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On a Conjecture of Chvátal

by

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ABSTRACT

The Sylvester-Gallai theorem asserts that every finite set S of points in two-dimensional Euclidean space includes two points, a and b, such that either there is no other point in S is on the line ab, or the line ab contains all the points in S. Recently, V. Chvátal extended the notion of lines to arbitrary metric spaces and made a conjecture that generalizes the Sylvester-Gallai theorem. We prove this conjecture for subspaces of ℓ_1^2, the two-dimensional space with the ℓ_1-metric.

The following problem was proposed by Sylvester in 1893 [5].

> *Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.*

Several proofs were found during the twentieth century; the first was due to Gallai [3].

In an arbitrary metric space (V, ρ), Menger [4] defined a ternary relation $\mathcal{B}(\rho)$ of *metric betweenness on* V by

$$(u, v, w) \in \mathcal{B} \Leftrightarrow u, v, w \text{ are all distinct and } \rho(u, w) = \rho(u, v) + \rho(v, w)$$

and Chvátal [1] defined the notion of a *line* in the following way. First, the ternary relation $\mathcal{B}(\rho)$ is transformed into a set $\mathcal{H}(\mathcal{B}(\rho))$ of three-point subsets of V:

$$\mathcal{H}(\mathcal{B}(\rho)) = \{\{a, b, c\} : (a, b, c) \in \mathcal{B}(\rho)\}.$$

Then a family $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ of subsets of V is defined by

$$\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho))) = \{S \subseteq V : \text{ no } T \text{ in } \mathcal{H}(\mathcal{B}(\rho)) \text{ has } |T \cap S| = 2\}.$$

(It is clear that $\mathcal{A}(\mathcal{H}(\mathcal{B}(\rho)))$ is closed under arbitary intersection.) For any subset W of V, a set $\mathcal{C}(W)$ as

$$\mathcal{C}(W) = \bigcap_{W \subseteq S \in \mathcal{A}} S.$$

Equivalently, if W is finite, then $\mathcal{C}(W)$ is the return value of the program

> $S = W$;
> **while** `some` T `in` $\mathcal{H}(\mathcal{B}(\rho))$ `has` $|T \cap S| = 2$ **do** $S = S \cup T$ **end**;
> `return` S;

Finally, for any two distinct points a and b of V, the *line* ab is defined as $\mathcal{C}(\{a, b\})$. With this definition, Chvátal [1] made the following conjecture.

Conjecture. *If* (V, ρ) *is a metric space such that* $1 < |V| < \infty$*, then* V *contains two distinct points* a *and* b *such that the line* ab *is* $\{a, b\}$ *or* V*.*

Chvátal [1] proved this conjecture for all metric spaces with at most nine points and for all (finite) metric spaces induced by connected graphs. (See [2] p.29 for the relevant definition.) The purpose of this article is to prove the conjecture for all (finite) subspaces of ℓ_1^2, the two-dimensional space with the ℓ_1-metric.

Theorem. *If* V *is a set of points in* $\mathbf{R}^2$ *such that* $1 < |V| < \infty$ *and if* ρ *is the* ℓ_1*-metric on* V*, then* V *contains two distinct points* a *and* b *such that the line* ab *defined by* (V, ρ) *is* $\{a, b\}$ *or* V*.*

Actually, we are going to describe a linear-time algorithm which, given a set V of points in $\mathbf{R}^2$ such that $1 < |V| < \infty$, finds distinct points a and b in V such that the line ab defined by V with the ℓ_1-metric is $\{a, b\}$ or V; the algorithm returns the message $\texttt{Nil}(a, b)$ in the former case and it returns the message $\texttt{Omni}(a, b)$ in the latter case. The description of the algorithm is spread over Figures 1 – 7. Throughout, we let $x(v)$ and $y(v)$ denote the two coordinates of a point v in $\mathbf{R}^2$ and we let $\mathcal{B}$ denote the metric betweenness induced on $\mathbf{R}^2$ by the ℓ_1-metric; in this notation, $(a, b, c) \in \mathcal{B}$ if and only if

$$\min\{x(a), x(c)\} \le x(b) \le \max\{x(a), x(c)\}$$

and

$$\min\{y(a), y(c)\} \le y(b) \le \max\{y(a), y(c)\}.$$

$x_{min} = \min\{x(v) : v \in V\}$;
$x_{max} = \max\{x(v) : v \in V\}$;
$y_{min} = \min\{y(v) : v \in V\}$;
$y_{max} = \max\{y(v) : v \in V\}$;
if $x_{min} = x_{max}$ or $y_{min} = y_{max}$
then $a, b =$ any two distinct points of V;
  return $\texttt{Omni}(a, b)$;
else **if** V includes at least two corners of $[x_{min}, x_{max}] \times [y_{min}, y_{max}]$
  then label these corners a and b;
    if the line ab is $\{a, b\}$
    then return $\texttt{Nil}(a, b)$;
    else return $\texttt{Omni}(a, b)$;
    end
  else proceed to CLASSIFICATION;
  end
end

Figure 1: The preamble

Correctness of the algorithm is for the most part obvious, but we shall spell out a few details all the same.

In CASE 1, we have $V \subseteq \mathcal{C}(\{a, b, c\})$; if $y(e) \ge y(d)$, then $(a, e, d) \in \mathcal{B}$, and so the line ae includes points d, b, and c; if $y(e) < y(d)$, then $(e, d, c) \in \mathcal{B}$, and so the line dc includes points e, b, and a.

In CASE 2, we have $(c, a, b), (a, b, d) \in \mathcal{B}$ and, if point e is found, then $V \subseteq \mathcal{C}(\{a, b, e\})$. Now if $(e, f, d) \in \mathcal{B}$, then $(a, b, f) \in \mathcal{B}$ by our choice of e, and so the line ab includes points d, f, and e; if $(f, e, d) \in \mathcal{B}$, then $(f, e, b) \in \mathcal{B}$, and so the line df includes points e, b, and a; if $(e, d, f) \in \mathcal{B}$, then $(d, f, b) \in \mathcal{B}$, and so the line df includes points e, b, and a.

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if    $V$ includes a corner of the $[x_{min}, x_{max}] \times [y_{min}, y_{max}]$
then  label this corner $a$;
      if necessary, rotate the plane to get $x(a) = x_{\min}$, $y(a) = y_{\max}$;
      $b, c =$ points of $V$ such that $y(b) = y_{\min}$, $x(c) = x_{\max}$;
      go to CASE $1(a, b, c)$;
else  if    $V$ includes two points on the same edge of
            $[x_{min}, x_{max}] \times [y_{min}, y_{max}]$
      then  label these two points $a$ and $b$;
            if necessary, rotate the plane to get $y(a) = y(b) = y_{\min}$;
            $c, d =$ points of $V$ such that $x(c) = x_{\min}$, $x(d) = x_{\max}$;
            if necessary, flip the plane and relabel the points
            to maintain $y(a) = y(b) = y_{\min}$, $x(c) = x_{\min}$, $x(d) = x_{\max}$
            and to get $x(a) < x(b)$, $y(c) \leq y(d)$;
            go to CASE $2(a, b, c, d)$;
      else  $a, b, c, d =$ (unique) points of $V$ such that
            $y(a) = y_{\max}$, $y(b) = y_{\min}$, $x(c) = x_{\min}$, $x(d) = x_{\max}$;
            if necessary, flip the plane and relabel the points
            to maintain $y(a) = y_{\max}$, $y(b) = y_{\min}$, $x(c) = x_{\min}$, $x(d) = x_{\max}$
            and to get $x(a) \leq x(b)$;
            if    $y(c) \geq y(d)$
            then  go to CASE $3(a, b, c, d)$;
            else  if    $V$ includes a point in $[x(a), x(b)] \times [y(c), y(d)]$
                  then  label this point $e$;
                        go to CASE $4(a, b, c, d, e)$;
                  else  go to CASE $5(a, b, c, d)$;
                  end
            end
      end
end
```

Figure 2: CLASSIFICATION

if V includes a point u other than b such that $y(v) = y_{\min}$
then return $\texttt{Omni}(b, u)$;
else **if** V includes a point v other than c such that $x(v) = x_{\max}$
 then return $\texttt{Omni}(c, v)$;
 else **if** the line bc is $\{b, c\}$
 then return $\texttt{Nil}(b, c)$;
 else $d =$ point of V such that $(b, d, c) \in \mathcal{B}$
  if the line ab is $\{a, b\}$
  then return $\texttt{Nil}(a, b)$;
  else $e =$ point of V such that $(a, e, b) \in \mathcal{B}$
   if $y(e) \geq y(d)$
   then return $\texttt{Omni}(a, e)$;
   else return $\texttt{Omni}(d, c)$;
   end
  end
 end
end

Figure 3: CASE $1(a, b, c)$

if V includes no point in $(x(a), x(b)) \times (y(d), y_{\max}]$
then return $\texttt{Omni}(a, b)$;
else $e =$ the point of $V \cap (x(a), x(b)) \times (y(d), y_{\max}]$ that
 minimizes $y(e)$ and, subject to this condition, maximizes $x(e)$;
 if the line ed is $\{e, d\}$
 then return $\texttt{Nil}(e, d)$;
 else $f =$ point of V such that
  $(f, e, d) \in \mathcal{B}$ or $(e, f, d) \in \mathcal{B}$ or $(e, d, f) \in \mathcal{B}$
  if $(e, f, d) \in \mathcal{B}$
  then return $\texttt{Omni}(a, b)$;
  else return $\texttt{Omni}(d, f)$;
  end
 end
end

Figure 4: CASE $2(a, b, c, d)$

```
if    the line ab is {a, b}
then  return Nil(a, b);
else  e = point of V such that (a, e, b) ∈ 𝓑
      if necessary, rotate the plane and relabel the points
      to maintain y(a) = y_max, y(b) = y_min, x(c) = x_min, x(d) = x_max
      and to get (a, e, d) ∈ 𝓑;
      if    the line ca is {c, a}
      then  return Nil(c, a);
      else  f = point of V such that (c, f, a) ∈ 𝓑
            if    y(e) ≤ y(f)
            then  return Omni(a, e);
            else  return Omni(c, f);
            end
      end
end
```

Figure 5: CASE $3(a, b, c, d)$

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if    the line ca is {c, a}
then  return Nil(c, a);
else  f = point of V such that (c, f, a) ∈ 𝓑
      if    y(f) ≤ y(d)
      then  return Omni(c, f);
      else  return Omni(b, e);
      end
end
```

Figure 6: CASE $4(a, b, c, d, e)$

In CASE 3, we have $V \subseteq \mathcal{C}(\{a, b, c, d\})$; if $y(e) \leq y(f)$, then $(b, e, f) \in \mathcal{B}$, and so the line ae includes points b, d, f, and c; if $y(e) > y(f)$, then $(c, f, e) \in \mathcal{B}$, and so the line cf includes points a, e, b, and d.

In CASE 4, we have $V \subseteq \mathcal{C}(\{a, b, c, d\})$ again and $(a, e, b), (c, e, d) \in \mathcal{B}$; if $y(f) \leq y(d)$, then $(c, f, d) \in \mathcal{B}$, and so the line cf includes points a, d, e, and b; if $y(f) > y(d)$, then $(f, e, b) \in \mathcal{B}$, and so the line be includes points a, f, c, and d.

if the line ab is $\{a, b\}$
then return $\texttt{Nil}(a, b)$;
else e = point of V such that $(a, e, b) \in \mathcal{B}$
    if necessary, rotate the plane and relabel the points
    to maintain $y(a) = y_{\max}$, $y(b) = y_{\min}$, $x(c) = x_{\min}$, $x(d) = x_{\max}$
    and to get $(a, e, d) \in \mathcal{B}$;
    if the line cd is $\{c, d\}$
    then return $\texttt{Nil}(c, d)$;
    else f = point of V such that $(c, f, d) \in \mathcal{B}$
        if $x(f) < x(a)$
        then return $\texttt{Omni}(c, f)$;
        else return $\texttt{Omni}(a, e)$;
        end
    end
end

Figure 7: CASE $5(a, b, c, d)$

In CASE 5, we have $V \subseteq \mathcal{C}(\{a, b, c, d\})$ once more. If $x(f) < x(a)$, then $(c, f, e) \in \mathcal{B}$, and so the line cf includes points d, e, a, and b. If $x(f) \geq x(a)$, then actually $x(f) > x(b)$ (else we would have ended up in CASE 4), and so $(a, e, f) \in \mathcal{B}$; now the line ae includes points b, d, f, and c.

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