DIMACS Technical Report 2002-47
November 2002

Ratio of generalized parameters

by

I. E. Zverovich [1]
RUTCOR–Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Rd.
Piscataway, NJ 08854-8003, U.S.A.
igor@rutcor.rutgers.edu

I. I. Zverovich
ASA Institute of Technology,
Brooklyn, New York, NY 11201, U.S.A.

[1]Supported by DIMACS Summer Research Program 2002.

ABSTRACT

Let $\mathcal{P}$ be a class of graphs. A *$\mathcal{P}$-set* in a graph G is a vertex subset X such that $G(X) \in \mathcal{P}$. We define the *$\mathcal{P}$-stability number* of a graph G, $\alpha_{\mathcal{P}}(G)$, as the maximum cardinality of a $\mathcal{P}$-set in G.
A *$\mathcal{P}$-dominating set* in a graph is a dominating $\mathcal{P}$-set. Accordingly, the *$\mathcal{P}$-domination number* of a graph G, $\gamma_{\mathcal{P}}(G)$, is the minimum cardinality of a $\mathcal{P}$-dominating set in G.
For each $r \geq 1$, we define a graph G to be an *$\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph* if $\alpha_{\mathcal{P}}(H)/\gamma_{\mathcal{Q}}(H) \leq r$ for each induced subgraph H of G.
We characterize all classes of $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graphs in terms of forbidden induced subgraphs for all hereditary classes $\mathcal{P}$ and $\mathcal{Q}$ containing all edgeless graphs, and for all real numbers $r \geq 1$. We propose a number of related open problems and conjectures.

Keywords: $\mathcal{P}$-stability number, $\mathcal{P}$-domination number, hereditary class, forbidden induced subgraphs.

Let $\mathcal{P}$ be a class of graphs. A $\mathcal{P}$*-set* in a graph G is a vertex subset X such that $G(X) \in \mathcal{P}$. We define the $\mathcal{P}$*-stability number* of a graph G, $\alpha_{\mathcal{P}}(G)$, as the maximum cardinality of a $\mathcal{P}$-set in G.

A $\mathcal{P}$*-dominating set* in a graph is a dominating $\mathcal{P}$-set. Accordingly, the $\mathcal{P}$*-domination number* of a graph G, $\gamma_{\mathcal{P}}(G)$, is the minimum cardinality of a $\mathcal{P}$-dominating set in G.

For a hereditary class $\mathcal{P}$, $\gamma_{\mathcal{P}}(G)$ exists for every graph if and only if the following condition holds

O-Condition $\mathcal{P}$ contains all edgeless graphs.

If $\mathcal{P}$ and $\mathcal{Q}$ are hereditary classes satisfying O-Condition, then for every maximal stable set S in a graph G we have $\alpha_{\mathcal{P}}(G) \geq |S| \geq \gamma_{\mathcal{Q}}(G)$, since S is simultaneously a $\mathcal{P}$-set and a dominating $\mathcal{Q}$-set. Therefore $\alpha_{\mathcal{P}}(G)/\gamma_{\mathcal{Q}}(G) \geq 1$ for every graph G.

Definition 1. *For each $r \geq 1$, we define a graph G to be an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$*-perfect graph *if $\alpha_{\mathcal{P}}(H)/\gamma_{\mathcal{Q}}(H) \leq r$ for every induced subgraph H of G. We denote by $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$ the class of all $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graphs.*

The classes $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$ are interesting because we can approximate both $\alpha_{\mathcal{P}}(G)$ and $\gamma_{\mathcal{Q}}(G)$ within them for each fixed r regardless of particular structure of the classes $\mathcal{P}$ and $\mathcal{Q}$. Indeed, let S be an arbitrary maximal stable set of an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph G.

Then $\alpha_{\mathcal{P}}(G)/|S| \leq \alpha_{\mathcal{P}}(G)/\gamma_{\mathcal{Q}}(G) \leq r$ and therefore $|S| \geq \alpha_{\mathcal{P}}(G)/r$. Since S is also a dominating $\mathcal{Q}$-set of G, $|S|/\gamma_{\mathcal{Q}}(G) \leq \alpha_{\mathcal{P}}(G)/\gamma_{\mathcal{Q}}(G) \leq r$ and $|S| \leq r\gamma_{\mathcal{Q}}(G)$.

We characterize the classes $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$ in terms of forbidden induced subgraphs. A vertex u is *dominating* in a graph G if u is adjacent to all other vertices of G. For an integer $p \geq 1$ and a hereditary class $\mathcal{P}$, we define two sets of graphs:

- $DI(\mathcal{P}, p)$ consists of all graphs in $\mathcal{P}$ of order p that have a dominating vertex, and
- $DO(\mathcal{P}, p)$ consists of all graphs G such that
  - $|V(G)| = p$,
  - G has a dominating vertex u,
  - $G - u \in \mathcal{P}$, and
  -
  - $G - v \notin \mathcal{P}$ for each vertex $v \neq u$.

The conditions $G - u \in \mathcal{P}$, and $G - v \notin \mathcal{P}$ imply that u is a unique dominating vertex in $G \in DO(\mathcal{P}, p)$.

Theorem 1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary classes satisfying* O-Condition, *and let $r \in [p, p+1)$ for an integer $p \geq 1$.*

(i) *A graph G is an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph if and only if it G does not contain all graphs in $DI(\mathcal{P}, p+1) \cup DO(\mathcal{P}, p+2)$ as induced subgraphs.*

(ii) *$DI(\mathcal{P}, p+1) \cup DO(\mathcal{P}, p+2)$ is the set of all minimal forbidden induced subgraphs for the class $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$.*

Proof. (i) Necessity. Let H be an induced subgraph of G. If $H \in DI(\mathcal{P}, p+1)$, then $\alpha_{\mathcal{P}}(H) = |V(H)| = p+1$ [since $H \in \mathcal{P}$] and $\gamma_{\mathcal{Q}}(H) = 1$ [since H has a dominating vertex]. Therefore $\alpha_{\mathcal{P}}(H)/\gamma_{\mathcal{Q}}(H) = p+1 > r$, i.e., H is not an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph.

If $H \in DO(\mathcal{P}, p+2)$, then $\alpha_{\mathcal{P}}(H) = |V(H)| + 1 = p+1$ [since $H - u \in \mathcal{P}$ for a dominating vertex u in H] and $\gamma_{\mathcal{Q}}(H) = 1$ [since H has a dominating vertex]. Again, $\alpha_{\mathcal{P}}(H)/\gamma_{\mathcal{Q}}(H) = p+1 > r$, i.e., H is not an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph.

Sufficiency. Let G be a minimal forbidden induced subgraph for the class $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$. Suppose that $G \notin DI(\mathcal{P}, p+1) \cup DO(\mathcal{P}, p+2)$. Let S be a maximum $\mathcal{P}$-set in G, and let D be a minimum dominating $\mathcal{Q}$-set in G.

Claim 1. *Each vertex in $S \cap D$ is adjacent to at most $p-1$ vertices of S.*

Proof. Suppose that it does not hold, i.e, there is a vertex $u \in S \cap D$ such that the set $X = N(u) \cap S$ contains at least p vertices. The subgraph H induced by $\{u\} \cup X \subseteq \mathcal{P}$ belongs to $DI(\mathcal{P}, p+1)$, since S is a $\mathcal{P}$-set. It follows that $H \notin \alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$. By minimality of G, $G = H$, a contradiction. □

Claim 2. *Each vertex in $D \setminus S$ is adjacent to at most p vertices of S.*

Proof. Suppose that it does not hold, i.e, there is a vertex $u \in D \setminus S$ such that the set $X = N(u) \cap S$ contains at least $p+1$ vertices. We denote by H the subgraph induced by $\{u\} \cup X$.

If $G - v \in \mathcal{P}$ for a vertex $v \in X$, then we can delete v and obtain an induced subgraph H' belonging to $\mathcal{P}$. Clearly, $H' \in DI(\mathcal{P}, p+1)$. By minimality of G, $G = H'$, a contradiction. Thus, $G - v \notin \mathcal{P}$, and therefore $H \in DO(\mathcal{P}, p+2)$. It follows that $G = H$, a contradiction. □

Let S_1 be the set of all vertices in S that are dominated by $S \cap D$. In particular, $S \cap D \subseteq S_1$. According to Claim 1,

$$|S_1| \leq p|S \cap D|. \tag{1}$$

Let S_2 be the set of all vertices in $S \setminus D$ that are dominated by $D \setminus S$. By to Claim 2,

$$|S_2| \leq p|D \setminus S|. \tag{2}$$

Each vertex of S is dominated by a vertex of D. Then (1) and (2) imply

$$|S| \leq |S_1| + |S_2| \leq p|S \cap D| + p|D \setminus S| = p|D|.$$

In other words, $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}} = |S|/|D| \leq p$, i.e., G is an $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph, a contradiction.

(ii) Let $F \in DI(\mathcal{P}, p+1)$. To show minimality of F as a forbidden induced subgraphs for the class $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$, it is sufficient to note that $\alpha_{\mathcal{P}}(F')/\gamma_{\mathcal{Q}}(F') \leq |V(F')| \leq |V(F)| - 1 = p \leq r$ for each proper induced subgraph F' of F.

Now let $F \in DO(\mathcal{P}, p+2)$, and let u be a unique dominating vertex of F. We consider an arbitrary proper induced subgraph F' of F. If $u \notin V(F')$ then either

- $\alpha_{\mathcal{P}}(F') \leq p+1$ and $\gamma_{\mathcal{Q}}(F') \geq 2$ [since u is a unique dominating vertex of F], or
- $\alpha_{\mathcal{P}}(F') \leq p$ and $\gamma_{\mathcal{Q}}(F') \geq 1$.

If $u \in V(F')$ then $\alpha_{\mathcal{P}}(F') \leq p$ and, since $G - v \notin \mathcal{P}$ for each vertex $v \neq u$ of F. In any case, $\alpha_{\mathcal{P}}(F')/\gamma_{\mathcal{Q}}(F') \leq p \leq r$.

Thus, all proper induced subgraphs of F are $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graphs. □

Corollary 1. *All classes of $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graph have finite forbidden induced subgraph characterizations independently of $\mathcal{P}$, $\mathcal{Q}$, and $r \geq 1$.*

Proof. The statement follows from Theorem 1(ii), since the sets $DI(\mathcal{P}, p+1) \cup DO(\mathcal{P}, p+2)$ are always finite. □

Thus, recognizing all classes $\alpha_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$ can be done in polynomial time [for a fixed r]. If $\mathcal{P}$ is the class $\mathcal{O}$ of all edgeless graphs, and $\mathcal{Q}$ is the class $\mathcal{GRAPH}$ of all graphs, then $\alpha_{\mathcal{P}}(G)$ is the usual stability number $\alpha(G)$, and $\gamma_{\mathcal{Q}}(G)$ is the usual domination number $\gamma(G)$. In this case we obtain the class of *$\alpha/\gamma(r)$-perfect graphs.*

Corollary 2 (Naidenkoand Orlovich [3]). *Let $r \in [p, p+1)$ for an integer $p \geq 1$. A graph G is an $\alpha/\gamma(r)$-perfect graph if and only if it does not contain the star $K_{1,p+1}$ as an induced subgraph.*

Proof. Indeed, $DI(\mathcal{O}, p+1) = \emptyset$, since there are no edgeless graphs of order $p+1 \geq 2$ with a dominating vertex. Also, $DO(\mathcal{O}, p+2) = \{K_{p+1}\}$. Now the result follows from Theorem 1. □

In connection with Corollary 1 we mention a related result of Berman [1], and Halldórsson [2]. For $p \geq 2$, there is a polynomial-time algorithm to approximate the stability number for $K_{1,p}$-free graphs within a factor of $\frac{p}{2}$, i.e. to determine a stable set S of a given $K_{1,p}$-free graph G such that $|S| \geq \frac{2}{p}\alpha(G)$.

Since the result of Theorem 1 does not depend on $\mathcal{Q}$, Corollary 1 is valid in more general form.

Corollary 3. *Let $\mathcal{Q}$ be a hereditary classes satisfying* O-Condition*, and let $r \in [p, p+1)$ for an integer $p \geq 1$. A graph G is an $\alpha/\gamma_{\mathcal{Q}}(r)$-perfect graph if and only if it does not contain the star $K_{1,p+1}$ as an induced subgraph.*

In case of $\mathcal{Q} = \mathcal{O}$, we have $\gamma_{\mathcal{Q}}(G) = \iota(G)$, the independent domination number of G. In other words, we obtain the class of *$\alpha/\iota(r)$-perfect graphs*, where $r \geq 1$.

Corollary 4. *Let $r \in [p, p+1)$ for some $p \geq 1$. A graph is an $\alpha/\iota(r)$-perfect graph if and only if it does not contain the star $K_{1,p+1}$ as an induced subgraph.*

If $\mathcal{P} \subseteq \mathcal{Q}$, then $\gamma_{\mathcal{P}}(G) \geq \gamma_{\mathcal{Q}}(G)$, or $\gamma_{\mathcal{P}}(G)/\gamma_{\mathcal{Q}}(G) \geq 1$ for every graph G. We introduce the classes of $\gamma_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graphs consisting of all graphs G such that $\gamma_{\mathcal{P}}(H)/\gamma_{\mathcal{Q}}(H) \leq r$ for each induced subgraph H of G.

Conjecture 1. *Let $\mathcal{P} \subseteq \mathcal{Q}$ be hereditary classes, both satisfying* O-Condition*. For every $r \geq 1$, the class of all $\gamma_{\mathcal{P}}/\gamma_{\mathcal{Q}}(r)$-perfect graphs has a finite forbidden induced subgraph characterization.*

Open Problem 1. *Characterize $\alpha_{\mathcal{P}}/\alpha_{\mathcal{Q}}(r)$-perfect graphs in terms of forbidden induced subgraphs.*

We specify an interesting particular case, where $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = \mathcal{GRAPH}$, that is the class of $\iota/\gamma(r)$-perfect graphs, $r \geq 1$.

Conjecture 2. *For every $r \geq 1$, the class of all $\iota/\gamma(r)$-perfect graphs has a finite forbidden induced subgraph characterization.*

Open Problem 2. *Characterize $\iota/\gamma(r)$-perfect graphs in terms of forbidden induced subgraphs.*

The case $r = 1$ of Open Problem 2 corresponds to the class of domination perfect graphs, i.e., graphs G such that $\gamma(H) = \iota(H)$ for every induced subgraph H of G. A result of Zverovich and Zverovich [4] implies that $\iota/\gamma(1)$-perfect graphs are characterized by 17 minimal forbidden induced subgraphs. Note that a related problem concerning ratio of the irredundance number and the domination number was investigated by Zverovich [5].

If $\mathcal{P} \supseteq \mathcal{Q}$, then $\alpha_{\mathcal{P}}(G) \geq \alpha_{\mathcal{Q}}(G)$, or $\alpha_{\mathcal{P}}(G)/\alpha_{\mathcal{Q}}(G) \geq 1$ for every graph G. Therefore it is natural to consider the classes of $\alpha_{\mathcal{P}}/\alpha_{\mathcal{Q}}(r)$-perfect graphs [for $r \geq 1$] defined in an obvious way.

Open Problem 3. *Let $\mathcal{P} \supseteq \mathcal{Q}$ be hereditary classes, both satisfying* O-Condition*. Does each class of $\alpha_{\mathcal{P}}/\alpha_{\mathcal{Q}}(r)$-perfect graphs, $r \geq 1$, have a finite forbidden induced subgraph characterization?*

Open Problem 4. *Characterize the classes of $\alpha_{\mathcal{P}}/\alpha_{\mathcal{Q}}(r)$-perfect graphs in terms of forbidden induced subgraphs.*

References

[1] P. Berman, A $d/2$ approximation for maximum weight independent set in d-claw free graphs, Lecture Notes Comput. Sci. **1851** (2000) 214–219

[2] M. M. Halldórsson, Approximating discrete collections via local improvements, in: *Proceedings of the 6th annual ACM-SIAM symposium on discrete algorithms* (1995) 160–169

[3] V. G. Naidenko and Yu. L. Orlovich, Independent and dominating sets in $K_{1,r}$-free graphs, Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz.-Mat. Navuk (2) (2000) 126–130, 144 (in Russian)

[4] I. E. Zverovich and V. E. Zverovich, An induced subgraph characterization of domination perfect graphs, J. Graph Theory **20** (3) (1995) 375–395

[5] V. E. Zverovich, The ratio of the irredundance number and the domination number for block-cactus graphs, J. Graph Theory **29** (3) (1998) 139–149