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On the band-, tree- and clique-width of graphs with bounded vertex degree

by

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ABSTRACT

The band-, tree- and clique-width are of primary importance in algorithmic graph theory due to the fact that many problems being NP-hard for general graphs can be solved in polynomial time when restricted to graphs where one of these parameters is bounded. It is known that for any fixed $\Delta \geq 3$, all three parameters are unbounded for graphs with vertex degree at most Δ. In this paper, we distinguish representative subclasses of graphs with bounded vertex degree that have bounded band-, tree- or clique-width. Our proofs are constructive and lead to efficient algorithms for a variety of NP-hard graph problems when restricted to those classes.

1 Introduction

The band-, tree- and clique-width are of primary importance in algorithmic graph theory due to the fact that many problems which are NP-hard for general graphs can be solved in polynomial time when restricted to graphs where one of these parameters is bounded. In fact, all problems expressible in monadic second order logic become polynomial time solvable when restricted to graphs of bounded tree-width [6, 14] and all problems expressible in monadic second order logic using quantifiers on vertices but not on edges become polynomial time solvable when restricted to graphs of bounded clique-width [9, 10]. This includes, for example, MAXIMUM CLIQUE, INDEPENDENT SET, MINIMUM DOMINATING or INDEPENDENT DOMINATING SET problems as well as k-COLORABILITY (for fixed k), MAXIMUM INDUCED MATCHING, INDUCED PATH, etc. Furthermore, graphs of bounded band-width are easily seen to be of bounded tree- and clique-width (cf. Proposition 1 below) and the mentioned problems are thus equally tractable.

It is known that for any fixed $\Delta \geq 3$, all three parameters are unbounded for graphs with vertex degree at most Δ. This is the case, for instance, for the so-called *walls* which are planar graphs of maximum vertex degree 3 (cf. [14]) and have arbitrarily large band-, tree- and clique-width [8, 14].

The objective of the present paper is to distinguish subclasses of graphs of bounded vertex degree that have bounded band-, tree- or clique-width. To this end, we study hereditary classes defined by forbidding large induced subgraphs. Two particular forbidden graphs play the critical role in our study. For integers $i, j, k \geq 0$, these are the two graphs $S_{i,j,k}$ and $T_{i,j,k}$ depicted in Figure 1(a) and 1(b), respectively. (Note that $S_{0,j,k}$ is a chordless path on $j + k + 1$ vertices and $T_{0,0,0}$ is simply a triangle.) By $\mathcal{S}$ we shall denote the class of graphs every component of which is of the form $S_{i,j,k}$, and by $\mathcal{T}$ the class of graphs every component of which is of the form $T_{i,j,k}$.

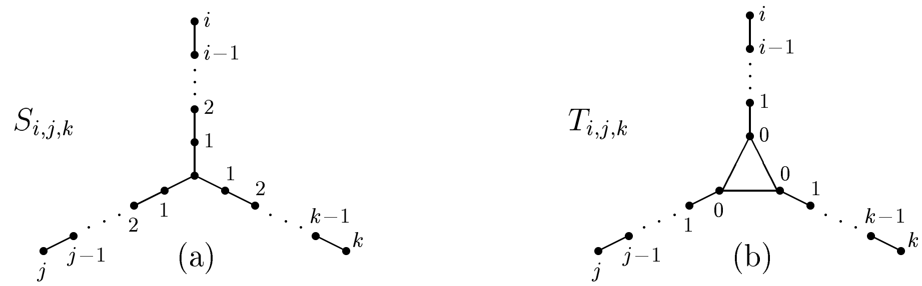

Figure 1: The graphs (a) $S_{i,j,k}$ and (b) $T_{i,j,k}$

We prove that in connected graphs of bounded vertex degree that do not contain S_{i_1,j_1,k_1} and T_{i_2,j_2,k_2} (for arbitrary values of the indices) as induced subgraphs we can find in polynomial time an induced path such that all vertices are within bounded distance of this path. This implies, in particular, that for any hereditary class of graphs $\mathcal{X}$ with bounded vertex degree that contains neither $\mathcal{S}$ nor $\mathcal{T}$, the tree-width and the clique-width are bounded by a constant. Under the assumption that $P \neq NP$, this result is best possible for the classes of graphs defined by finitely many forbidden induced subgraphs. This is a consequence of the following two facts: first, if a class of graphs $\mathcal{X}$ defined by finitely many forbidden induced subgraphs contains $\mathcal{S}$ or $\mathcal{T}$, then the independent dominating set problem is NP-hard in the class $\mathcal{X}$ [3], and second, this problem is polynomially solvable for graphs with bounded tree-width or clique-width. For the band-width, we discover some stronger conditions under which this parameter is bounded. All our results are algorithmical in the sense that they not only prove boundedness but also imply the existence of simple polynomial time algorithms for the respective widths problems.

The outline of our presentation is as follows. In the next section we provide formal definitions and prove preliminary results. In Section 3 we establish structural properties of graphs of bounded vertex degree that do not contain $S_{k,k,k}$ and $T_{k,k,k}$ as induced subgraphs. Section 4 contains our main results. In Section 5 we illustrate the results by applying them to several well-know classes such as asteroidal triple-free graphs (that includes comparability, permutation, and interval graphs) and chordal bipartite graphs.

2 Definitions and preliminary results

For standard graph-theoretical terminology, the reader is referred to any of the classical textbooks on graph theory. The set of vertices and set of edges of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. Given a vertex $v \in V(G)$, $N_G(v)$ stands for the neighborhood of v in the graph G, i.e. the set of vertices of G adjacent to v, and $N_G[v]$ for the closed neighborhood of v, i.e. $N_G[v] = N_G(v) \cup \{v\}$. For a subset of vertices $U \subseteq V(G)$, $G-U$ is the subgraph of G obtained by deleting the vertices of U, $N_G(U)$ is the neighborhood of U, i.e. the set of vertices of G outside U that have a neighbor in U, and $N_G[U]$ is the closed neighborhood of U, i.e. $N_G[U] = N_G(U) \cup U$.

A graph G will be called *H-free* if G does not contain H as an induced subgraph. The class of all graphs containing no induced subgraphs in a set M will be denoted $\mathrm{Free}(M)$. We shall write $\mathrm{Free}_k(M)$ to denote the graphs with vertex degree at most k in $\mathrm{Free}(M)$. It is well-known that a class of graphs $\mathcal{X}$ is *hereditary* (i.e. closed under deletion of vertices) if and only if $\mathcal{X} = Free(M)$ for a certain set M (possibly infinite).

For a class of graphs $\mathcal{Y}$, we denote by $[\mathcal{Y}]_k$ the class of graphs G such that $G-U$ belongs to $\mathcal{Y}$ for a subset $U \subseteq V(G)$ of cardinality at most k. Also, given a class of graphs $\mathcal{Y}$, $[\mathcal{Y}]_B$ denotes the class of graphs every block of which belongs to $\mathcal{Y}$.

We now define the three width parameters that we consider in the paper. Let L be an ordering of the vertices of a graph G, i.e. L is a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$. The *width* of the ordering is defined as

$$\max\{|L(u) - L(v)| \ \mid uv \in E(G)\}.$$

The *band-width* $\mathrm{bw}(G)$ of G [11] is the minimum integer k for which there exists an ordering L of width k for G.

A *tree decomposition* [2, 14, 15] of a graph G is a pair $(T, \mathcal{W})$ where T is a tree and $\mathcal{W}$ assigns a set $W_t \subseteq V(G)$ to each vertex t of T such that

(i) $V(G) = \bigcup_{t \in V(T)} W_t$,

(ii) for every edge $uv \in E(G)$, there is some $t \in V(T)$ such that $u, v \in W_t$ and

(iii) for every vertex $u \in V(G)$, the set $\{t \in V(T) \mid u \in W_t\}$ induces a subtree of T.

The *width* of a tree decomposition $(T, \mathcal{W})$ is $\max_{t \in V(T)} |W_t| - 1$ and the *tree-width* $\operatorname{tw}(G)$ of G is the minimum width of a tree decomposition of G.

Finally, the *clique-width* $\operatorname{cw}(G)$ [7] of a graph G is the minimum number of labels needed to construct G using the following four operations:

(i) Create a new vertex v with label i (denoted $i(v)$).

(ii) Form the disjoint union of two labeled graphs G and H (denoted $G \oplus H$).

(iii) Join all vertices with label i to all vertices with label j ($i \neq j$, denoted $\eta_{i,j}$).

(iv) Change the label of all vertices with label i to j (denoted $\rho_{i \to j}$).

Every graph can be defined by an algebraic expression using these four operations. For instance, the cycle on five consecutive vertices a, b, c, d, e can be defined as follows:

$$\eta_{4,1}(\eta_{4,3}(4(e) \oplus \rho_{4\to 3}(\rho_{3\to 2}(\eta_{4,3}(4(d) \oplus \eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))))).$$

Such an expression is called a *k-expression* if it uses at most k different labels. Thus the clique-width of G is the minimum k for which there exists a k-expression defining G.

After these definitions we proceed to some auxiliary results.

Proposition 1 *Let G be a graph and $U \subseteq V(G)$. Given an ordering of width k of the vertices of $G - U$, one can construct in polynomial time*

(i) a tree decomposition of G of width at most $2\lceil \frac{k}{2} \rceil + |U|$ and
(ii) a $(k + 2 + |U|)$-expression defining G.

Proof. Let $n = |V(G) \setminus U|$. Let L be the ordering of width k of the vertices of $G - U$ and let $v_i \in V(G) \setminus U$ be such that $L(v_i) = i$ for $1 \leq i \leq n$.

Let T be such that $V(T) = V(G) \setminus U$ and

$$E(T) = \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\}.$$

For $1 \leq i \leq n$, let

$$W_{v_i} = U \cup \left\{ v_j \mid 1 \leq j \leq n,\ |j - i| \leq \left\lceil \frac{k}{2} \right\rceil \right\}.$$

Now it is easy to check that $(T,\mathcal{W})$ is a tree decomposition of G of width at most $2\lceil\frac{k}{2}\rceil+|U|$ which implies (i).

To prove (ii) we describe a procedure to construct G using the four operations described above and $k+2+|U|$ different labels.

First, for all vertices $u \in U$ apply $i_u(u)$ such that $i_u \neq i_v$ for all $u,v \in U$ with $u \neq v$. Next, for all edges $uv \in E(G)$ with $u,v \in U$ apply η_{i_u,i_v}.

Let I be a set of $(k+1)$ different labels such that $I \cap \{i_u \mid u \in U\} = \emptyset$. Let i_0 be a label not in $I \cup \{i_u \mid u \in U\}$.

For r from 1 up to n, apply the following sequence of operations: Let i_s be the label presently assigned to vertex v_s with $\max\{1, r-k\} \leq s \leq r$. Let

$$i_r \in I \setminus \{i_s \mid \max\{1, r-k\} \leq s \leq r\}.$$

Apply $i_r(v_r)$. For all $u \in N_G(v_r) \cap U$, apply η_{i_r,i_u}. For all $v_s \in N_G(v_r) \cap \{v_1, v_2, \ldots, v_{r-1}\}$, apply η_{i_r,i_s}. Finally, apply $\rho_{i_{r-k}\to i_0}$ and proceed to $r+1$.

It is obvious that this procedure constructs G using $k+2+|U|$ different labels which completes the proof. ■

Corollary 1 *If G is a graph and $U \subseteq V(G)$, then*

(i) $\mathrm{tw}(G) \leq 2\lceil\frac{\mathrm{bw}(G-U)}{2}\rceil + |U|$ *and*

(ii) $\mathrm{cw}(G) \leq \mathrm{bw}(G-U) + 2 + |U|$.

Proposition 2 *If the tree-width of graphs in a class $\mathcal{Y}$ is bounded by p, and a tree decomposition of width at most p for any graph in $\mathcal{Y}$ can be constructed in polynomial time, then one can construct in polynomial time a tree-decomposition of width at most $p+k$ for any graph in $[\mathcal{Y}]_k$ and a tree-decomposition of width at most p for any graph in $[\mathcal{Y}]_B$.*

Proof. For the graphs in $[\mathcal{Y}]_k$, one can proceed as in the proof of Proposition 1. Now let $G \in [\mathcal{Y}]_B$ and let $B_1, B_2, \ldots, B_l$ be the blocks of G. Let $(T(i), \mathcal{W}(i))$ be a tree decomposition of B_i for $1 \leq i \leq l$. We assume that all graphs $G, T(1), T(2), \ldots, T(l-1)$ and $T(l)$ are vertex disjoint. Let C be the set of cutvertices of G. For every cutvertex $u \in C$ and every block B_i such that $u \in V(B_i)$, we fix an arbitrary vertex $t(u,i) \in V(T(i))$ such that $u \in W(i)_{t(u,i)}$.

Let T be the tree such that $V(T) = C \cup (\bigcup_{i=1}^{l} V(T(i)))$, $\bigcup_{i=1}^{l} E(T(i)) \subseteq E(T)$ and $E(T)$ contains an edge $ut(u,i)$ for every cutvertex $u \in C$ and every block B_i such that $u \in V(B_i)$. Let $W_t = W(i)_t$ for all $t \in V(T(i))$, $1 \leq i \leq l$ and let $W_u = \{u\}$ for all $u \in B$.

It is straightforward to verify that $(T,\mathcal{W})$ is a tree decomposition of width at most p of G and the proof is completed. ■

Proposition 3 *If the clique-width of graphs in a class $\mathcal{Y}$ is bounded by p, and a p-expression for any graph in $\mathcal{Y}$ can be constructed in polynomial time, then one can construct in polynomial time a $2^k(p+1)$-expression for any graph in $[\mathcal{Y}]_k$ and a $(p+2)$-expression for any graph in $[\mathcal{Y}]_B$.*

Proof. To prove the result for graphs in $[\mathcal{Y}]_k$, it suffices to consider the case $k = 1$. Let $G \in [\mathcal{Y}]_1$ and $G - \{v\} \in \mathcal{Y}$. Given a p-expression F for $G - \{v\}$, we first construct a $2p$-expression F' for $G - \{v\}$ in such a way that every labeled vertex $l(u)$ in F changes its label either to $l'(u)$ or to $l''(u)$ in F' depending on whether or not it is adjacent to v. Now we only need one additional label to obtain a $(2p + 1)$-expression for G from F'. Thus, $\mathrm{cw}(G) \leq 2\mathrm{cw}(G - \{v\}) + 1$ and the proposition holds for graphs in $[\mathcal{Y}]_k$ by induction.

To prove the result for graphs in $[\mathcal{Y}]_B$, consider a graph $G \in [\mathcal{Y}]_B$. By assumption, the clique-width of every block of G is bounded by p. We show by induction on the number of blocks that $\mathrm{cw}(G) \leq p+2$. Two additional labels needed to construct a $(p+2)$-expression for G will be denoted α and β. First, let G be a block and v be an arbitrary vertex in G. Any p-expression $F(G)$ defining G can trivially be modified into a $(p + 1)$-expression $F'_v(G)$ in which v is the only vertex labeled with α. We then transform $F'_v(G)$ into a $(p+2)$-expression $F_v(G)$ by re-labeling every vertex different from v with β. Now let H be a graph with $b > 1$ blocks and G be a block in H with a single cutvertex v. By deleting from H all the vertices of G except v, we obtain a graph with $b - 1$ blocks. Such a graph can be defined by a $(p + 2)$-expression T due to the inductive hypothesis. Assume the vertex v is created in T with label j. Then substituting $j(v)$ with $\rho_{\alpha\to j}(F_v(G))$ in T we obtain a $(p + 2)$-expression defining H. To see this, it is enough to notice that the label β is never renamed or used in any η-operation in T by the inductive hypothesis. This completes the proof. ■

3 Properties of $(S_{k,k,k}, T_{k,k,k})$-free graphs of bounded vertex degree

In this section we establish a sequence of useful structural and algorithmical properties of graphs that do not contain $S_{k,k,k}$ and $T_{k,k,k}$ as induced subgraphs and have bounded vertex degree. Throughout this section we let k and Δ be fixed positive integers.

Lemma 1 *If G is a $S_{k,k,k}$-free and $T_{k,k,k}$-free graph of vertex degree at most Δ, then G does not contain two vertex disjoint induced paths*

$$P : x_0x_1 \ldots x_{4\Delta k}$$

and

$$Q : y_0y_1 \ldots y_k$$

such that $y_0x_{2\Delta k} \in E(G)$ and all edges between vertices of P and vertices of Q are incident to y_0 (see Figure 2).

Proof. For contradiction, we assume that two paths P and Q as in the statement exist. If $x_{2\Delta k}$ is the only neighbor of y_0 on P, then G is obviously not $S_{k,k,k}$-free, which is a contradiction. Hence, for some $2 \leq l \leq \Delta - 1$, let

$$N_G(y_0) \cap V(P) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_l}\}$$

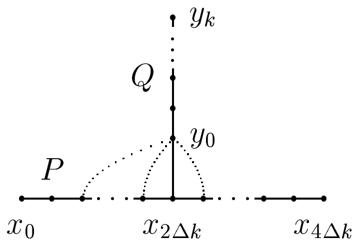

Figure 2: P and Q

such that $i_j < i_{j+1}$ for each $j = 1, \ldots, l-1$. If $i_{j+1} - i_j \geq 2k$ for some $1 \leq j \leq l-1$, then the subgraph of G induced by the set

$$V(Q) \cup \{x_{i_j}, x_{i_j+1}, \ldots, x_{i_j+(k-1)}\} \cup \{x_{i_{j+1}}, x_{i_{j+1}-1}, \ldots, x_{i_{j+1}-(k-1)}\}$$

is isomorphic to $S_{k,k,k}$, which is a contradiction. Hence $i_{j+1} - i_j < 2k$ for each $1 \leq j \leq l-1$. Since $x_{2\Delta k} \in N_G(y_0)$, this implies that

$$\begin{array}{rcl} i_1 & > & 2\Delta k - (\Delta - 2)2k = 4k, \\ i_l & < & 2\Delta k + (\Delta - 2)2k = 4\Delta k - 4k \end{array}$$

and hence the subgraph induced by the set

$$V(Q) \cup \{x_{i_1}, x_{i_1-1}, \ldots, x_0\} \cup \{x_{i_l}, x_{i_l+1}, \ldots, x_{4\Delta k}\}$$

contains either $S_{k,k,k}$ or $T_{k,k,k}$ as an induced subgraph. This contradiction completes the proof. ■

Lemma 2 *Let G be a connected $(S_{k,k,k}, T_{k,k,k})$-free graph of vertex degree at most Δ, and*

$$P : x_0 x_1 \ldots x_{\Delta k}$$

an induced path in G. Then the subgraph $G - N_G[V(P)]$ does not contain an induced cycle

$$C : y_1 y_2 \ldots y_l y_1$$

of length $l \geq 2(\Delta - 1)(k + 1)$.

Proof. For contradiction, we assume that $G - N_G[V(P)]$ contains an induced cycle C as in the statement. Since G is connected, we may consider a shortest path

$$Q : z_0 z_1 \ldots z_r$$

connecting P to C such that $z_0 \in V(P)$ and $z_r \in V(C)$. Note that $r \geq 2$.

The vertex z_1 has at most $(\Delta - 1)$ neighbors in $V(P)$. These neighbors divide P into at most Δ edge-disjoint segments, one of which has length at least $\frac{\Delta k}{\Delta} = k$. Without loss of generality, we assume that

$$N_G(z_1) \cap \{x_i, x_{i+1}, \ldots, x_{i+k-1}\} = \{x_i\}$$

for some $0 \leq i \leq \Delta k - k$.

If z_r is the only neighbor of z_{r-1} in $V(C)$, then G is obviously not $S_{k,k,k}$-free, which is a contradiction. Therefore, we assume that z_{r-1} has at least two neighbors in $V(C)$. Clearly, z_{r-1} has at most $(\Delta - 1)$ neighbors in $V(C)$. These neighbors divide C into at least two and at most $(\Delta - 1)$ edge-disjoint segments, one of which has length at least $\frac{2(\Delta-1)(k+1)}{\Delta-1} = 2k+2$. Without loss of generality, we assume that

$$N_G(z_{r-1}) \cap \{y_j, y_{j+1}, \ldots, y_{j+s}\} = \{y_j, y_{j+s}\}$$

for some $1 \leq j \leq 2(\Delta - 1)(k + 1) - s$ and some $s \geq 2k + 2$. Note that $y_j y_{j+s} \in E(G)$ is possible ($s = l - 1$). Now the subgraph of G induced by the set (see Figure 3)

$$\{z_1, \ldots, z_{r-1}\} \cup \{x_i, x_{i+1}, \ldots, x_{i+k-1}\} \cup \{y_j, y_{j+1}, \ldots, y_{j+k}\} \cup \{y_{j+s}, y_{j+s-1}, \ldots, y_{j+s-k}\}$$

contains either $S_{k,k,k}$ or $T_{k,k,k}$ as an induced subgraph, which is a contradiction and the proof is completed. ■

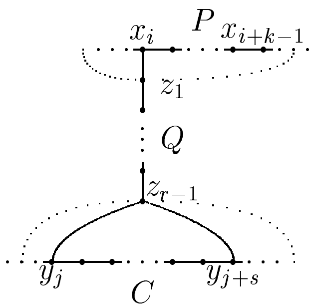

Figure 3: Parts of P, Q and C

Corollary 2 *If G is a connected $(S_{k,k,k}, T_{k,k,k})$-free graph of vertex degree at most Δ, then there is a set $U \subseteq V(G)$ of at most $c_1 = c_1(k, \Delta)$ vertices such that $G - U$ contains no induced cycle of length at least $2(\Delta - 1)(k + 1)$. Furthermore, such a set can be found in polynomial time.*

Proof. If we define $c_1' = c_1'(k, \Delta) = 1 + \sum_{i=0}^{\Delta k-1} (\Delta - 1)^i \Delta$, then the graph G either has order $|V(G)| \leq c_1'$ or contains an induced path of length Δk.

If $|V(G)| \leq c_1'$, then let $V' = V(G)$. If $|V(G)| \geq c_1' + 1$, then, by considering the pairwise distances of the vertices of G, we can find in polynomial time an induced path P of length Δk. Since G has maximum vertex degree at most Δ, the set $N_G[V(P)]$ contains at most $c_1'' = c_1''(k, \Delta) = (\Delta k + 1)(\Delta - 1) + 1$ vertices. Let $U = N_G[V(P)]$. By Lemma 2, the graph $G - U$ contains no induced cycle of length at least $2(\Delta - 1)(k + 1)$. The desired result now follows with $c_1 = \max\{c_1', c_1''\}$. ■

Lemma 3 *If G is a connected $(S_{k,k,k}, T_{k,k,k})$-free graph of vertex degree at most Δ, then G contains an induced path P such that for every vertex $u \in V(G)$,*

$$\operatorname{dist}_G(u, V(P)) = \min\{\operatorname{dist}_G(u,v) \mid v \in V(P)\} \leq 4\Delta k.$$

Moreover, such a path can be found in polynomial time.

Proof. A procedure that computes a path with the desired property can be roughly described as follows.

(0) Find an arbitrary induced path P which is maximal under inclusion.

(1) Denote by l the length of P, and by $x_0x_1\ldots x_l$ the vertices of P. For each vertex $u \in V(G)$, compute $\operatorname{dist}_G(u, V(P))$. If all these distances are at most $4\Delta k$, then STOP: P is the path we sought for.

(2) Let $u \in V(G)$ be a vertex of G with $\operatorname{dist}_G(u, V(P)) > 4\Delta k$. If $l \leq 4\Delta k$, then a shortest path P' from u to P is longer than P. In this case, set $P := P'$ and go to step (1).

(3) By considering the pairwise distance between u and the vertices of P, determine the set
$$D = \{v \in V(P) \mid \operatorname{dist}_G(u,v) = \operatorname{dist}_G(u, V(P))\}.$$
Up to symmetry, we let r be the smallest index such that $x_r \in D$ and $r \geq l - 2\Delta k + 1$ (if there is no such r, define r to be the largest index such that $x_r \in D$ and $r \leq 2\Delta k - 1$). Let $Q : u = y_0y_1\ldots y_s = x_r$ be a shortest path connecting u to x_r, where $s = \operatorname{dist}_G(u, V(P)) > 4\Delta k$. Set
$$P := x_{2\Delta k}x_{2\Delta k+1}\ldots x_r y_{s-1}y_{s-2}\ldots y_1 u$$
and go to step (1).

Obviously, every step of the procedure can be implemented in polynomial time. Now let us show that the loop (1)-(3) repeats at most $|V(G)|$ times. First, notice that since $l > 4\Delta k$ in step (3),

$$D \cap \{x_{2\Delta k}, x_{2\Delta k+1}, \ldots, x_{l-2\Delta k}\} = \emptyset,$$

otherwise G would contain two vertex disjoint paths as described in Lemma 1, which is a contradiction. Therefore,

$$D \subseteq \{x_0, x_1, \ldots, x_{2\Delta k-1}\} \cup \{x_l, x_{l-1}, \ldots, x_{l-2\Delta k+1}\}.$$

Clearly, $D \neq \emptyset$ and hence in step (3) of the procedure we find a new induced path of length

$$r - 2\Delta k + s > l - 2\Delta k + 1 - 2\Delta k + 4\Delta k + 1 > l.$$

Summarizing, both in step (2) and step (3) we go to step (1) with a larger induced path. Therefore, after at most $|V(G)|$ loops the procedure terminates. This proves the correctness of the procedure and a polynomial time bound. ∎

4 Main Results

We now proceed to our main results.

Theorem 1 *If G is a $S_{k,k,k}$-free and $T_{k,k,k}$-free graph of vertex degree at most Δ that does not contain an induced cycle of length at least $2(\Delta-1)(k+1)$, then the band-width of G is bounded by a constant $c_2 = c_2(k,\Delta)$, and an ordering of the vertices of G of width at most c_2 can be determined in polynomial time.*

Proof. Clearly, we can deal with different components of G separately. Hence we may assume that G is connected.

By Lemma 3, we can find in polynomial time an induced path $P : x_0x_1 \dots x_l$ such that $\mathrm{dist}_G(u, V(P)) \le 4\Delta k$ for every vertex $u \in V(G)$.

We will now define a partition

$$V(G) = \bigcup_{i=0}^{l} V_i$$

assigning every vertex of G to some vertex of the path P. For each $i = 0, \dots, l$, the set V_i will consist of the vertices assigned to $x_i \in V(P)$.

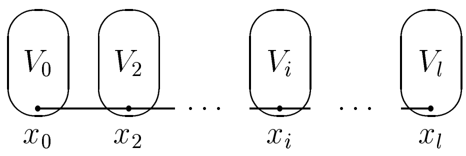

Figure 4: The partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_l$

Let $u \in V(G)$. As in the proof of Lemma 3, we determine in polynomial time the set

$$D(u) = \{v \in V(P) \mid \mathrm{dist}_G(u,v) = \mathrm{dist}_G(u,P)\}.$$

Since G has vertex degree at most Δ and $\mathrm{dist}_G(u, V(P)) \le 4\Delta k$, we have

$$|D(u)| \le c_2' = c_2'(k,\Delta) = 1 + \sum_{\nu=0}^{4\Delta k} (\Delta-1)^{\nu}\Delta.$$

We choose an arbitrary vertex $x_i \in D(u)$ and assign u to x_i, i.e. $u \in V_i$. The main property to establish the boundedness of the band-width is expressed in the following claim.

Claim If $u \in V_i$ and $v \in V_j$ for some $0 \le i < j \le l$ such that

$$j - i \ge 2(\Delta-1)(k+1) + 4(\Delta-1)(k+1)c_2',$$

then $uv \notin E(G)$.

Proof of the claim: For contradiction, we assume that $uv \in E(G)$.

Let $x_{i'}$ be the vertex in $D(u)$ with the maximum index, then

$$i' - i \leq 2(\Delta - 1)(k+1)c_2'.$$

To show this, consider two consecutive vertices x_s and x_t in $D(u)$, i.e. for each $r \in \{s+1,\dots,t-1\}$, $x_r \notin D(u)$. Then $t - s < 2(\Delta-1)(k+1)$, since otherwise the vertices of two shortest paths connecting u to x_s and x_t together with the vertices $x_s,\dots,x_t$ would induce a graph containing a chordless cycle of length at least $2(\Delta-1)(k+1)$.

Analogously, if $x_{j'}$ is the vertex in $D(v)$ with the minimum index, then

$$j - j' \leq 2(\Delta - 1)(k+1)c_2'.$$

Consequently,

$$j' - i' \geq j - i - 4(\Delta-1)(k+1)c_2' \geq 2(\Delta-1)(k+1).$$

But now the vertices of two shortest paths connecting u and v respectively to $x_{i'}$ and $x_{j'}$ together with the vertices $x_{i'}, x_{i'+1},\dots,,x_{j'}$ induce in G a graph containing a chordless cycle of length at least $2(\Delta-1)(k+1)$. This contradiction completes the proof of the claim.

Since G has vertex degree at most Δ and by Lemma 3, we have $|V_i| \leq c_2' = c_2'(k,\Delta)$ for every $0 \leq i \leq l$. Let $v_1v_2\dots v_n$ be an ordering of the vertices of G in which u comes before v whenever $u \in V_i$ and $v \in V_j$ with $i < j$. By the above claim, it is obvious that $v_iv_j \in E(G)$ implies

$$|j - i| \leq c_2 = (2(\Delta-1)(k+1) + 4(\Delta-1)(k+1)c_2')c_2'$$

and the proof is completed. ■

Whereas band-width is very sensitive to the deletion of a bounded number of vertices from the graph (consider e.g. the star $K_{1,n-1}$), the tree-width and the clique-width are not. Combining Proposition 1, Corollary 2 and Theorem 1, we obtain the following corollary.

Corollary 3 *If G is a $S_{k,k,k}$-free and $T_{k,k,k}$-free graph of vertex degree at most Δ, then*

(i) $\mathrm{tw}(G) \leq 2\lceil \frac{c_2}{2} \rceil + c_1$

(ii) $\mathrm{cw}(G) \leq c_2 + 2 + c_1$.

Furthermore, a tree decomposition of G of width at most $2\lceil \frac{c_2}{2} \rceil + c_1$ and a $(c_2 + 2 + c_1)$-expression of G can be found in polynomial time.

Note that given the conclusion (i) of Corollary 3, it follows from [1] or [13] that a tree decomposition of small width can be found efficiently. Nevertheless, the procedure we proposed in the proof of Proposition 1 is much simpler.

With the help of Propositions 2 and 3, a stronger result can be derived from Corollary 3 for the tree- and clique-width of graphs with bounded vertex degree. Recall that $\mathcal{S}$ denotes the class of graphs every component of which is of the form $S_{i,j,k}$, and $\mathcal{T}$ the class of graphs every component of which is of the form $T_{i,j,k}$.

Theorem 2 *Let $\mathcal{X}$ be a hereditary class of graphs with bounded vertex degree. If $\mathcal{S} \not\subseteq \mathcal{X}$ and $\mathcal{T} \not\subseteq \mathcal{X}$, then*

(i) the tree-width of graphs in $\mathcal{X}$ is bounded by a constant c_3, and a tree decomposition of width at most c_3 can be constructed in polynomial time and

(ii) the clique-width of graphs in $\mathcal{X}$ is bounded by a constant c_4, and a c_4-expression can be constructed in polynomial time.

Proof. If $\mathcal{S} \not\subseteq \mathcal{X}$ and $\mathcal{T} \not\subseteq \mathcal{X}$, we may consider two graphs $H \in \mathcal{S} \setminus \mathcal{X}$ and $J \in \mathcal{T} \setminus \mathcal{X}$ such that every component of H is of the form $S_{k,k,k}$ and every component of J is of the form $T_{k,k,k}$ for some fixed $k \geq 1$.

Denote by s and t the number of components of H and J, respectively. Also, let H_i and J_i denote, respectively, the subgraphs of H and J induced by the vertices of exactly i components. In particular, $H_1 = S_{k,k,k}$, $J_1 = T_{k,k,k}$, $H_s = H$ and $J_t = J$.

For every graph G with vertex degree at most Δ, there are at most $p = p(k, \Delta)$ vertices in the closed neighborhood of set of vertices that induces a subgraph isomorphic to either $S_{k,k,k}$ or $T_{k,k,k}$. Therefore, the following inclusion holds.

$$\begin{aligned} \mathrm{Free}_\Delta(H_s, J_t) \subseteq\ & \mathrm{Free}_\Delta(H_1, J_1) \cup [\mathrm{Free}_\Delta(H_{s-1}, J_1)]_p \\ & \cup [\mathrm{Free}_\Delta(H_1, J_{t-1})]_p \cup [\mathrm{Free}_\Delta(H_{s-1}, J_{t-1})]_{2p}. \end{aligned}$$

From this inclusion and Propositions 2 and 3 we conclude by induction on s and t that the tree-width and clique-width of graphs in the class $\mathcal{X} \subseteq \mathrm{Free}_\Delta(H, J)$ is bounded. The basis of induction is provided by Corollary 3. Notice that s and t are constants associated with the class $\mathcal{X}$. ∎

5 Examples

An *asteroidal triple* in a graph is an independent set of three vertices such that for any two vertices in the set, there is a path between these two vertices avoiding the neighborhood of the third vertex [5]. A graph is asteroidal triple-free, or AT-free for short, if it contains no asteroidal triple. Asteroidal triple-free graphs include several well-known graph classes such as comparability graphs, permutation graphs, or interval graphs. All three graph parameters studied in this paper are unbounded in the class of AT-free graphs (since they are unbounded even for permutation graphs [12]), and several fundamental graph problems remain NP-hard in that class. However, according to our main results, the situation changes crucially whenever we deal with AT-free graphs with bounded vertex degree. Formally,

Proposition 4 *The band-, tree- and clique-width of AT-free graphs of vertex degree at most Δ is bounded by a constant $c_5 = c_5(\Delta)$.*

Proof. The proposition follows immediately from Theorem 1, Proposition 1 and the observation that $S_{2,2,2}$, $T_{2,2,2}$ and induced cycles of length at least 6 contain an asteroidal triple. ■

As another important example, let us mention chordal bipartite graphs, i.e. bipartite graphs without induced cycles of length at least 6. This class has applications in the study of linear programming, as the bipartite adjacency matrices of chordal bipartite graphs are totally balanced. Note that the class of chordal bipartite graphs is not a subclass of AT-free graphs, since $S_{2,2,2}$ is not forbidden in this class.

The clique-width (and hence the band- and tree-width) of chordal bipartite graphs in unbounded, since it is unbounded even for bipartite permutation graphs [4], a proper subclass of chordal bipartite graphs. We use Propositions 2, 3 and Theorem 2 in order to show that chordal bipartite graphs with vertex degree at most three have bounded tree- and clique-width.

Proposition 5 *The tree- and clique-width of chordal bipartite graphs with vertex degree at most three is bounded by a constant.*

Proof. We shall prove the proposition by showing that every chordal bipartite graph G with vertex degree at most 3 containing $S_{2,2,2}$ as an induced subgraph has a cutvertex. We denote by x the center of $S_{2,2,2}$, by y_1, y_2 and y_3 the vertices of degree 2, and by z_1, z_2 and z_3 the respective vertices of degree 1 in $S_{2,2,2}$.

Assume y_1 is not a cutvertex in G, and consider a shortest path $P : p_0p_1 \dots p_k$ connecting the vertex $z_1 = p_0$ to a vertex $p_k \in \{y_2, y_3, z_2, z_3\}$ in the subgraph $G - \{y_1\}$. If $p_k = z_2$, then k is even. Notice that the even-indexed vertices of P different from p_k are not adjacent to y_2, since otherwise P is not a shortest path. Let j be the largest index with $y_1p_j \in E(G)$ (possibly $j = 0$). But now $y_1p_j \dots p_ky_2xy_1$ is an induced cycle of length at least 6 in G, a contradiction. Analogously, $p_k \neq z_3$.

If $p_k = y_2$, then $k = 3$. Indeed, because of the degree constraint the cycle $C : p_1 \dots p_kxy_1p_1$ contains at most one chord, and this chord is of the form y_1p_j with an even j. Therefore, if $k > 3$, then the vertices of C induce a chordless cycle of length at least 6. Thus $k = 3$, and to avoid an induced cycle C_6, $y_1p_2 \in E(G)$ and $p_1z_2 \notin E(G)$. But now y_3 is a cutvertex in G. Indeed, if not, there must exist a path in $G - \{y_3\}$ connecting z_3 to a vertex in the set $\{z_1, p_1, z_2\}$ missing the vertices of the set $B = \{x, y_1, y_2, p_2\}$ (since all vertices in B have degree three in G), but then the vertices of the path together with some vertices in B and y_3 would induce a forbidden cycle.

Thus, the chordal bipartite graphs with vertex degree at most 3 form a subclass of the class $[\mathrm{Free}_3(S_{2,2,2}, T_{0,0,0})]_B$. In the latter class the tree- and clique-width are bounded according to Propositions 2, 3 and Theorem 2. ■

References

[1] H.L. Bodlaender, A linear-time algorithm for finding tree-decompositions of small treewidth, *SIAM J. Comput.* 25 (1996), 1305–1317.

[2] H.L. Bodlaender, Treewidth: algorithmic techniques and results, in *Lecture Notes in Comput. Sci.* 1295, pages 19–36, Springer, Berlin, 1997.

[3] R. Boliac and V. Lozin, Independent domination in finitely defined classes of graphs, *Theoretical Computer Science*, accepted for publication, (preprint is available online at http://rutcor.rutgers.edu/pub/rrr/reports2002/6_2002.ps).

[4] A. Brandstädt and V.V. Lozin, On the linear structure and clique-width of bipartite permutation graphs, *Ars Combinatoria*, to appear (preprint is available online at http://rutcor.rutgers.edu/pub/rrr/ reports2001/29_2001.ps).

[5] D.G Corneil, S. Olariu and L. Stewart, Asteroidal triple-free graphs, *SIAM J. Discrete Math.* 10 (1997), 399–430.

[6] B. Courcelle, The monadic second-order logic of graphs. III. Tree-decompositions, minors and complexity issues, *RAIRO Inform. Théor. Appl.* 26 (1992), 257–286.

[7] B. Courcelle, J. Engelfriet and G. Rozenberg, Handle-rewriting hypergraph grammars, *J. Comput. System Sci.* 46 (1993), 218–270.

[8] B. Courcelle and S. Olariu, Upper bounds to the clique-width of a graph, *Discrete Applied Math.* 101 (2000), 77–114.

[9] B. Courcelle, J.A. Makowsky and U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, *Theory Comput. Systems* 33 (2000), 125–150.

[10] B. Courcelle, J.A. Makowsky and U. Rotics, On the fixed parameter complexity of graph enumeration problems definable in monadic second-order logic, *Discrete Appl. Math.* 108 (2001), 23–52.

[11] J. Díaz, J. Petit and M.J. Serna, A survey of graph layout problems, *ACM Computing Surveys* 34 (2002), 313–356.

[12] M.C. Golumbic and U. Rotics, On the clique-width of some perfect graph classes, *International J. Foundations of Computer Science* 11 (2000), 423–443.

[13] L. Perković and B. Reed, An improved algorithm for finding tree decompositions of small width, *Internat. J. Found. Comput. Sci.* 11 (2000), 365–371.

[14] B. Reed, Tree width and tangles: a new connectivity measure and some applications, in *London Math. Soc. Lecture Note Ser.* 241, pages 87–162, Cambridge Univ. Press, Cambridge, 1997.

[15] N. Robertson and P.D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, *J. Algorithms* 7 (1986), 309–322.