphs is strongly related to perfect graphs, and has several applications in combinatorics and game theory. These and related results are the subject of this survey. Though some of these results are independent of the Strong Perfect Graph Conjecture, the recent solution of this conjecture and the efficient recognition of perfect graphs have several important implications, in particular in game theory.

Key words: kernel, kernel-solvable graphs, perfect graphs, normal hypergraphs, Strong Perfect Graph Conjecture, Berge-Duchet conjecture; cooperative games, coalition, stable family of coalitions, game form, effectivity function, stable effectivity functions, Scarf theorem.

1 Introduction

We assume that the reader is familiar with standard terminology of graph theory. As customary, we shall denote for a given graph $G = (V, E)$ by $\omega(G)$ its clique number (the maximum size of a clique), by $\chi(G)$ its chromatic number (the minimum number of colors in a vertex coloring), etc. For clarity, we always refer to undirected graphs as $G = (V, E)$, while we use $D = (V, A)$ for directed graphs, and we write $V = V(G) = V(D)$, $E = E(G)$ and $A = A(D)$. A directed arc oriented from vertex u to v is denoted by $[u, v)$, while an undirected edge between the same vertices is written as (u, v).

A graph $G = (V, E)$ is called *Berge* if it does not contain an induced odd hole (chordless odd cycle of length ≥ 5) or odd anti-hole (complement of an odd hole). It is called *perfect* if $\omega(G') = \chi(G')$ for all induced subgraphs G' of G. The *Strong Perfect Graph Conjecture* (SPGC in short) introduced by Berge [7, 8] claims the equivalence of these two properties[1], or in other words that Berge graphs and perfect graphs are the same.

A subset S of the vertices of a directed graph $D = (V, A)$ is called *absorbing* if for every vertex $u \notin S$ there exists a vertex $v \in S$ such that $[u, v) \in A$. A vertex subset S is called a *kernel* (of D), if it is independent and absorbing.

An *orientation* of an undirected graph $G = (V, E)$ is a directed (multi-) graph $D = (V, A)$ obtained from G by orienting every edge of it either one way or both ways. Such an orientation is called *clique-acyclic* if in every clique C of G there exists an *absorbing* vertex, that is a vertex $v \in C$ such that $[u, v) \in A$ for all $u \in C \setminus \{v\}$, or in other words, if every clique has a kernel. Finally a graph $G = (V, E)$ is called *kernel solvable* if every clique acyclic orientation of it has a kernel.

Kernel solvable graphs were considered by Berge and Duchet [16], who conjectured that they are the same as perfect graphs. One direction, namely that kernel solvable graphs are perfect follows from SPGC, since it is easy to see that graphs containing induced odd holes or odd anti-holes are not kernel solvable. The other half of the conjecture, namely that perfect graphs are kernel solvable, was proved in [31]. Modulo SPGC all three properties, Berge, perfect and kernel solvable, are equivalent, showing possibly three different faces of the same class of graphs.

Despite the seemingly complicated definitions, the notions of kernels and kernel solvability have many applications and several interesting relations to other areas, most notably to game theory. These notions and the related results are the subject of our short survey. Though some of these results are independent of the Strong Perfect Graph Conjecture, the recent solution of SPGC and the efficient recognition of perfect graphs have several important implications, in particular in game theory.

[1] As of the writing of this survey, a remarkable proof for SPGC was announced by M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas. A polynomial time recognition algorithm for Berge graphs was also announced by M. Chudnovsky and P. Seymour, and a simplified version of it by G. Cornuejols, Liu and K. Vuskovic.

2 Kernels

The general notion of a kernel was introduced originally by von Neumann and Morgenstern [101] as an abstract generalization of their concept of solution for cooperative games (so-called NM-solutions). Since then kernels have many applications in both positional and cooperative games, as well as in logic (see Berge [11,12]).

Let us consider a directed graph $D = (V, A)$, and let $N^+(v) = \{v' \mid [v, v') \in A\}$ denote the *out-neighborhood* of vertex v. Let us also recall that a subset $S \subseteq V$ is a kernel of D if it is independent and absorbing (i.e., if $N^+(v) \cap S = \emptyset$ iff $v \in S$). Clearly, only a maximal independent set can be a kernel, since a non-maximal independent set is not absorbing. However, kernels may fail to exist. For example, a chordless odd directed cycle has no kernel. On the other hand, an even directed cycle has two different kernels. More generally, we can state the following.

Theorem 2.1 *Let $k(D)$ denote the number of kernels in a directed graph D. Then the following claims hold true:*

(i) if all directed cycles in D are odd then $k(D) \leq 1$;

(ii) if all directed cycles in D are even then $k(D) \geq 1$;

(iii) if D is acyclic then $k(D) = 1$.

Claim (i) is an easy exercise. Claim (ii) was proved by Richardson [110] (see [20,83] for simpler proofs), while (iii) is an immediate corollary of (i) and (ii).

Claim (iii) was independently, and much earlier, proved by von Neumann and Morgenstern [101]. This rather simple statement plays an important role in the theory of *combinatorial or Nim-type games* (i.e., two-person positional games with perfect information and without random moves, see [102] for many interesting examples and a complete bibliography on the subject). Given an acyclic directed (multi-) graph $D = (V, A)$ and a token in one of its vertices, let two players move this token along the arcs of D, alternating. In each step the token is moved from the current vertex v to some $v' \in N^+(v)$. Since the graph is finite and acyclic, the token sooner or later will arrive to a dead end (i.e., to a vertex v for which $N^+(v) = \emptyset$). The player who's turn would be to move from a dead end is the looser of the game. According to (iii) of Theorem 2.1, the acyclic directed graph D has a unique kernel S. It is easy to see that the player who can start in a vertex $v \notin S$ has a winning strategy. Indeed, the player who moves from a vertex $v \notin S$ can always move the token into S, since S is absorbing. His opponent then either cannot move (being in a dead end, and hence loosing the game) or is forced to leave S again, since S is independent.

Let us mention that as early as in 1902 Bouton [35] applied analogous arguments to a special directed graph which describes the game of Nim and obtained an elegant characterization of the winning positions and strategies.

Let us add that the unique kernel S in an acyclic directed graph D can be constructed efficiently, by recursively, adding all dead ends to S and deleting those vertices and their neighbors from D (this procedure is also known in game theory as "backward induction", see e.g., [100]). In general, finding a kernel in a directed graph, or recognizing if there is none, seems to be much more difficult, as shown by Chvátal [38]:

Theorem 2.2 ([38]) *It is NP-complete to recognize whether a directed graph D has a kernel, or not.*

Later, Fraenkel [55] proved that the problem remains NP-complete even for planar directed graphs of degree at most 3, in which in- and out-degrees are at most 2.

Early investigations to find sufficient conditions for the existence of a kernel were motivated by applications in game theory, but later the subject developed on its own. Numerous related results can be found in the literature (see e.g., [2, 4, 9–27, 31, 32, 36, 37, 45–51, 58–65, 67–71, 73, 84–86, 95–97, 101, 110]).

Several of these results strengthen Richardson's theorem ((ii) of Theorem 2.1). In a directed graph $D = (V, A)$ an arc $[u, v)$ is called *reversible* if both $[u, v) \in A$ and $[v, u) \in A$. If $C = ([v_0, v_1), \ldots, [v_{2k-1}, v_{2k}), [v_{2k}, v_0))$ is an odd directed cycle in D, then the arcs $[v_i, v_{i+2})$, and $[v_{i+1}, v_{i+3}))$ for some $i = 0, ..., 2k$ (and where additions are taken modulo $2k$) are called *crossing consecutive* chords of C.

Theorem 2.3 *A directed graph $D = (V, A)$ has a kernel if each of its odd directed cycles has at least*

(iv) two reversible arcs;

(v) two crossing consecutive chords;

(vi) two chords whose heads are two consecutive vertices of the cycle.

These claims were shown respectively by Duchet [45], Duchet and Meyniel [49], and by Galeana-Sanches and Neumann-Lara [65]. A stronger claim that if all odd directed cycles have at least two chords then the graph has a kernel, was conjectured by Meyniel in 1976, and was disproved later by Galeana-Sanches [58], who showed that for every $k \geq 2$ there exists a kernel-less directed graph in which every directed cycle contains at least k chords.

Another sufficient condition for the existence of a kernel was shown recently in terms of unions of partial orders by Fleiner [54]:

Theorem 2.4 ([54]) *Let $D_i = (V, A_i)$, $i = 1, 2$ be transitively closed, acyclic directed graphs on the same vertex set V (i.e. representing partial orders on V). Then, $D = (V, A_1 \cup A_2)$ has a kernel.*

The analogous statement does not hold for the union of more than two partial orders (see also [1]).

Let us remark that the above sufficient conditions for the existence of a kernel, when hold, in fact imply the existence of a kernel in every induced subgraph of D. Such directed graphs are called *kernel perfect* in the literature. Kernel-less directed graphs in which every proper induced subgraph has a kernel are called *critically kernel imperfect.* It is an easy observation that such a critical kernel imperfect graph must be strongly connected. Let us further call a directed graph $D = (V, A)$ *strongly kernel perfect* if every (not necessarily induced) subgraph of it $D' = (V, A')$, $A' \subseteq A$ has a kernel.

Observation 2.5 *A directed graph is strongly kernel perfect if and only if it does not contain an odd directed cycle.*

Proof: This follows easily by the definition and by Richardson's theorem ((ii) of Theorem 2.1). □

For the criticality of arc removal, let us note first that any proper subgraph of a directed cycle has a kernel. Duchet [45] conjectured that odd directed cycles are in fact the only connected arc minimal kernel-less directed graphs, i.e., in which a kernel appears after the removal of any arc. This conjecture, if true, would imply that in every connected kernel-less directed graph not only there exists an odd directed cycle (by Richardson's theorem), but it can be obtained by a successive elimination of arcs such that no kernel appears on the way. This appealing conjecture however was disproved by Apartsin, Ferapontova and Gurvich [4], who showed that the circulant graph $C_{43}(1, 7, 8)$ is a counterexample (i.e., the directed graph on vertices $V = \mathbb{Z}_{43}$ with arc set $A = \{[i, i + d) \mid d \in \{1, 7, 8\}\}$, where additions are taken modulo 43). Later, Boros and Gurvich [32] suggested and proved a corrected version of the above conjecture, involving the notion of kernel solvability (see Theorem 3.10).

Let us finally recall a generalization of the notion of kernels. Given a directed graph $D = (V, A)$, a subset $S \subseteq V$ of the vertices is called a *semi-kernel* (or sometimes it is called *quasi-kernel*) if S is an independent set and for every vertex $v \notin S$ there exists vertex $u \in S$ such that either $[v, u) \in A$ or there exists $w \in V \setminus S$ such that $[v, w), [w, u) \in A$. This notion was considered by Chvátal and Lovász [42], who proved that

Theorem 2.6 ([42]) *Every directed graph has a semi-kernel.*

Jacob and Meyniel [87] improved this result showing that if D is a kernel-less directed graph, then it has at least three semi-kernels.

3 Kernel solvable graphs

Given an undirected graph $G = (V, E)$, the directed graph $D = (V, A)$ is called an *orientation* (sometimes also called a *super-orientation*) of G, if A is obtained from G by replacing each edge $(u, v) \in E$ by one or both of the directed arcs $[u, v)$ and $[v, u)$. Let us denote by $A^- \subseteq A$ the set of arcs which are not reversible in D, and let $D^- = (V, A^-)$. The directed graph $D^- = (V, A^-)$ is called a *sub-orientation* of G. Clearly, there is a one-to-one correspondence between sub- and (super-)orientations of a given undirected graph G. We say that D is a *clique acyclic* orientation of G, if there is no directed cycle in D^- the vertices of which would form a clique in G (in the literature we also find other terms, such as *normal* or *admissible* for the same notion).

Let us first observe that if D is a kernel perfect orientation of G, then D must be clique acyclic. This is quite obvious, since the vertex set of directed cycle of D^- within a clique of G would clearly induce a subgraph of D with no kernel.

An interesting problem, which generated a lot of research, involves the characterization of graphs for which every clique acyclic orientation is also kernel perfect.

Let us call a graph $G = (V, E)$ *kernel solvable* if every clique acyclic orientation of it has a kernel (this family of graphs are also called *solvable* or *quasi-perfect*).

Let us first observe that kernel solvability is a hereditary property:

Observation 3.1 *Induced subgraphs of a kernel solvable graph are kernel solvable.*

Proof: Let us assume that $G' = (V', E')$ is a subgraph of the kernel solvable graph $G = (V, E)$ induced by $V' \subseteq V$, and let $D' = (V', A')$ be a clique acyclic orientation of G'. Let us imagine now the vertices of G ordered along a line, first starting from the left the vertices belonging to $V \setminus V'$, and then on the right end we place the vertices of V'. Let us next extend the orientation D' by orienting all edges of $E \setminus E'$ from left-to-right according to the above ordering, and let $D = (V, A)$ denote the obtained orientation of G. Due to the above construction, all directed cycles in D belong to D'. Thus, since D' is clique acyclic, D is clique acyclic, too. Since G is kernel solvable, D must have a kernel $S \subseteq V$, and by the above construction of D, the set $S \cap V'$ must be a kernel of D'. $\square$

It follows immediately from the above observation that:

Corollary 3.2 *Every clique acyclic orientation of a kernel solvable graph is also kernel perfect.*

Kernel solvable graphs were introduced by Berge and Duchet [16]. They noticed the above claim, and that odd holes and odd anti-holes are not kernel solvable (see also [20] Propositions 2.1-2.4). Motivated by these observations, they conjectured that kernel solvable and perfect graphs are the same:

Conjecture 3.3 (Berge and Duchet [16])

(BD1) *Perfect graphs are kernel solvable.*

(BD2) *Kernel solvable graphs are perfect.*

Both conjectures were proved for line graphs by Maffray [95,97], and for line multi-graphs by Borodin, Kostochka and Woodall [30].

BD1 was further shown for several special classes of perfect graphs, including chordal graphs, i-triangulated (also known as Gallai) graphs (Jacob [86] and Maffray [96]), and the complements of strongly perfect graphs (Blidia, Duchet, and Maffray [26]). Finally, in 1996 BD1 was shown for all perfect graphs by Boros and Gurvich [31].

Theorem 3.4 (Boros and Gurvich [31]) *Perfect graphs are kernel solvable.*

The proof of this theorem is based on results from cooperative game theory, by Bondareva [28,29], Shapley [113], Scarf [111], Gurvich and Vasin [81,82], Moulin and Peleg [99], Kaneko and Wooders [88], Keiding [89], and Danilov and Sotskov [43, 44] (see Section 8). A simpler proof was given by Aharoni and Holzman [2], based on a generalization of kernels (see Section 5).

Unlike BD1, the reverse of BD2, for which no direct proof is known yet, follows easily from SPGC, as Observation 3.1 shows, since odd holes and odd anti-holes are not kernel solvable (see e.g., [16, 20, 31], or Section 6).

Given a graph $G = (V, E)$, a vertex $w \in V$, and another graph $H = (W, F)$ ($W \cap V = \emptyset$), let us define a new graph $G' = ((V \setminus \{w\}) \cup W, E')$, where $(u, v) \in E'$ if $u, v \in V \setminus \{w\}$ and $(u, v) \in E$, or if $u, v \in W$ and $(u, v) \in F$, or if $u \in V \setminus \{w\}$, $v \in W$ and $(u, w) \in E$. We say that G' is obtained from G by *substituting* vertex w with H.

Theorem 3.5 ([31]) *Substituting a vertex by an independent set preserves kernel solvability.*

Interestingly, substitution by cliques is not known to preserve kernel solvability.

Conjecture 3.6 ([31]) *Substituting a vertex by an edge preserves kernel solvability.*

In fact this conjecture was shown in [31] to be equivalent with BD2. Let us remark that the same operation trivially keeps a Berge graph Berge. Fulkerson [56] reduced the weak perfect graph conjecture to the analogous property for perfect graphs, that is that substituting a vertex by an edge (or equivalently, by a clique) preserves perfectness, which was proved later by Lovász [93]. Let us add finally that SPGC implies both BD2 and Conjecture 3.6, yet no independent proof is known. The closest result toward this conjecture, independently of SPGC, is the following:

Theorem 3.7 ([31]) *If $G = (V, E)$ is not perfect, then there are pairwise vertex disjoint cliques K_v, $v \in V$, such that substituting each vertex $v \in V$ by the corresponding clique K_v results in a non kernel solvable graph.*

The proof of this claim relies heavily on game theory, see Section 8. The sizes of the cliques K_v, $v \in V$ needed for this statement to work maybe exponential in the size of G.

Another remarkable property, parallel to analogous properties of perfect and Berge graphs concerns complementary graphs.

Conjecture 3.8 ([31]) *A graph G is kernel solvable if and only if its complement, $\overline{G}$ is kernel solvable.*

This conjecture was also shown in [31] to be equivalent with BD2, and thus both are implied by SPGC. Let us remark that complements of Berge graphs are trivially Berge, and complements of perfect graphs are perfect by the perfect graph theorem, which was proved by Lovász [93]. A combination of BD2 and Conjecture 3.8 was stated also as an open problem in Blidia, Duchet, Jacob, Maffray and Meyniel [24]:

Conjecture 3.9 ([24]) *A graph G is perfect if and only if both G and $\overline{G}$ are kernel solvable.*

One direction of this follows from BD1 (Theorem 3.4) and the perfect graph theorem. The reverse direction can be shown by Theorems 3.7 and 3.5 to be equivalent with Conjecture 3.8, and hence with BD2. The reverse direction is obviously implied by SPGC, since kernel solvable graphs cannot contain odd holes, by Observation 3.1.

Let us also consider edge-critical characterizations. It is an easy exercise that odd holes are the only connected non-Berge graphs from which the removal of any edge yields a Berge graph. Olaru [103] proved that odd holes are the only connected imperfect graphs, which after the removal of any edge become perfect (see also Olaru and Sachs [104]).

Theorem 3.10 ([32]) *Odd holes are the only connected non kernel solvable graphs which become kernel solvable after the removal of any edge.*

This statement was shown to correct a conjecture of Duchet [45], and follows easily from BD1 (Theorem 3.4), and from a result of Meyniel [98], claiming that a graph is perfect if each of its odd cycles has at least two chords.

Let us consider finally two related special types of orientations.

An orientation $D = (V, A)$ of a graph $G = (V, E)$ is called *M-clique-acyclic* if for every directed cycle of length 3 at least 2 of its 3 arcs are reversible. The graph G is called *kernel-M-solvable* if every M-clique-acyclic orientation of it has a kernel. Duchet [46] proved that parity graphs are kernel-M-solvable. Later, Blidia and Engel [27] proved that perfectly orderable graphs, and in a sense, almost all perfect graphs are kernel-M-solvable.

Theorem 3.11 *Perfect graphs are kernel-M-solvable.*

Proof: Originally Meyniel conjectured this statement, which now follows from BD1 (Theorem 3.4), via the easy fact that every M-clique-acyclic orientation is clique-acyclic, and hence, each kernel solvable graph is kernel-M-solvable. □

Conjecture 3.12 (Jaeger, Meyniel, see [24]) *A graph G is perfect if and only if both G and $\overline{G}$ are kernel-M-solvable.*

One direction of this conjecture follows easily by Theorem 3.11 and the perfect graph theorem. The other direction is implied by SPGC, since kernel-M-solvable graphs cannot contain odd holes, as the construction in the proof of Observation 3.1 shows.

Observation 3.13 *The anti-hole $\overline{C}_7$ is kernel-M-solvable, but not kernel solvable.*

Proof: Clearly, $\overline{C}_7$ is not kernel solvable, as Observation 3.1 shows. On the other hand, it has only two non-isomorphic clique acyclic orientations without kernels, as Observation 6.2 shows, neither of which are M-clique-acyclic. $\square$

An orientation $D = (V, A)$ of $G = (V, E)$ is called *fraternal* if $[u, w) \in A$ and $[v, w) \in A$ imply that $[u, v) \in A$ or $[v, u) \in A$. This concept was introduced by Skrien [114] and a characterization was obtained by Gavril and Urrutia [74] for the case when D has no reversible arcs. They also proved that triangulated graphs and circular arc graphs are fraternally orientable. Many properties of fraternally orientable graphs were obtained by J. Bang-Jensen, J.Huang, and E. Prisner [6]. Galeana-Sanches [61, 62] characterized those perfect graphs which have clique-acyclic fraternal orientation and proved BD1 for these graphs. Gavril, Toledano, and de Werra [73] gave a polynomial algorithm to find a kernel in a subclass of these graphs.

4 Line graphs of bipartite graphs

We show two well-known examples based on the simple fact that line graphs of bipartite graphs are kernel solvable, which follows from BD1 (Theorem 3.4), and which of course follows independently from much earlier results.

Let us show first that BD1 generalizes the Stable Marriage Theorem by Gale and Shapley [57]. Let us consider a complete bipartite graph $B = (M \cup W, E)$, $M \cap W = \emptyset$, $E = M \times W$, where vertices are interpreted as men and women, and where each edge $(m, w) \in E$, $m \in M$, $w \in W$ is viewed as a possible couple (a potential marriage). Let us also assume that each man $m \in M$ has a complete preference order $\succ_m$ over all women, and each woman $w \in W$ has a complete preference order $\succ_w$ over all men. Given a matching $F \subseteq E$, an edge $(m, w) \in E \setminus F$ is called a *breaking couple* (with respect to F), if they prefer each other more than their respective partners in the matching, i.e., if $(m, w') \in F$ and $(m', w) \in F$, then $m \succ_w m'$ and $w \succ_m w'$ both hold. The matching F is called *stable* is no breaking couple exist with respect to it. The celebrated result of Gale and Shapley, which can easily be derived from BD1, states that

Theorem 4.1 (Gale and Shapley [57]) *A stable matching always exists.*

Proof: Let us consider the line graph $G = L(B)$, with vertex set $V(G) = M \times W$, and edge set $E(G) = \{((m, w), (m', w')) \mid$ either $m = m'$ or $w = w'$, but not both$\}$, and let us consider the orientation D of G, in which $[(m, w), (m, w')) \in A(D)$ iff $w' \succ_m w$, and $[(m, w), (m', w)) \in A(D)$ iff $m' \succ_w m$. It is easy to see that D is a clique acyclic orientation of G, due to the fact that the preferences are complete orders for each of the individuals. Since the line graph of a bipartite graph is perfect (follows e.g., by the König-Hall theorem), BD1 implies the existence of a kernel $F \subseteq V(G) = M \times W$. It is straightforward to check that F is indeed a stable matching in B. $\Box$

Let us note that in the proof we used BD1 only in the special case of line graphs of (complete) bipartite graphs. It is easy to show that conversely, Theorem 4.1 readily implies the kernel solvability of line graphs of bipartite graphs.

Theorem 4.2 (Maffray [97]) *Line graphs of bipartite graphs are kernel solvable.*

Proof: Let us assume first that B is a complete bipartite graph and G is its line graph, as in the previous statement. Clearly, the sets $C_m = \{(m, w) \mid w \in W\}$ for $m \in M$ and $C_w = \{(m, w) \mid m \in M\}$ for $w \in W$ are the only maximal cliques of B. Furthermore, if D is an arbitrary clique acyclic orientation of G, then it can be extended to a complete linear order on each of these cliques, i.e., preferences $\succ_m$ for $m \in M$ and $\succ_w$ for $w \in W$ can be associated to D (not necessarily uniquely). With these preferences we can apply Theorem 4.1, from which the existence of a stable matching $F \subseteq E(B) = M \times W$ follows. It is again straightforward, by the definitions that F is a kernel in D. Since this hold true for all cliques acyclic orientations of G, the kernel solvability of G follows. Then Observation 3.1 completes the proof of our statement, since the line graph of a bipartite graph is an induced subgraph of the line graph of the corresponding complete bipartite graph. $\Box$

In other words, there is a correspondence between preferences (resp., stable matchings) in B and clique acyclic orientations (resp., kernels) of G, and in fact Theorems 4.1 and 4.2 are equivalent.

Another example involves list edge colorings and a conjecture of Dinitz.

Given a graph $G = (V, E)$ let us consider a list $\mathcal{L} = \{C_e \mid e \in E\}$ of sets of "colors" assigned to the edges. An $\mathcal{L}$-*list edge coloring* of G is a mapping $c : E \longrightarrow \bigcup_{e \in E} C_e$ of the edge set to the set of colors such that $c(e) \neq c(e')$ whenever e and e' have a common endpoint, and $c(e) \in C_e$ for all $e \in E$. The *list chromatic index* of G is the smallest integer k for which an $\mathcal{L}$-list edge coloring of G exists whenever $|C_e| \geq k$ for all $e \in E$.

Dinitz in 1979 conjectured (see e.g. Erdös [52]) that the list edge chromatic index of the complete bipartite graph $K_{n,n}$ is n. Galvin [72] found an amazingly simple proof for this conjecture by deriving it essentially from Theorem 4.2.

Theorem 4.3 (Galvin [72]) *For every $n \geq 1$ the list chromatic index of the complete bipartite graph $K_{n,n}$ is n.*

Proof: Let us denote by X and Y the two classes of vertices of $K_{n,n}$, $|X| = |Y| = n$, and let us denote by $E = E(K_{n,n})$ its edge set, which is also the vertex set of its line graph $L(K_{n,n})$.

In order to obtain the simple proof, we need to recall two key ideas, one by Maffray [97], and the another by Bondy, Boppana and Siegel (see [3][Remark 2.4, p. 129] or Galvin [72]).

Lemma 4.4 (Maffray [97]) *The line graph $L(K_{n,n})$ of the complete bipartite graph $K_{n,n}$ has an orientation $D = (E, A)$, which is kernel perfect and for which $|N_D^+(e)| \geq n - 1$ for all edges $e \in E$ (i.e., for all vertices of the line graph $L(K_{n,n})$).*

Proof: Indeed, since $L(K_{n,n})$ is perfect, it has a coloring with n colors, since its largest cliques contain n vertices (formed by edges incident with a vertex of $K_{n,n}$). Let $c : E \longrightarrow \{1, 2, ..., n\}$ be such a coloring, and let us define the orientation D by setting $A = A_X \cup A_Y$, where

$$\begin{aligned} A_X &= \{[(u,v),(u,v')) \mid u \in X,\ v,v' \in Y,\ c(u,v) < c(u,v')\}, \text{ and} \\ A_Y &= \{[(u,v),(u',v)) \mid u,u' \in X,\ v, \in Y,\ c(u,v) > c(u',v)\} \end{aligned}$$

It is easy to check that this orientation is regular, and $|N^+_D(e)| = n - 1$ for all edges $e = (u,v) \in E$. It is also immediate to observe that D is clique acyclic, since the only cliques of $L(K_{n,n})$ are those formed by edges incident with a particular vertex, and these cliques form directed paths by the above definition. Thus, by Theorem 4.2 and Corollary 3.2 the orientation D must be kernel perfect. □

Lemma 4.5 (Bondy, Boppana and Siegel) *Let $D = (E, A)$ be a kernel perfect orientation of the line graph of $G = (V, E)$, and let $\mathcal{L} = \{C_e \mid e \in E\}$ be a list of colors, such that $|C_e| > |N^+_D(e)|$ for all $e \in E$. Then, G has an $\mathcal{L}$-list edge coloring.*

Proof: Let us prove this by induction by the number of colors in $\bigcup_{e\in E} C_e$. Clearly, if this union contains only one color, then E must form a matching in G, and the statement is trivially true.

In the general case, let $c \in \bigcup_{e\in E} C_e$, and let $E_c = \{e \in E \mid c \in C_e\}$, and let D_c be the subgraph of D induced by E_c. Since D is kernel perfect, D_c has a kernel $F \subseteq E_c$, which is a matching in G. Let us color all edges of F by c, let $G' = (V, E \setminus F)$, $C'_e = C_e \setminus \{e\}$ for $e \in E \setminus F$, and let D' be the subgraph of D' induced by $E \setminus F$. We claim that the conditions hold for G', D' and $\mathcal{L}'$, and thus the lemma will follow by induction. Indeed, for $e \notin E_c$ we have $|C'_e| = |C_e|$ and $|N^+_{D'}(e)| \leq |N^+_D(e)|$, while for $e \in E_c \setminus F$ we have $|C'_e| = |C_e| - 1$ and $|N^+_{D'}(e)| < |N^+_D(e)|$, since F is a kernel in the subgraph of D induced by E_c. Finally, D' is indeed kernel perfect, since it is an induced subgraph of the kernel perfect D. □

Now, the theorem follows readily by applying the above lemma for $K_{n,n}$ with the kernel perfect orientation of $L(K_{n,n})$ from Lemma 4.4. □

Let us remark that the conjecture by Dinitz is a special case of the still open *list edge coloring conjecture*, claiming that if the edges of a graph $G = (V, E)$ can be colored by k colors, and $\mathcal{L} = \{C_e \mid e \in E\}$ is a list of colors such that $|C_e| \geq k$ for all $e \in E$, then G has an $\mathcal{L}$-list edge coloring.

5 Fractional kernels

A very useful generalization of kernels was considered by Aharoni and Holzman [2].

Let us consider a graph $G = (V, E)$ and an orientation $D = (V, A)$ of it, as before. A nonnegative real vector $x \in \mathbb{R}^V_+$ is called *fractionally independent* if $x(K) \leq 1$ for all cliques

K of G, where $x(S) = \sum_{v \in S} x(v)$. It is called *fractionally dominating*, if $x(N_D^+(v)) \geq 1$ for all vertices $v \in V$, and *fractionally strongly dominating* if for every vertex v there is a clique $K_v \subseteq N_D^+(v)$ for which $x(K_v) \geq 1$. Finally, it is called a *strong fractional kernel* of D if it is both fractionally independent and strongly dominating. It is clear that the characteristic vector of a kernel of D is a strong fractional kernel, and conversely, an integer strong fractional kernel must be a kernel, too.

Given a graph $G = (V, E)$ let us denote by $\mathcal{C}_G$ the family of its maximal cliques, for a subset $S \subseteq V$ let χ^S denote its characteristic vector, and let

$$P(G) = \{x \in \mathbb{R}_+^V \mid x(K) \leq 1 \text{ for all } K \in \mathcal{C}_G\}$$

be the so called *clique polytope* of G.

Lemma 5.1 (Aharoni and Holzman [2]) *The strong fractional kernels of an orientation $D = (V, A)$ of $G = (V, E)$ form a (possibly empty) face of $P(G)$.*

Proof: Assume that $x, x', x'' \in P(G)$, x is a strong fractional kernel and $x = \lambda x' + (1 - \lambda)x''$ for some $0 < \lambda < 1$. Then, we claim that both x' and x'' are strong fractional kernels, from which the lemma follows readily. Indeed by the definition, for every vertex $v \in V$ there exists a clique $K_v \in N_D^+(v)$ of G for which $x(K_v) = 1$, i.e., for which we have $\lambda x'(K_v) + (1 - \lambda)x''(K_v) = 1$. Since $x', x'' \in P(G)$ we also have $x'(K_v) \leq 1$ and $x''(K_v) \leq 1$, and thus $x'(K_v) = x''(K_v) = 1$ follows for all vertices $v \in V$. □

In fact for clique acyclic orientations the strong fractional kernels form a non-empty face.

Theorem 5.2 (Aharoni and Holzman [2]) *Every clique acyclic orientation D of an arbitrary graph G has a strong fractional kernel.*

The above strong result implies Theorem 3.4. To see this we need to recall a polyhedral result.

Theorem 5.3 (Chvátal [39]) *If G is a perfect graph then*

$$\text{conv } \{\chi^I \mid I \textit{ is independent in } G\} = P(G).$$

Moreover, it follows from the Perfect Graph Theorem (Lovász [93,94]) that the above equality holds if and only if G is perfect.

Proof of Theorem 3.4. (Aharoni and Holzman [2]): Let G be a perfect graph, and D be an arbitrary clique acyclic orientation of it. By Theorem 5.2, the directed graph D has a strong fractional kernel $x \in \mathbb{R}_+^V$, and thus the strong fractional kernels of D from a non-empty face of $P(G)$ by Lemma 5.1. Then, Theorem 5.3 implies that the vertices of this face are characteristic vectors of independent sets of G, and thus each of those must be a kernel. □

The elegance of the above proof is enhanced by the fact that Theorem 5.2 itself has a short proof from a lemma of Scarf [111], which we recall here for completeness and because the same lemma seems to be instrumental in all known proofs of Theorem 3.4. For a vector $x \in \mathbb{R}_+^m$ we denote by $Supp(x) = \{j \mid x_j > 0\}$ its *support*.

Lemma 5.4 (Scarf [111]) *Let $m < n$ be integers, $b \in \mathbb{R}_+^m$, and $B = (b_{ij})$ and $C = (c_{ij})$ be $m \times n$ matrices such that the first m columns of B form an identity matrix and $c_{ii} \geq c_{ik} \geq c_{ij}$ for all $1 \leq i \neq j \leq m$, and $k > m$. Then, there exists a vector $y \in \mathbb{R}_+^m$ such that $By = b$ and such that for every column index $1 \leq k \leq n$ there is a row index $1 \leq i \leq m$ for which $c_{ik} \geq c_{ij}$ holds for all $j \in Supp(y)$.*

Proof of Theorem 5.2. (Aharoni and Holzman [2]): Let $G = (V, E)$ be a graph and $D = (V, A)$ be a clique acyclic orientation of it. Let us define $D' = (\mathcal{C}_G \cup V, A')$, where $A' = A \cup \{[K, v) \mid K \in \mathcal{C}_G \text{ and } v \in K\}$. Clearly, D' is also clique acyclic, thus in particular, there exist linear orders $\succ_K$ for every clique $K \in \mathcal{C}_G$ such that $[u, v) \in A'$ implies $v \succ_K u$. Let us further define a matrix $C = (c_{K,v}) \in \mathbb{R}^{\mathcal{C}_G \times (\mathcal{C}_G \cup V)}$, where $c_{K,v}$ is the height of v in the linear order $\succ_K$ if $v \in K$, and $c_{K,v} = 0$ for $v \in \mathcal{C}_G \cup (V \setminus K)$ (i.e., $c_{K,K}$ is the largest in the row of C labelled by $K \in \mathcal{C}_G$). Let us also define $B = (b_{K,v}) \in \mathbb{R}^{\mathcal{C}_G \times (\mathcal{C}_G \cup V)}$, where $b_{K,v} = 1$ if $v = K$ or if $v \in K$, and $b_{K,v} = 0$ otherwise. Finally let $b = (1, 1, ..., 1) \in \mathbb{R}^{\mathcal{C}_G}$. It is easy to check that the conditions of Lemma 5.4 hold for B, b and C. Let us then consider the vector $y \in \mathbb{R}_+^{\mathcal{C}_G \cup V}$ guaranteed by the above lemma, and let $x = (y_v \mid v \in V)$. It is straightforward to see that x is a strong fractional kernel in D. $\square$

6 Rejecting orientations

According to BD2, a non-perfect graph G has a clique acyclic orientation which does not have a kernel. Though this claim follows from SPGC, no direct, constructive proof of it is known, yet.

It is easy to see that a cyclic orientation of the edges of an odd hole is such an orientation, since for every independent set S of a cyclic orientation of the odd hole H_{2k+1} there must exist two consecutive vertices v_i and v_{i+1} which do not belong to S, and hence S cannot be reached by an arc from v_i, showing that S cannot be a kernel.

In general, let us say that in an orientation $D = (V, A)$ of the graph $G = (V, E)$ a vertex v *rejects* an independent set S, if $v \notin S$, and for all vertices $u \in S$ for which $(u, v) \in E$ we have $[u, v) \in A^-$. In other words, if all edges between v and S are not reversible, and oriented toward v. Clearly, if there is no kernel in a directed graph D, then each maximal independent set of it must be rejected by some vertex. Let us call such a directed graph *rejecting*, and let us call a clique acyclic rejecting orientation simply a CAR. Thus by definition, a graph G is not kernel solvable if and only if it has (or sometimes we say admits) a CAR.

Let us call a directed graph *strongly rejecting*, if all independent sets (not only maximal ones) are rejected by some vertices.

Theorem 6.1 ([31]) *A connected graph is not kernel solvable if and only if it has a strongly rejecting clique acyclic orientation.*

In fact odd anti-holes admit several strongly rejecting clique acyclic orientations, and their structure is not as trivial as those for odd holes (see [31] for several examples). Perhaps this structural difficulty of rejecting orientations is behind the fact that no direct proof for BD2 is known, yet.

Observation 6.2 ([31]) *The anti-hole $\bar{C}_7$ has two non-isomorphic CARs:*

In the first orientation let $\{[0,2),[1,3),[2,4),[3,5),[4,6),[5,0),[6,1)\}$ be the set of one way oriented edges, and let the edges in $\{(0,3),(1,4),(2,5),(3,6),(4,1),(5,2),(6,3)\}$ be oriented both ways.

In the second orientation edges $\{(2,4),(3,5),(2,5),(1,6)\}$ are both ways oriented, while $\{[2,6),[3,6),[3,0),[4,0),[4,1),[5,1),[0,2),[0,5),[1,3),[6,4)\}$ are oriented only one way (see also Figures 1 and 2 in [31]).

Proof: It is not difficult to verify that a maximum independent set of $\bar{C}_7$ can be rejected in a CAR only by a vertex from the unique maximum clique which is disjoint from it. This fact and some case analysis proves the observation. $\square$

More generally, for the anti-hole $\bar{C}_{2k+1}$, a maximum independent set S can be rejected in a CAR only by a vertex from the unique maximum clique disjoint from S. Still more generally:

Theorem 6.3 (cf. Theorem 9.30 in Preissmann and Sebő [109])
Let $G=(V,E)$ be a minimal imperfect graph and $D=(V,A)$ be a CAR of it. Then for every maximum independent set S of G there exists a unique vertex $v=v_S$ at which S is rejected in D; $v_S \neq v_{S'}$ if $S \neq S'$; moreover v_S belongs to the vis-a-vis clique $C=C(S)$ and $C(S) \subseteq N_D^+(v_S)$.

Proof: By Theorem 5.2 and Lemma 5.1 the strong fractional kernels of D form a non-empty face of $P(G)$. Since D is a CAR of G, D does not have a kernel, and thus this face cannot contain the characteristic vector of an independent set. By a result of Padberg [105] $P(G)$ has exactly one non-integral vertex for a minimal imperfect graph, namely the vector $x=(\frac{1}{\omega},...,\frac{1}{\omega})$, where $\omega=\omega(G)$ is the size of a maximum clique in G. Thus, x is the only strong fractional kernel of D. By the definition of a fractional kernel, for every vertex $v \in V$ there exists a clique $K_v \subseteq N_D^+(v)$ such that $x(K_v)=1$, implying that K_v must be a maximum clique of size ω. If $v=v_S$ for a maximum independent set S, then $S \cap N_D^+(v)=\emptyset$, and thus $S \cap K_v=\emptyset$ is implied. Hence S must be the unique vis-a-vis independent set of K_v. The rest of the statement follows by simple counting. $\square$

Let us recall (see e.g., Padberg [105]) that each partitionable (α,ω)-graph has exactly $n=\alpha\omega+1$ vertices, n maximum independent sets of size α each, n maximum cliques of size

ω each. Furthermore, each vertex belongs exactly to ω maximum cliques and to α maximum independent sets of G. Each maximum clique C intersects each maximum independent set, except only one, which we denote by $S(C)$ and call the vis-a-vis to C; analogously, each maximum independent set S intersects each maximum clique, except only one which we denote by $C(S)$ and call it the vis-a-vis to S.

Let us call a CAR of a partitionable graph G *uniform* if each maximum independent set S of G is rejected at a unique vertex $v = v(S) \in C(S)$ belonging to the vis-a-vis clique, and $v(S) \neq v(S')$ if $S \neq S'$.

An odd hole has a unique CAR which is obviously uniform. Another three examples of uniform CARs can be found in [31]: two for the anti-hole $\bar{C}_7$, and one for the partitionable $(3,3)$-graph distinct from the $(3,3)$-web (see Chvátal, Graham, Perold and Whitesides [41]). Theorem 6.3 claims that in minimal imperfect graphs all CARs are uniform. We can also hope for the existence of such orientations even in a wider class of graphs.

Conjecture 6.4 *Each partitionable graph has a uniform CAR.*

Let us note that Conjecture 6.4 if true, would imply BD2, but not conversely.

The following relaxation seems also interesting. An orientation of an (α,ω)-graph we will call an (α,ω)-*CAR* if all n maximum (but not necessarily all maximal) independents sets are rejected, and there is no clique-cycle in a maximum clique (but there may be one in a maximal but not maximum clique).

Conjecture 6.5 *Each partitionable (α,ω)-graph has a uniform (α,ω)-CAR.*

7 Graphs and games

There is another, strong connection between kernels of graphs and games, less obvious than the connection mentioned in Section 2. In fact this connection has led to the first proofs of Theorems 3.4 and 3.7.

Let us denote by $\mathbf{I}$ a finite set of *players* (or *voters*), and by $\mathbf{A}$ a finite set of *outcomes* (or *candidates*). Each player $i \in \mathbf{I}$ has a set of strategies X_i, and let $\mathbf{X} = \times_{i\in\mathbf{I}} X_i$.

A game is described by two mappings, $g : \mathbf{X} \to \mathbf{A}$ and $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$. The mapping g specifies the outcome of the game for every possible combinations of the strategies of the players. (Note that g is supposed to be surjective but not usually injective.) The real function u, called the *utility* (or *payoff*) describes the "value" of an outcome for an individual player, and $u(i,a) \neq u(i,a')$ is assumed for all players $i \in \mathbf{I}$ and pairs of different outcomes $a \neq a' \in \mathbf{A}$. We say that player $i \in \mathbf{I}$ *prefers* outcome $a \in \mathbf{A}$ to $a' \in \mathbf{A}$ if $u(i,a) > u(i,a')$. We shall call the quadruple $\langle \mathbf{I}, \mathbf{A}, \mathbf{X}, g \rangle$ a *game form*, and the quintuple $\langle \mathbf{I}, \mathbf{A}, \mathbf{X}, g, u \rangle$ a *game in normal form*.

Subsets of the players and outcomes are called *coalitions* and *blocks*, respectively. We say that a coalition $K \subseteq \mathbf{I}$ is *effective* for a block $B \subseteq \mathbf{A}$ if the players of K can guarantee the outcome of the game to belong to B, or in other words, if the players in K can choose strategies such that for every possible strategies of the rest of the players the outcome of the game belongs to B.

In general, we call a Boolean mapping $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ an *effectivity function*. Obviously, every game form $\Gamma = \langle \mathbf{I}, \mathbf{A}, \mathbf{X}, g \rangle$ defines uniquely an effectivity function, denoted by $\mathcal{E}_\Gamma$, by setting $\mathcal{E}_\Gamma(K,B) = 1$ for a coalition $K \subseteq \mathbf{I}$ and a block $B \subseteq \mathbf{A}$ if and only if K is effective for B in Γ, that is if there are strategies $x_K \in \times_{i \in K} X_i$ of the players of K such that $g(x_K, x_{\mathbf{I} \setminus K}) \in B$ for every strategy $x_{\mathbf{I} \setminus K} \in \times_{i \in \mathbf{I} \setminus K} X_i$ of the rest of the players.

Of course, not every effectivity function (Boolean mapping) $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ corresponds to a game form. We call $\mathcal{E}$ *playing* if $\mathcal{E} = \mathcal{E}_\Gamma$ for a game form $\Gamma = \langle \mathbf{I}, \mathbf{A}, \mathbf{X}, g \rangle$.

Theorem 7.1 (Moulin and Peleg [99]) *An effectivity function $\mathcal{E} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ is playing if and only if it satisfies the following conditions:*

Monotonicity: *If $K \subseteq K' \subseteq \mathbf{I}$ and $B \subseteq B' \subseteq \mathbf{A}$ then $\mathcal{E}(K,B) \leq \mathcal{E}(K',B')$.*

Super-additivity: *If $\mathcal{E}(K_1,B_1) = \mathcal{E}(K_2,B_2) = 1$ and $K_1 \cap K_2 = \emptyset$ then $\mathcal{E}(K_1 \cup K_2, B_1 \cap B_2) = 1$.*

Boundary conditions: *We have $\mathcal{E}(\mathbf{I},B) = \mathcal{E}(K,\mathbf{A}) = 1$ for all $\emptyset \neq B \subseteq \mathbf{A}$ and $\emptyset \neq K \subseteq \mathbf{I}$; furthermore $\mathcal{E}(\mathbf{I},\emptyset) = \mathcal{E}(\emptyset,\mathbf{A}) = 0$.*

Though the knowledge of $\mathbf{X}$ and g provides a lot of information, many properties of a game can already be represented in terms of its effectivity function. We shall call the quadruple $\langle \mathbf{I}, \mathbf{A}, \mathcal{E}, u \rangle$ a *game in effectivity function form*.

In many cases it is convenient to represent a Boolean mapping by explicitly listing its true points. Given a list $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$ of coalition - block pairs, $K_v \subseteq \mathbf{I}$, and $B_v \subseteq \mathbf{A}$ for $v \in V$, we can associate to it an effectivity function $\mathcal{E}_\mathcal{L} : 2^{\mathbf{I}} \times 2^{\mathbf{A}} \to \{0,1\}$ by defining $\mathcal{E}_\mathcal{L}(K,B) = 1$ if and only if $(K,B) = (K_v,B_v)$ for some $v \in V$.

Not all lists of coalition - block pairs (K_v,B_v), $v \in V$ can however be viewed as a partial list of the effective pairs of a game form. Let us call an effectivity function $\mathcal{E}$ a *playing minor* if there exists a game form $\Gamma = \langle \mathbf{I}, \mathbf{A}, \mathbf{X}, g \rangle$ for which $\mathcal{E}_\Gamma(K,B) = 1$ whenever $\mathcal{E}(K,B) = 1$, that is if $\mathcal{E}$ is a minorant (as a Boolean mapping) of a playing effectivity function.

Theorem 7.2 (Gurvich and Vasin [82]) *Given the sets of players $\mathbf{I}$, outcomes $\mathbf{A}$, and a list $\mathcal{L} = \{(K_v,B_v) \mid v \in V\}$ of coalition - block pairs, the corresponding effectivity function $\mathcal{E}_\mathcal{L}$ is a playing minor if and only if $\bigcap_{v \in S} B_v \neq \emptyset$ for every subset $S \subseteq V$ for which the corresponding coalitions are pairwise disjoint, i.e., for which $K_v \cap K_{v'} = \emptyset$ for all $v \neq v' \in S$.*

The reader can find a short proof in English in [31], where Theorem 7.2 is derived from Theorem 7.1.

Theorems 7.1 and 7.2 allows for a meaningful association between graphs and games. Given a graph $G = (V,E)$ let us denote, respectively by $\mathbf{I}_G$ and $\mathbf{A}_G$ the families of maximal cliques and maximal independent sets of G, and view them as players and outcomes of an associated game. For a vertex $v \in V$ let us define K_v and B_v, respectively as the families of maximal cliques and maximal independent sets containing vertex v. Let $\mathcal{L}(G) = \{(K_v, B_v) \mid v \in V\}$, and let $\mathcal{E}_G = \mathcal{E}_{\mathcal{L}(G)}$ be the associated effectivity function.

Lemma 7.3 (Boros and Gurvich [31]) *For every graph $G = (V, E)$ the corresponding effectivity function $\mathcal{E}_G$ is a playing minor.*

Proof: Indeed, by the above definitions we have $B_v \cap B_{v'} \neq \emptyset$ if and only if $(v, v') \notin E$ if and only if $K_v \cap K_{v'} = \emptyset$. Thus Theorem 7.2 implies the statement. □

Since by Theorem 7.1 the conjunction of two playing effectivity functions is also a playing effectivity function, there exists a unique minimal playing effectivity function $\widehat{\mathcal{E}}_G \geq \mathcal{E}_G$. In fact it follows from Theorems 7.1 and 7.2 that this can be defined by $\widehat{\mathcal{E}}_G(K, B) = 1$ if and only if there exists an independent set $S \subseteq V$ in G for which $K \supseteq \bigcup_{v \in S} K_v$ and $B \supseteq \bigcap_{v \in S} B_v$.

8 Cores of games and kernels of graphs

Given the sets of players $\mathbf{I}$, outcomes $\mathbf{A}$ and a utility function $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$, we say that a coalition - block pair (K, B), $K \subseteq \mathbf{I}$ and $B \subseteq \mathbf{A}$, *can reject* an outcome $a \in \mathbf{A}$ if $u(k, b) > u(k, a)$ for every player $k \in K$ and every outcome $b \in B$. In other words, (K, B) can reject a if every outcome in B is strictly and unanimously preferred to a by all players of K. Let

$$R(K, B, u) = \{a \in \mathbf{A} | \forall k \in K\, \forall b \in B\, u(k, b) > u(k, a)\}.$$

Given a game $\langle \mathbf{I}, \mathbf{A}, \mathcal{E}, u \rangle$ in effectivity function form, and a subfamily $\mathcal{K} \subseteq 2^{\mathbf{I}}$ of its coalitions, the $\mathcal{K}$-*core* of the game is defined as

$$C(\mathcal{E}, u, \mathcal{K}) = \mathbf{A} \setminus \bigcup_{\substack{K \in \mathcal{K}, B \subseteq \mathbf{A} \\ \mathcal{E}(K,B)=1}} R(K, B, u),$$

that is as the set of those outcomes which are not rejected by either of the coalitions $K \in \mathcal{K}$. An effectivity function $\mathcal{E}$ is called *stable* if $C(\mathcal{E}, u, 2^{\mathbf{I}}) \neq \emptyset$ for every utility function $u : \mathbf{I} \times \mathbf{A} \to \mathbb{R}$.

Let us note that for an effectivity function $\mathcal{E}_{\mathcal{L}}$ corresponding to a list $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$, its core can equivalently be written as

$$C(\mathcal{E}_{\mathcal{L}}, u, 2^{\mathbf{I}}) = \mathbf{A} \setminus \bigcup_{v \in V} R(K_v, B_v, u).$$

To simplify notation, we shall write $\mathcal{C}(\mathcal{L}, u) = C(\mathcal{E}_{\mathcal{L}}, u, 2^{\mathbf{I}})$ in this case.

Finding necessary and/or sufficient conditions for stability is an important problems of game theory (see e.g., [100, 107]). The first surprising connection with graph theory is that the notion of kernel solvability is exactly the corresponding graph theoretical notion to stability.

Theorem 8.1 (Boros and Gurvich [31]) *A graph $G = (V, E)$ is kernel solvable if and only if $\mathcal{E}_G$ is stable.*

The proof of this statement is elementary, and is based on a simple characterization of stability by Keiding [89].

Given an effectivity function $\mathcal{E}_{\mathcal{L}}$ corresponding to the list $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$, let us consider the following extended table of triples $\mathcal{R} = \{(K_v, B_v, R_v \mid v \in V\}$, where $R_v \subseteq \mathbf{A}$ for all $v \in V$. We shall call $\mathcal{R}$ a *rejecting table* of $\mathcal{E}_{\mathcal{L}}$, if $R_v \cap B_v = \emptyset$ for all $v \in V$, and $\mathbf{A} = \cup_{v \in V} R_v$.

An ordered subset $\{v_0, v_1, ..., v_{s-1}\} \subseteq V$ is called a *common player cycle* of the rejecting table $\mathcal{R}$ if $R_{v_l} \cap B_{v_{l+1 \mod s}} \neq \emptyset$ for $l = 0, ..., s-1$, and $\emptyset \neq \cap_{l=0}^{s-1} K_{v_l}$.

Theorem 8.2 (Keiding [89]) *An effectivity function $\mathcal{E}_{\mathcal{L}}$, corresponding to the list $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$, is stable if and only if every rejecting table of it contains a common player cycle.*

Proof: Though the original claim was stated with a slightly more general definition of a rejecting table, this somewhat stronger claim can also be seen easily. Namely, if u is a utility function for which $C(\mathcal{L}, u) = \emptyset$ then let us consider the rejecting table $\mathcal{R}$ in which $R_v = R(K_v, B_v, u)$ for all $v \in V$. It is easy to see that $\mathcal{R}$ cannot have a common player cycle, since otherwise such a common player would have cyclical preference over a subset of the outcomes. The reverse direction is even simpler, since if $\mathcal{R}$ is a rejecting table without common player cycles, then we can define a utility function u satisfying the inequalities $u(i, a) > u(i, b)$ for all $a \in B_v$, $b \in R_v$ and $i \in K_v$, since the lack of common player cycles in $\mathcal{R}$ assures that these requirements are cycle free. For this utility we obviously have $C(\mathcal{L}, u) = \mathbf{A} \setminus (\bigcup_{v \in V} R_v) = \emptyset$. □

Proof of Theorem 8.1: The claim follows by observing that for any graph G, there is a one-to-one correspondence between CARs of G and common player cycle free rejecting tables of $\mathcal{E}_G$. Indeed, if $\mathcal{R}$ is a rejecting table for $\mathcal{E}_G$, then for every maximal independent set $S \in \mathbf{A}$ there is a vertex $v \in V$ such that $S \in R_v$. Since $B_v \cap R_v = \emptyset$, $v \notin S$ follows, and thus there are some edges in G between S and v. Let us orient all these edges toward v. By doing this for all maximal independent sets, we orient some of the edges of G. Let us finally orient the rest of the edges both ways. It is immediate to see that this orientation will be clique acyclic if and only if $\mathcal{R}$ is common player cycle free, and consequently, it is a CAR of G if and only if $\mathcal{R}$ was a common player cycle free rejecting table of $\mathcal{E}_G$. Conversely, if D is a clique acyclic orientation of G in which every maximal independent set is rejected at some of the vertices. Let us then define R_v for $v \in V$ as the set of those maximal independent sets which are rejected at vertex v. Then it is easy to see that $\mathcal{R} = \{(K_v, B_v, R_v) \mid v \in V\}$ is a common player cycle free rejecting table for $\mathcal{E}_G$. □

Let us remark that by interchanging the role of players and outcomes (as well as coalitions and blocks), we can associate to every list $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$ a *dual* list $\mathcal{L}^d = \{(B_v, K_v) \mid$

$v \in V\}$, and consequently, to every effectivity function $\mathcal{E}_{\mathcal{L}}$ its *dual* $\mathcal{E}^d_{\mathcal{L}} = \mathcal{E}_{\mathcal{L}^d}$. It is then immediate to see that we have $\mathcal{E}^d_G = \mathcal{E}_{\overline{G}}$, where $\overline{G}$ is the complement of G. Thus, in view of Conjecture 3.8 and Theorem 8.1, an interesting consequence of SPGC is the following game theoretic conjecture (which is also equivalent with BD2).

Conjecture 8.3 *Given a graph $G = (V, E)$, the corresponding effectivity function $\mathcal{E}_G$ is stable if and only if $\mathcal{E}_{\overline{G}}$ is stable, too.*

Not all coalitions can exercise their power in most situations, thus characterizations of the non-emptiness of the $\mathcal{K}$-core for a given family $\mathcal{K}$ of coalitions is also an important problem. Interestingly, useful sufficient conditions can be formulated for this, based only on the structure of the hypergraph $\mathcal{K}$.

Given the set of players $\mathbf{I}$, a family $\mathcal{K} \subseteq 2^{\mathbf{I}}$ of coalitions is called *stable* if $C(\mathcal{E}, u, \mathcal{K}) \neq \emptyset$ for every utility function u and playing effectivity function $\mathcal{E}$.

Given a graph $G = (V, E)$ let us recall that $\mathbf{I}_G$ denotes a family of cliques of G, including all maximal ones and possibly some others, and it is interpreted as the set of players. For $v \in V$, let K_v denote the collection of all those cliques from $\mathbf{I}_G$ which contain vertex v, as before. Furthermore, let $\mathcal{K}_G = \{K_v \mid v \in V\}$. Then, a second surprising connection between games and perfect graphs can be stated as follows:

Theorem 8.4 (Boros and Gurvich [31]) *Given a graph (V, E), the family $\mathcal{K}_G$ is stable if and only if G is perfect.*

This theorem provides an interesting characterization of perfect graphs in terms of an important game theoretical property. The proof has several components, and quite involved. Here below we recall only some of the basic ideas, and refer the reader for the details of a complete proof to [31, 33, 34].

Following the convenient definition by Berge [13], a hypergraph $\mathcal{H} \subseteq 2^W$ is called *normal* if it has the *Helly property* (i.e., $H_1 \cap H_2 \cap H_3 \neq \emptyset$ whenever $H_i \in \mathcal{H}$, $i = 1, 2, 3$ and $H_i \cap H_j \neq \emptyset$ for all pairs $1 \leq i < j \leq 3$), and if its line graph $L(\mathcal{H}) = (\mathcal{H}, \{(H, H') \mid H, H \in \mathcal{H},\ H \cap H' \neq \emptyset\})$ is perfect. Clearly, for every $w \in W$ the set $H_w = \{H \mid H \in \mathcal{H},\ w \in H\}$ is a clique of $L(\mathcal{H})$, by definition. Lovász [93] characterized normal hypergraphs, and showed that $\mathcal{H}$ is normal, if and only if $L(\mathcal{H})$ is a perfect graph and the sets H_w, $w \in W$ contain all maximal cliques of $L(\mathcal{H})$.

Corollary 8.5 (Lovász [93]) *The family $\mathcal{K}_G$ is normal if and only if G is perfect.*

Furthermore, Lovász [93] showed that the polyhedron

$$P_{\mathcal{H}} = \{x \in \mathbb{R}^{\mathcal{H}}_{+} \mid x(H_w) \leq 1 \text{ for all } w \in W\}$$

has integral vertices whenever $\mathcal{H}$ is normal. A related notion of game theory is that of *partitionable* hypergraphs: a nonnegative integral weight function $w : \mathcal{H} \to \mathbb{Z}_+$ is called *balanced* if for some $m \in \mathbb{Z}_+$ we have $\sum_{H\in\mathcal{H}, H\ni w} w(H) = m$ for all $w \in W$. It is called a *partition* if $m = 1$. Finally, $\mathcal{H}$ is called *partitionable* if for every balanced weighting w of $\mathcal{H}$ there exists a partition w' for which $w'(H) \leq w(H)$ for all $H \in \mathcal{H}$ (see e.g., Bondareva [28,29], Peleg [106] and Shapley [113]). It is easy to see that this can equivalently be stated saying that the hypergraph $\mathcal{H} \subseteq 2^W$ is *partitionable* if the polyhedron

$$Q_{\mathcal{H}} = \{x \in \mathbb{R}_+^{\mathcal{H}} \mid x(H_w) = 1 \text{ for all } w \in W\}$$

has integral vertices.

Corollary 8.6 ([31, 91, 92]) *A hypergraph $\mathcal{H} \subseteq 2^W$ is normal if and only if the extension of $\mathcal{H}$ by all its single element subsets $\mathcal{H} \cup W$ is partitionable.*

Though this statement follows almost immediately from the above definitions, it was only fairly recently recognized in the game theory literature, perhaps due to the somewhat disguised form of the original definition of a partitionable hypergraph (see e.g., [31,34,91,92]).

The last, and least trivial component in the proof of Theorem 8.4 is the following result.

Theorem 8.7 (Gurvich and Vasin [82]) *Given the set of players $\mathbf{I}$ and a family $\mathcal{K} \subseteq 2^{\mathbf{I}}$ of coalitions, $\mathcal{K}$ is stable if and only if $\mathcal{K} \cup \mathbf{I}$ is partitionable.*

This theorem is based on a theorem by Scarf [111], characterizing the core of *games with non-transferable utilities* (so called NTU-games, see e.g., [5,34] for definitions). Interestingly, the key claim in this result is the same Lemma 5.4, which we cited as pivotal in the result about fractional kernels. For a proof of Theorem 8.7 in English we refer the reader to [34].

Now, Theorem 8.4 follows readily by putting together Corollaries 8.5, 8.6 and Theorem 8.7.

Let us remark that, by Corollary 8.6 and Theorem 8.7, a family $\mathcal{K}$ of coalitions is stable if and only if it is normal. This latter condition can be tested in polynomial time, due to the recent result on efficient recognition of perfect graphs, and thus the stability of $\mathcal{K}$ can also be tested in polynomial time.

Let us add that the characterization of stable families of coalitions for cooperative games has a long history in game theory.

For *games with transferrable utilities* (so called TU-games, see e.g., [5,34] for definitions) a family $\mathcal{K}$ of coalitions is known to be stable if and only if it is a partitionable hypergraph (see [81]). This criterion is derived from a characterization of the core of a TU-game by Bondareva [28,29] and Shapley [113], based directly on Farkas' Lemma.

The same criterion of stability as in Theorem 8.7 was obtained independently for both TU- and NTU-games by Kaneko and Wooders [88], under the additional assumption that $\mathbf{I} \subseteq \mathcal{K}$. Let us remark that this assumption is standard in cooperative game theory. The main "reason" to reduce the family of the coalitions from 2^I to a subfamily $\mathcal{K}$ is that there are too many coalitions and some of them have no chances to realize or to exercise their joint power.

Yet, nothing can prevent an individual player to organize a coalition which consists only of him- or herself. However, from the combinatorial point of view the above two assumptions "$\mathcal{K}$ is partitionable" and "$\mathcal{K} \cup \mathbf{I}$ is partitionable" differ a lot. The latter assumption is stronger and it has a direct interpretation in terms of perfect graphs, as Theorem 8.7 shows.

Returning to the connection between graphs and games, we can see that Theorems 8.1 and 8.4 maps kernel solvability and perfectness of a graph G in a one-to-one way to the stability of $\mathcal{E}_G$ and $\mathcal{K}_G$, respectively. Thus, to obtain a proof for BD1, it is enough to use the following simple game theoretical statement:

Theorem 8.8 (Boros and Gurvich [31]) *Given a graph $G = (V, E)$, the effectivity function $\mathcal{E}_G$ is stable whenever the family $\mathcal{K}_G$ is stable.*

Proof: Let us assume that $\mathcal{K}_G$ is stable. By Lemma 7.3 the effectivity function $\mathcal{E}_G$ is a playing minor, and hence it has a playing majorant $\mathcal{E}^* \supseteq \mathcal{E}_G$. Let us then observe that

$$C(\mathcal{E}_G, u, 2^{\mathbf{I}_G}) = C(\mathcal{E}_G, u, \mathcal{K}_G) \supseteq C(\mathcal{E}^*, u, \mathcal{K}_G) \neq \emptyset$$

holds for every utility function $u : \mathbf{I}_G \times \mathbf{A}_G \longrightarrow \mathbb{R}$. Here the first equation holds because $\mathcal{E}_G(K, B) = 0$ unless $(K, B) = (K_v, B_v)$ for some $v \in V$, i.e., unless $K \in \mathcal{K}_G$. The second relation holds by the definition of the core. Finally, the right hand side is not empty for an arbitrary utility function u, since $\mathcal{K}_G$ is assumed to be stable, and $\mathcal{E}^*$ is a playing effectivity function. Thus, by definition, $\mathcal{E}_G$ is stable, too. $\square$

Let us remark that, as an interesting game theoretical consequence, the reverse implication in the above theorem also holds, by Theorems 8.1, 8.4 and SPGC.

9 Criteria of stability in terms of perfect graphs

Let us finally look at the obtained results from the point of view of game theory. We got a characterization of stability for effectivity functions of the form $\mathcal{E}_G$ corresponding to graphs. Yet, only very special effectivity functions are corresponding to graphs. Here we provide a general criterion of stability for an arbitrary effectivity function, surprisingly also in terms of perfect graphs.

An arbitrary effectivity function $\mathcal{E}_\mathcal{L}$, $\mathcal{L} = \{(K_v, B_v) \mid v \in V\}$, where $K_v \subseteq \mathbf{I}$ and $B_v \subseteq \mathbf{A}$ for $v \in V$, can be interpreted as a pair of hypergraphs $\mathcal{K} = \{K_v \mid v \in V\}$ and $\mathcal{B} = \{B_v \mid v \in V\}$ on the vertex sets $\mathbf{I}$ and $\mathbf{A}$, respectively, whose edges are indexed by a common set V. Let us consider the dual (transposed) hypergraphs $\mathcal{C} = \{C_i \mid i \in \mathbf{I}\}$ and $\mathcal{S} = \{S_a \mid a \in \mathbf{A}\}$ on the common vertex set V defined by

$$C_i = \{v \in V \mid i \in K_v\} \quad \text{and} \quad S_a = \{v \in V \mid a \in B_v\}$$

for $i \in \mathbf{I}$ and $a \in \mathbf{A}$. The effectivity function $\mathcal{E} = \mathcal{E}_\mathcal{L}$ can be equivalently defined by the hypergraphs $\mathcal{C}$ and $\mathcal{S}$, thus we shall also use the notation $\mathcal{E} = \{C_i, S_a; I, A; V\}$.

The following theorem provides a criterion of stability in terms of perfect graphs.

Theorem 9.1 ([33]) *An effectivity function $\mathcal{E} = \{C_i, S_a; \mathbf{I}, \mathbf{A}; V\}$, as well as its dual $\mathcal{E}^d = \{S_a, C_i; \mathbf{A}, \mathbf{I}; V\}$, are both stable if there exists a perfect graph $G = (V, E)$ such that*

(i) every clique of G is a subset of some set C_i, $i \in \mathbf{I}$, and

(ii) every independent set of G is a subset of some set S_a, $a \in \mathbf{A}$.

The notion of a CAR can be naturally extended to arbitrary effectivity functions as follows. Given an effectivity function $\mathcal{E} = \{C_i, S_a; \mathbf{I}, \mathbf{A}; V\}$, let us consider the graph $G_{\mathcal{C}} = (V, E)$, where $(u, v) \in E$ if $\{u, v\} \subseteq C_i$ for some $i \in \mathbf{I}$. Clearly, all sets C_i, $i \in \mathbf{I}$ are cliques of $G_{\mathcal{C}}$, but not all cliques may correspond to such a set. We say that an orientation D of $G_{\mathcal{C}}$ is *$\mathcal{C}$-acyclic* if there is no directed cycle of one way oriented edges within a clique C_i for some $i \in \mathbf{I}$. We say that D is *$\mathcal{S}$-rejecting* if all subsets $S_a \subseteq V$, $a \in \mathbf{A}$ are rejected in D, and we use the same abbreviation CAR for *$\mathcal{C}$-acyclic* and *$\mathcal{S}$-rejecting* orientations $G_{\mathcal{C}}$.

Proposition 9.2 ([33]) *An effectivity function $\mathcal{E} = \{C_i, S_a; \mathbf{I}, \mathbf{A}; V\}$ is stable if and only if the graph $G_{\mathcal{C}}$ has no CARs.*

In its turn, Theorem 9.1 follows directly from Proposition 9.2.

The next theorem provides necessary conditions for stability.

Theorem 9.3 ([33]) *If an effectivity function $\mathcal{E} = \{C_i, S_a; \mathbf{I}, \mathbf{A}; V\}$ or its dual $\mathcal{E}^d$ is stable then*

(i) for each $V' \subseteq V$ such that $|C_i \cap V'| \leq 1$ for all $i \in \mathbf{I}$ there exists an $a \in \mathbf{A}$ such that $V' \subseteq S_a$, and

(ii) for each $V'' \subseteq V$ such that $|S_a \cap V''| \leq 1$ for all $a \in \mathbf{A}$ there exists an $i \in \mathbf{I}$ such that $V'' \subseteq C_i$.

Corollary 9.4 *If an effectivity function $\mathcal{E} = \{C_i, S_a; \mathbf{I}, \mathbf{A}; V\}$ is stable then every two distinct vertex $u \neq v \in V$ must belong either to a set C_i for some $i \in \mathbf{I}$ or to an S_a for some $a \in \mathbf{A}$. In other words, $\mathcal{E}$ is not stable whenever $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$ and $K_1 \cap K_2 = B_1 \cap B_2 = \emptyset$ for some coalitions $K_1, K_2 \subseteq \mathbf{I}$ and blocks $B_1, B_2 \subseteq \mathbf{A}$.*

It can be derived from the above results that verifying whether an effectivity function $\mathcal{E}$ is stable (or playing-minor) is an NP-complete problem (see [33]). Let us contrast this with the fact that the stability of $\mathcal{E}_G$ and $\mathcal{K}_G$ are equivalent with the perfectness of the graph G, and hence it can be tested in polynomial time, according to the recently announced solution of SPGC and efficient recognition of Berge graphs. Let us also add that the stability of a family of coalition $\mathcal{K}$, in general, can also be tested in polynomial time, due to Corollary 8.6, Theorem 8.7 and the recently announced polynomial time recognition of perfect graphs.

Let us demonstrate finally that the conditions of Theorem 9.3 are only necessary but not sufficient for stability and the conditions of Theorem 9.1 are only sufficient but not necessary.

For each positive integer k let us define an effectivity function $\mathcal{E}_k$ by setting $V = \{1, \ldots, k\}$, $\mathbf{I} = \mathbf{A} = \{(p,q) \mid 1 \leq p < q \leq k\}$, and $C_{(p,q)} = S_{(p,q)} = \{(p,q)\}$ for all $(p,q) \in \mathbf{I} = \mathbf{A}$. Direct computations show that $\mathcal{E}_k$ is stable if and only if $k \leq 6$. A CAR for $\mathcal{E}_7$ is given by the following three directed cycles: $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 1$, $1 \to 3 \to 5 \to 7 \to 2 \to 4 \to 6 \to 1$, and $1 \to 5 \to 2 \to 6 \to 3 \to 7 \to 4 \to 1$.

The conditions of Theorem 9.3 hold automatically for each k but $\mathcal{E}_k$ is not stable if $k \geq 7$.

On their turn, the conditions of Theorem 9.1 hold only for $k \leq 4$, yet, $\mathcal{E}_5$ and $\mathcal{E}_6$ are still stable. For example, if $k = 5$ then the only graph satisfying the conditions (i) and (ii) of Theorem 9.1 is C_5 but it is not perfect. As it is well-known, any graph G or its complement G^c on 6 or more vertices must contain a triangle, in a contradiction with (i) and (ii) of Theorem 9.1. Thus this theorem cannot be applied for $\mathcal{E}_5$ and $\mathcal{E}_6$ and yet they are stable.

Let us remark that the conditions of both Theorems 9.1 and 9.3 hold (or do not hold) for dual effectivity functions $\mathcal{E}$ and $\mathcal{E}^d$ simultaneously. However, in general it may happen that one of them is stable while the other one is not, as an example of Peleg [108] shows. Other similar examples satisfying some additional properties can also be found in [33].

Let us also note that, according to BD2, such examples cannot exist for effectivity functions generated by graphs. Indeed, if the graphs G and G^c are perfect then, as we know, both $\mathcal{E}_G$ and $\mathcal{E}_G^d = \mathcal{E}_{G^c}$ are stable. If these graphs are not perfect then, by BD2, both effectivity functions are not stable. Moreover, it was also shown in [33] that BD2 can be equivalently reformulated in the game theoretic terms as follows:

Conjecture 9.5 *If an effectivity function $\mathcal{E}$ is stable and its dual $\mathcal{E}^d$ is not, then there exist coalitions $K_1, K_2 \subseteq \mathbf{I}$ and blocks $B_1, B_2 \subseteq \mathbf{A}$ such that $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$, $K_1 \cap K_2 \neq \emptyset$, and $B_1 \cap B_2 \neq \emptyset$. In other words, then there exists a player $i \in \mathbf{I}$ and an outcome $a \in \mathbf{A}$ such that $|C_i \cap S_a| \geq 2$.*

Obviously, such an effectivity function can not be generated by a graph.

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