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On Generating Three-connected Planar Graphs

by

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ABSTRACT

In this paper, a new method of generating three connected planar graphs in terms of generators is provided. Also it is shown that a planar graph is three connected planar graph if, and only if, it has a wheel or a pseudo-wheel as a generator.

1 Introduction

As known, 3-connected planar graphs play an important role in the enumeration of maps [5] and in characterizing the criterion of planarity of graphs [2]. In [8], W.T.Tutte provided a method of obtaining a 3-connected graph from a smaller one.

Proposition 1.1. *A graph G is 3-connected if, and only if, G is a wheel or can be obtained from a wheel by a seguence of operations of the following types*:

(a) *The addition of an edge.*

(b) *Replacing a vertex v, having $d(v) \geq 4$, by two adjacent vertices u and w in such a way that each vertex formerly adjacent to v is adjacent to exactly one of u and w, and in the resulting graph $d(u) \geq 3$ and $d(w) \geq 3$.*

Also in [7], by way of contracting edges of a 3-connected planar graph, one can prove that every planar 3-connected graph contains K_4 as a minor. In this paper, we provide another method of generating a 3-conneted graphs from a smaller one only in the way of adding a new vertex to the outer face joined (with subdivided edges) to the boundary of this face. Also it is shown that a planar graph is three connected planar graph if, and only if, it has a wheel or a pseudo-wheel as a generator. Another motivation is in the study of small circuit double cover (SCDC) problem [1,3] for planar graphs, the following question is arised.

Question 1.1. *When does an embedding of a planar graph with at least one interior vertex contain a generator and what is the generator?*

By a *generator* of a 3-connected planar graph G, we mean a 2-connected planar graph G_0, from which G can be obtained by a series of the following operations, *i.e.*, adding a vertex v in the outer face of G_0 with its incident disjoint paths to some vertices on the boundary of G_0, such that the resulting graph in each step remains to be 2-connected. The procedure of obtaining G from its generator G_0 is referred to as *generating* G. Each incident path produced in generating is also called a *subdivision edge* of the resulting graph. A generator G_0 of G which does not have a proper subgraph G_0' of G_0 such that G_0' is also a generator of G is said to be *minimal.* For the purposes of this paper when we use the term generator we mean a minimal generator that contains an interior vertex. Of course generators are a type of minors[6].

In this paper, we shall answer the above question for three connected planar graphs.

2 Definitions and notations

Graphs are always assumed to be simple (with neither multi-edge nor loop), undirected and connected. If a graph $G = (V, E)$, where V is the set of vertices of G and E is the set of its edges, has an embedding $\mu(G)$ in the plane, then it is said to be *planar*. For brevity, here, G and $\mu(G)$ are not distinguished. The boundary of a face f of G is denoted by ∂f. If a vertex v of G does not belong to the outer face of G, then v is called an *interior vertex* of

G. If G is a planar graph and it has no interior vertex, it is called an *outer planar graph*. A graph F is called a *subdivision* of G, written $F =_{Top} G$, if F can be obtained from G by *subdividing* some edges of G, *i.e.*, by a sequence of operations of replacing an edge $e = xy$ of G by a path xPy having exactly two vertices $\{x, y\}$ in common with the current graph. The *topological length* $l_G(P)$ of a path P in G is the number of inner vertices of P of valency at least 3 in G, while the length of P is the number of edges on it.

Terminologies and notations that are not explained here can be seen in [3].

Let v be a k-vertex ($k \geq 2$) of the planar graph G. The operation that transforms v to an edge v_1v_2, such that each incident edge of v in G is either incident to v_1 or v_2 in the resulting planar graph G', but not both, such that $d_G(v) = d_{G'}(v_1) + d_{G'}(v_2) - 2$ and $d_{G'}(v_i) \geq 2$ ($i = 1, 2$), is said to be *vertex-splitting*. Clearly, if $k = 2$, then the operation is a subdivision of an edge of G.

Let u be a d-vertex ($d \geq 3$) of planar graph G. The operation that transforms u to an l-circuit ($3 \leq l \leq d$) C_l, such that each incident edge of u in G is incident with some vertex of C_l in the resulting planar graph G' and

$$\forall v \in V(C_l),\ d_{G'}(v) \geq 3,$$

is said to be an *expansion* of vertex u into a circuit.

A *wheel* W_{n+1} of order $n+1$, is formed from an n-circuit C_n by adding a single new vertex u joining u each vertex of C_n by a new edge. We call u the *center* of the wheel W_{n+1}, C_n its *rim*, and the new edges its *spokes*.

Let W_n be a wheel of order n, and the center of the wheel is not allowed to be on the boundary. If $G =_{Top} W_n$ and $|V(G)| \geq n + 1$, then G is said to be a *pseudo-wheel of* **Type 0**, denoted by PW_0. The vertex (of valency not less than 3) that is not on the boundary of PW_0 is said to be the *pseudo-center* of PW_0. If G is obtained from a pseudo-wheel PW_0 of **Type 0** by splitting the pseudo-center of valency k ($k \geq 3$) or further splitting the vertices (of valency ≥ 3) not on the boundary of PW_0, then G is said to be a *pseudo-wheel of* **Type 1**, denoted by PW_1, shown in Fig.1. The vertices (of valency ≥ 3) not on the boundary of PW_1 are called *pseudo-centers* of PW_1. If G is obtained from a wheel (or a pseudo-wheel) PW_1 of **Type 1** with a center (or only one pseudo-center) u, by expanding u into an l-circuit, then G is said to be a *pseudo-wheel of* **Type 2**, denoted by PW_2, shown in Fig.2. If G can be obtained from a wheel (or a pseudo-wheel) PW_0 with the center (or only one pseudo-center) u by a series of subdivisions on a spoke (or pseudo spoke) incident with u by introducing a vertex v, and then joining edges ux and vx for a fixed vertex x on the boundary, shown in Fig.3, then G is said to be a pseudo-wheel of **Type 3**, denoted by PW_3. Any pseudo-wheel of **Type 0**, **Type 1**, **Type 2** or **Type 3** is denoted by PW.

3 Main results

Let G be a 2-connected planar graph with an interior vertex u. Let

$$F_u = \{f_u^i \mid i = 1, 2, \cdots, d(u),\ u \in \partial f_u^i\}$$

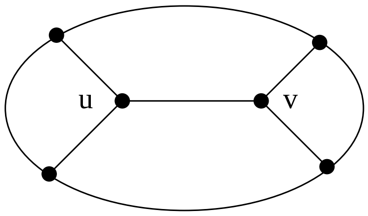

Fig. 1

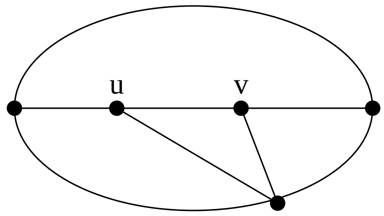

Fig. 2

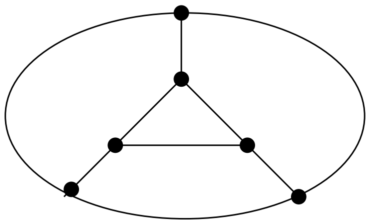

Fig. 3

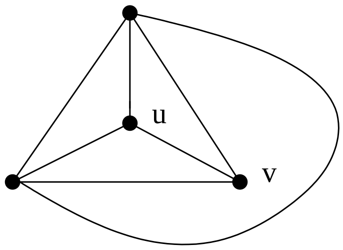

Fig. 4

and

$$C_u = \partial f_u^1 \bigtriangleup \partial f_u^2 \bigtriangleup \cdots \bigtriangleup \partial f_u^{d(u)},$$

where f_u^i is a face incident to u and $d(u)$ is the valency of u. It can be seen that if G is 3-connected, then C_u is a single close curve in the plane. Let $G[C_u]$ be the subgraph of G induced by all the edges on C_u and all the edges interior to C_u.

Lemma 3.1. *Let G be a 3-connected planar graph with at least one interior vertex u. Then $G[C_u]$ is a wheel or a pseudo-wheel of* **Type 0** *with only one interior vertex.*

Proof. ¿From what has been mentioned, it suffices to prove that there is only one interior vertex in $G[C_u]$. Since G is 3-connected, any f_u^i and f_u^j have at most one edge in common on their boundaries. So, $G[C_u]$ has only one interior vertex which is u itself. □

Theorem 3.1. *Let G be a 3-connected planar graph with at least one interior vertex. If every inner face incident with an interior vertex of G is of valency 3, then G contains a wheel as its generator.*

Proof. Let u be one of interior vertices with minimum valency among all interior vertices of G. Then it can be seen that $G[C_u]$ is a wheel. Let $V(G)\backslash V(G[C_u]) = V_0$. If $V_0 = \emptyset$, then $G = G[C_u]$, the theorem follows. If $V_0 \neq \emptyset$. Then, we have the following claim.

Claim *There exists a vertex v_0 in V_0 on the outer boundary of G such that the resulting graph G_1 obtained from deleting the vertex v_0 with its incident edges and subdivision edges is also a graph that every inner face incident with an interior vertex is of valency 3.*

Proof. Omitted because of the ease in contradiction and exhaustion. □

So if $V_0 \neq \emptyset$, by the claim, there must exist a vertex v_0 in V_0 on the outer boundary of G such that the resulting graph G_1 obtained from deleting the vertex v_0 with its incident edges and subdivision edges is also a graph that every inner face incident with an interior vertex is of valency 3. If $G_1 = G[C_u]$, then the Theorem holds. Otherwise, let $V_1 = V(G_1)\backslash V(G[C_u])$. We continue the above process for G_1. By the finiteness of G, there exists an m, $0 \leq m \leq |V(G)|$, such that $G_m = G[C_u]$. Hence, the theorem holds. □

A *planar near-triangulation* is such a connected planar graph whose inner faces are all triangles but the outer face may be not. Then from Theorem 3.1, it is easy to obtain the following Corollary. By using the method of generating a graph, in [3] we proved that every 2-connected planar near-triangulation admits an SCDC.

Corollary 3.1. *Every* 3*-connected planar near-triangulation with at least one interior vertex contains a wheel as its generator.*

Lemma 3.2. *Let G be a* 3*-connected planar graph with exactly one interior vertex, then G is a wheel.*

Proof. Let u be the interior vertex. From Lemma 3.1, $G[C_u]$ contains exactly one interior vertex. Let $V(G)\backslash V(G[C_u]) = V_0$. If $V_0 \neq \emptyset$, then by 3-connectedness there are At least three vertex-disjoint paths from $v_0 \in V_0$ (and u as well) to three distinct vertices, *e.g.*, v_1, v_2 and v_3, on the outer face boundary of $G[C_u]$. However, by planarity, v_2 has to be an interior vertex rather than u. The contradiction to the preassumed condition shows that $V_0 = \emptyset$, *i.e.*, $V(G) = V(G[C_u])$. Further, since u is the only interior vetex, G has no edge in the outer face of $G[C_u]$ and hence $E(G) = E(G[C_u])$. Thus, by 3-connectedness, the lemma follows. □

Lemma 3.3. *Let G be a* 3*-connected planar graph with exactly two interior vertices. Then, G contains a wheel, or a pseudo-wheel, as a generator.*

Proof. Let u and v be the two interior vertices of G. Let f_G be the outer face of G. Then, $uv \in E(G)$. Otherwise, assume that $uv \notin E(G)$. Since $|V(G[C_u]) \cap V(G[C_v]) \cap V(\partial f_G)| \leq 2$ by planarity, say $V_1 = V(G[C_u]) \cap V(G[C_v]) \cap V(\partial f_G)$, we have that V_1 forms a cut-set of cardinality less than 3. This is a contradiction to the 3-connectedness of G. Now, two cases for u and v are considered.

Case 1. $|C_u| \neq |C_v|$.

Let $|C_u| = \min\{|C_u|,\ |C_v|\}$ without loss of generality. Since G is 3-connected and planar, $G[C_u]$ is a wheel, or a pseudo-wheel of **Type 0**. Let x be a vertex of valency k ($k \geq 3$) on the boundary of f_G, such that it does not belong to $G[C_u]$(It does exist under the given conditions!). If the graph $G-x$ is not $G[C_u]$, denote it by G_1, then the connectivity $\kappa(G_1) \geq 2$. By the similar procedure to the proof of Theorem 3.1, we repeat the process until the last resulting graph is $G[C_u]$. ¿From the finiteness of G, it can be seen that $G[C_u]$ is a generator of G.

Case 2. $|C_u| = |C_v|$.

Claim 1. $|C_u| = |C_v| \geq 4$.

Proof. Suppose $|C_u| = |C_v| < 4$, *i.e.*, $|C_u| = |C_v| = 3$ because of 3-connectedness. Since for any $x \in V(G)$, its valency $d(x) \geq 3$ by 3-connectedness as well, both $G[C_u]$ and $G[C_v]$ are complete graph of order 4. On the other hand, from $uv \in E(G)$, there must exist multiple edges in G, as shown in Fig.4. This is contrary to the hypothesis that G is simple. Hence, $|C_u| = |C_v| \geq 4$. □

Subcase 1 $|C_u| = |C_v| = 4$.

Let G_0 be a graph with the minimum order n and size m among all 3-connected planar graphs considered. Since $uv \in E(G_0)$ and for any $x \in V(G)$, $d(x) \geq 3$, it can be seen that $n = 5$ and $m = \frac{1}{2}(3 \times 4 + 4 \times 1) = 8$. By exhausting all the graphs with the above conditions, the unique graph is as shown in Fig.3, which is a pseudo-wheel of **Type 3**. For a planar graph G with exactly two interior vertices u and v under the condition, if $n = 5$ and $m \geq 9$, then G is not simple. Further, if $n \geq 6$, then $V(G[C_u]) \neq V(G)$ (or $V(G[C_v]) \neq V(G)$). In this case, there must exist at least one vertex on the boundary of f_G, which does not belong to $G[C_u]$ (or $G[C_v]$). Then by applying the procedure as shown in Case 1, we conclude that G has a wheel, or a pseudo-wheel, as a generator.

Subcase 2 $|C_u| = |C_v| = 5$.

Similarly as Subcase 1, let G_1 be a graph with the minimum order n and size m of all 3-connected planar graph considered. Then, it can be seen that $n = 6$ and $m = 9$ and hence the unique graph is obtained from the wheel of order 5 with center x by splitting x into an edge uv, shown in Fig.1, which is a pseudo-wheel of **Type 1**. Because the condition: $n = 6$ and $m \geq 10$ causes $|C_u| \neq |C_v|$, only $n \geq 7$ should be observed. Since $|C_u| = |C_v| = 5$, there must exist at least one vertex y on the boundary of f_G, such that $y \notin V(G[C_u])$ (or $y \notin V(G[C_v])$). ¿From the procedure of Case 1, $G[C_u]$ (or $V(G[C_v])$) is a generator of G, which is a wheel, or a pseudo-wheel, by Lemma 3.1.

Subcase 3 $|C_u| = |C_v| \geq 6$.

¿From the above discussions of Subcases 1–2, $V(G[C_v]) \neq V(G)$ and there exists at least one vertex w on the boundary of f_G such that $w \notin V(G[C_u])$ (or $w \notin V(G[C_v])$). Hence, by the procedure as shown in case 1, $G[C_u]$ (or $V(G[C_v])$) is a generator of G, which is a wheel, or a pseudo-wheel.

Based on the discussions of Case 1 and Case 2, the lemma follows. □

Lemma 3.4. *Let G be a 3-connected planar graph with exactly three interior vertices. Then, G has a wheel, or a pseudo-wheel as a generator.*

Proof. Let v_1, v_2 and v_3 be the three interior vertices of G. Write

$$F(v) = \{v \mid |C_v| = \min\{|C_{v_i}| \mid i = 1, 2, 3\}\}.$$

Case 1 $|F(v)| = 1$.

Without loss of generality, let $|C_{v_1}| = \min\{|C_{v_i}| \mid i = 1, 2, 3\}$. Then there exists some vertices on the boundary of f_G, which do not belong to $V(G[C_{v_1}])$. The procedure as shown in the proof of Case 1 of Lemma 3.3 can be executed for the interior vertex v_1. Thus, $G[C_{v_1}]$ is a generator of G, which is a wheel, or pseudo-wheel by Lemma 3.1.

Case 2 $|F(v)| = 2$.

Without loss of generality, let $|C_{v_2}| = |C_{v_3}|$. If $V(G[C_{v_2}]) \neq V(G[C_{v_3}])$, then $G[C_{v_2}]$ and $G[C_{v_3}]$ can be considered as Case 1. Therefore, $G[C_{v_2}]$ or $G[C_{v_3}]$ is a generator of G, which is a wheel, or a pseudo-wheel. Otherwise, from the proof of Case 2 of Lemma 3.3, it can be seen that $G[C_{v_2} \cup C_{v_3} - \{v_2, v_3\}]$ is the pseudo-wheel of **Type3**. By similarly considering as Case 1, $G[C_{v_2} \cup C_{v_3} - \{v_2, v_3\}]$ is a generator of G.

Case 3 $|F(v)| = 3$.

Subcase 1 $V(G[C_{v_i}]) \neq V(G)$, $(i = 1, 2, 3.)$

If for any $i, j (i \neq j$ and $i, j \in \{1, 2, 3\})$, $V(G[C_{v_i}]) \neq V(G[C_{v_j}])$, then we can choose any $G[C_{v_i}]$ $(i = 1, 2, 3)$ as a generator of G as Case 1. Otherwise, by the same reason as in Case 2, only that there is some i, say $i = 1$, such that $V(G[C_{v_1}]) \neq V(G[C_{v_2}])$ and $V(G[C_{v_1}]) \neq V(G[C_{v_3}])$ should be observed. However, $G[C_{v_1}]$ itself is a generator of G. By Lemma 3.1, it is a wheel, or a pseudo-wheel.

Subcase 2 $V(G[C_{v_i}]) = V(G)$ $(i = 1, 2, 3.)$

We consider the graph G satisfied the above hypothesis with the minimum number of vertices and edges. Since G is a 3-connected and planar graph with exactly three interior vertices, it has that $|V(G)| \geq 6$ and for any x in $V(G)$, $d(x) \geq 3$. So, G have to be a 3-regular planar graph of order 6. It can be checked that G is obtained from the wheel of order 4 by expanding the center into a 3-circuit from the number of the non-isomorphic graphs[5, 0r 9]. Therefore, G is a pseudo-wheel of **Type 2**, denoted by $K_4 + C_3$, shown in Fig.2.

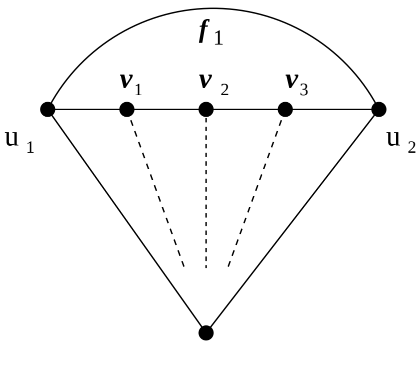

Fig. 5

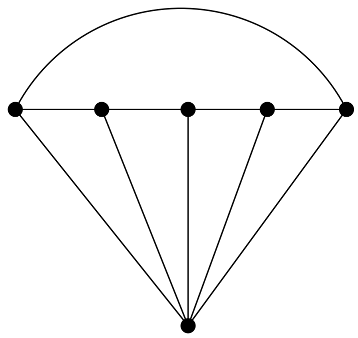

Fig. 6

Now, we observe the graph G of order 6 and size 10. Since $V(C_{v_i}) = V(G)$ $(i = 1, 2, 3)$, it has the structure as shown in Fig.5. By the conditions considered, no edge (v_i, u_j) in f_1 for $i = 1, 2, 3;$ $j = 1, 2$, occurs. So, G is a fan-shaped graph as shown in Fig.6. From the definition of PW, it is a pseudo-wheel of **Type 3**. Moreover, if $\nu(G) = 6$ and $|E(G)| \geq 11$, then from above discussion, G does not satisfy all the given conditions.

Furthermore, because all the vertices in $V(G)\backslash\{v_1, v_2, v_3\}$ are on the outer face boundary of G, for $\nu(G) \geq 7$, no G is with the given coditions.

Therefore, from Subcase 1-2, we deduce that if $|F(v)| = 3$, then G has a wheel, or a pseudo-wheel as a generator.

Based on Cases 1-3, the lemma follows. $\square$

Lemma 3.5. *Let G be a 3-connected planar graph with $k(\geq 4)$ interior vertices. Then there is only one graph G such that $V(G[C_{v_i}]) = V(G)$ $(i = 1, 2, \cdots, k)$ and G is a pseudo-wheel of order $k + 3$ and size $2(k + 2)$.*

Proof. Based on the discussion in Subcase 2 of Case 3 in the proof of Lemma 3.4, G here is also with the fan-shaped form as shown in Fig.5. By simply counting the valences of vertices, it soon be found that G has $k + 3$ vertices and $2(k + 2)$ edges. Further, it is easily checked that G is a pseudo-wheel. So, the lemma holds. $\square$

Theorem 3.2. *Let G be a 3-connected planar graph with at least one interior vertex. Then G has a generator that is a wheel, or a pseudo-wheel.*

Proof. ¿From the proofs of Lemmas 3.2–5, it clearly suffices to prove the theorem for 3-connected planar graphs with $k(\geq 4)$ interior vertices, named by $v_1, v_2, \cdots, v_l \in F(v) = \{v \mid |C_v| = \min\{|C_u|\ u \text{ is an interior vertex}\}\}$ for $4 \leq |F(v)| = l$.

If there exists an integer i_0, such that for $i_0 \neq j$, $i_0, j \in \{1, 2, \cdots, k\}$,

$$V(G[C_{v_{i_0}}]) \neq V(G[C_{v_j}]).$$

Then, it can be seen that $G[C_{v_{i_0}}]$ is an generator of G.

If there exists integers i_1 and i_2 such that $V(G[C_{v_{i_1}}]) = V(G[C_{v_{i_2}}])$ and for v_i $(i \in \{1, 2, \cdots, l\}\backslash\{i_1, i_2\})$, $V(G[C_{v_{i_1}}]) \neq V(G[C_{v_i}])$, then from the proof of Lemma 3.3, we have $G[G[C_{v_{i_1}}] \cup G[C_{v_{i_2}}] - \{v_1, v_2\}] =_{Top} G_1$, or G_2, where G_1, G_2 are shown in Fig. 1 and Fig.2 respectively. So, in this case, the theorem holds.

If there exists integers j_1, j_2 and j_3 such that $V(G[C_{v_{j_1}}]) = V(G[C_{v_{j_2}}]) = V(G[C_{v_{j_3}}])$ and for v_i $(i \in \{1, 2, \cdots, l\}\backslash\{j_1, j_2, j_3\})$, $V(G[C_{v_{j_1}}]) \neq V(G[C_{v_i}])$, then from the proof of Lemma 3.4, we have $G[G[C_{v_{j_1}}] \cup G[C_{v_{j_2}}] \cup G[C_{v_{j_3}}] - \{vj_i | i = 1, 2, 3\}] = G_1$, or $=_{Top} G_2$, where G_1, G_2 are shown in Fig.2, Fig.6 respectively. It can be seen that $G[G[C_{v_{j_1}}] \cup G[C_{v_{j_2}}] \cup G[C_{v_{j_3}}] - \{vj_i | i = 1, 2, 3\}]$ is a generator of G, which is a wheel, or a pseudo-wheel.

If $V(G[C_{v_{l_1}}]) = V(G[C_{v_{l_2}}]) = \cdots = V(G[C_{v_{l_m}}])$, then from Lemma 3.5, G has a pseudo-wheel of **Type 3** as a generator.

In consequence, the theorem follows. $\square$

Corollary 3.2. *Every simple 3-connected planar graph can be obtained from one of a wheel and a pseudo-wheel by a series of the following operation: adding a vertex v in the outer face with its incident disjoint paths.*

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