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Unique domination and domination perfect graphs

by

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ABSTRACT

We review a characterization of domination perfect graphs in terms of forbidden induced subgraphs obtained by Zverovich and Zverovich [12] using a computer code. Then we apply it to a problem of unique domination in graphs recently proposed by Fischermann and Volkmann.

1 Domination perfect graphs

Let G be a graph. A set $D \subseteq V(G)$ is a *dominating set* of G if each vertex of G either belongs to D or it is adjacent to a vertex of D. A set $S \subseteq V(G)$ is *stable* [or *independent*] if it induces an edgeless subgraph $G(S)$. An *independent dominating set* is vertex subset that is both stable and dominating, or equivalently, is maximal stable. The *domination number* $\gamma(G)$ is the minimum cardinality taken over all dominating sets of G, and the *independent domination number* $\iota(G)$ is the minimum cardinality taken over all maximal stable sets of vertices of G.

Let $X \subseteq V(G)$. For a vertex $x \in X$, the set

$$PN(x, X) = N[x] \backslash N[X \backslash \{x\}]$$

is called the *private neighborhood* of x [with respect to X]. A dominating set of a graph G is *minimal* if it does not contain another dominating set of G. It is well-known and easy to see that a dominating set D is minimal if and only if $PN(x, D) \neq \emptyset$ for each $x \in D$ [Berge's lemma].

Definition 1 (Sumner and Moore [8]). *A graph G is called* domination perfect *or* γ-perfect *if $\gamma(H) = \iota(H)$ for every induced subgraph H of G. A graph G is called* minimal domination imperfect *if G is not domination perfect and $\gamma(H) = \iota(H)$ for every proper induced subgraph H of G.*

We shortly describe a history of the forbidden induced subgraph characterization of domination perfect graphs [12]. First Allan and Laskar noted that $K_{1,3}$-free graphs are domination perfect. A more particular results that line graphs and middle graphs are domination perfect was obtained by Gupta [see Theorem 10.5 in [5]].

Theorem 1 (Sumner and Moore [8]). *A graph G is domination perfect if and only if $\gamma(H) = \iota(H)$ for every induced subgraph H of G with $\gamma(H) = 2$.*

Let $\mathcal{A} = \{H : |V(H)| \leq 8, \gamma(H) = 2$, and $\iota(H) > 2\}$. Sumner and Moore [8] proved a series of results on domination perfect graphs:

- A chordal graph is domination perfect if and only if it is G_1-free (Figure 3).
- If G is an $\mathcal{A}$-free graph and also G does not contain an induced copy of the graph S in Figure 1= , then G is domination perfect.
- A planar graph is domination perfect if and only if it is $\mathcal{A}$-free.

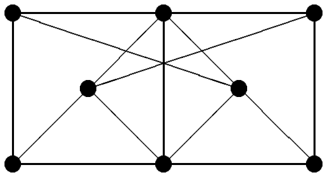

Figure 1. Graph S.

Bollobás and Cockayne [2] extended the result of Allan and Laskar: if G has no induced subgraph isomorphic to $K_{1,k}$ with $k \geq 3$, then

$$\iota(G) \leq \gamma(G)(k-2) - (k-3). \tag{1}$$

Zverovich and Zverovich [10] generalized this result by showing that (1) holds for a wider class of graphs. A graph G if *k-biclaw-free* if G does not contain two induced stars $K_{1,k}$, $k \geq 3$, having different centers and exactly one edge in common

Theorem 2 (Zverovich and Zverovich [10]). *The inequality (1) holds for k-biclaw-free graphs.*

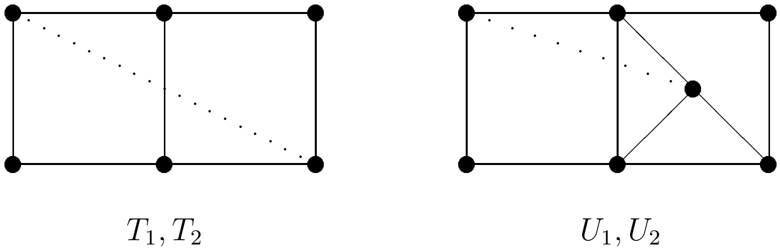

Figure 2. Graphs T_1, T_2 and U_1, U_2 [dotted lines denote potential edges].

Theorem 2 can be applied to domination perfect graphs if $k = 3$, since (1) becomes an equality: if G is a $(G_1, G_2, G_3, G_4, T_1, T_2)$-free graph (see Figure 3 and Figure 2), then G is domination perfect.

A result of Zverovich and Zverovich [10] is essentially that a triangle-free graph G is domination perfect if and only if G is (G_1, G_2, G_3, G_4)-free (see Figure 3). Then Fulman [4] found the next two minimal domination imperfect graphs G_5 and G_6, and finally Topp and Volkmann [9] extracted the minimal domination imperfect graphs $G_1, G_2, \ldots, G_{13}$ from $\mathcal{A}$.

The following theorem gives a characterization of domination perfect graphs and shows that there are exactly seventeen minimal domination imperfect graphs, contrary to a conjecture of Sumner [7] that there are infinitely many minimal forbidden induced subgraphs for the class of all domination perfect graphs.

Theorem 3 (Zverovich and Zverovich [12]). *A graph G is domination perfect if and only if G does not contain any of the graphs $G_1, G_2, \ldots, G_{17}$ in Figure 3 as an induced subgraph.*

Necessity is trivial, since $\gamma(G_i) = 2$ and $\iota(G_i) = 3$ for $i = 1, 2, \ldots, 17$.

We sketch the idea of our proof, stressing the important role of our computer code. Let G be a minimal forbidden induced subgraph for the class of all domination perfect graphs. Assume that $G \notin \{G_1, G_2, \ldots, G_{17}\}$. We choose a minimum dominating set D in G such that the induced subgraph $G(D)$ contains minimal possible number of edges. Since G is not a domination perfect graph, there are adjacent vertices u and v in $G(D)$.

Let P be the private neighborhood of $\{u, v\}$ with respect to D, that is $P = \{w \in V(G) : N_G[w] \cap D \subseteq \{u, v\}\}$. We denote by H the induced subgraph $G(P \cup \{u, v\})$. By minimality of D, $\gamma(H) \geq 2$. Since $\{u, v\}$ is a dominating set in H, we have $\gamma(H) = 2$. If $\{x, y\}$ is an independent dominating set in H, then $D' = (D \backslash \{u, v\}) \cup \{x, y\}$ is a minimum dominating set in G with $|E(G(D'))| < |E(G(D))|$, a contradiction to the choice of D. Thus, $\iota(H) \geq 3$, and H is a domination imperfect graph. It follows that $G = H$ and $D = \{u, v\}$.

The inequality $\iota(H) \geq 3$ implies that there exist distinct non-adjacent vertices a, b in $N(u) \backslash N(v)$, and there exist distinct non-adjacent vertices c, d in $N(v) \backslash N(u)$. Let $m \in \{0, 1, 2, 3, 4\}$ be the number of edges between the sets $\{a, b\}$ and $\{c, d\}$. We consider all possible variants for the subgraph F induced by the set $\{u, v, a, b, c, d\}$:

- if $m = 0$ then F is isomorphic to G_1, a contradiction,
- if $m = 1$ then the set X induces G_2, a contradiction,
- if $m = 2$ and the two edges between $\{a, b\}$ and $\{c, d\}$ have a common vertex, then the set X induces G_3, a contradiction,
- if $m = 2$ and the two edges between $\{a, b\}$ and $\{c, d\}$ constitute a matching, then $\iota(F) = 2$,
- if $m = 3$ then then $\iota(F) = 2$, and
- if $m = 4$ then F is isomorphic to G_4, a contradiction.

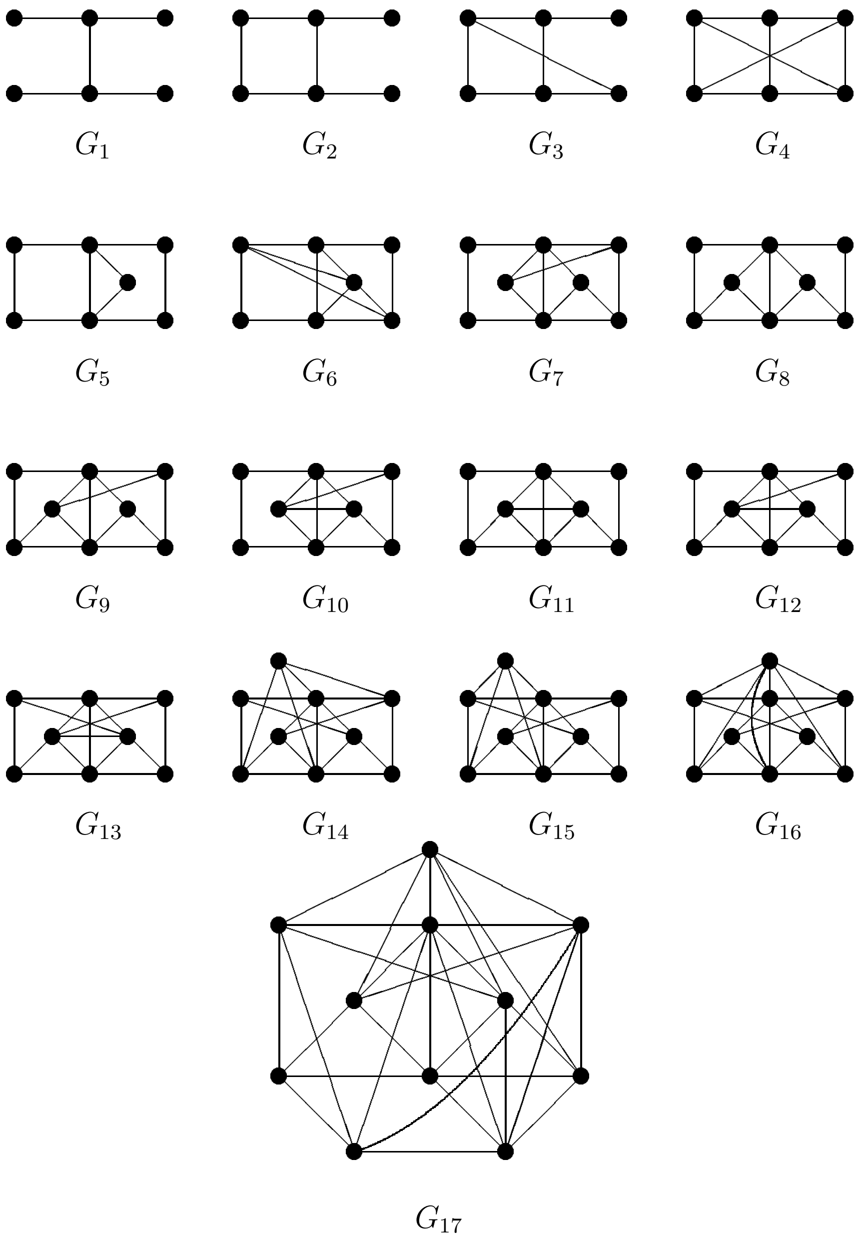

Figure 3. The minimal domination imperfect graphs.

Thus, we have two variants for F, where $\iota(F) = \gamma(F) = 2$. Let I be a minimum independent dominating set in F. Since G is a domination imperfect graph $\iota(G) > \gamma(G) = 2$. It follows that there exists a vertex $e \in V(G) \backslash V(F)$ such that e is non-adjacent to the set I, and e is adjacent to at least one of u, v [since $V(G) = V(H) = N(u) \cup N(v)$].

Considering arising variants with the new vertex e, we can find that the subgraph G' induced by $\{u, v, a, b, c, d, e\}$ is either isomorphic to G_5, G_6, or contains induced G_i for some $i \in \{1, 2, 3, 4\}$, or $\iota(G') = \gamma(G') = 2$. In the latter case we should continue the construction by adding a new vertex $f \notin V(G')$.

This process was programmed to analyze millions arising variants. Theoretically speaking, there are four possibilities for finishing our process depending on the set Min of all minimal domination imperfect graphs and on our method:

(I) Min is an infinite set,

(F1) Min is finite, but it is too large to be found by our program,

(F2) Min is finite, our program can construct it, but the proof is incomplete, and

(F3) Min is finite, our program can construct Min, and the proof is complete [no further variants].

Clearly, the cases (I), (F1) and even (F2) do not produce the desired result. The only positive result is (F3). As it appeared, our program was unable to produce the characterization, since computer resources were exhausted soon. Programming new and new lemmas extracted from available intermediate results made it possible to increase orders of graphs under consideration. In turn, this progress permitted further improvement of our computer program. Such an interactive process was terminated with the set Min and a complete proof [variant (F3)].

Finally, a pure mathematical proof was created [12]. Although the proof is tedious enough, it can be understood without computer programs in reasonable amount of time.

Our result also has interesting consequences concerning complexity, see details in [12]. Recently Rautenbach and Zverovich [6] found an interesting subclass of domination perfect graph Here we show that the finite forbidden induced subgraph characterization of domination perfect graphs is useful for unique domination in graphs.

2 An application to unique domination

A γ-set in a graph G is a minimum dominating set, and an ι-set is a minimum independent dominating set.

Definition 2 (Fischermann and Volkmann [3]). *Let $\mathcal{UNDOM}$ be a hereditary class consisting of all graphs G such that for every induced subgraph H of G, H has a unique ι-set if and only if H has a unique γ-set.*

The following result gives a solution to a problem proposed by Fischermann and Volkmann [3].

Theorem 4. *A graph G belongs to $\mathcal{UNDOM}$ if and only if G is a (B_1, B_2, B_3)-free graph (see Figure 4).*

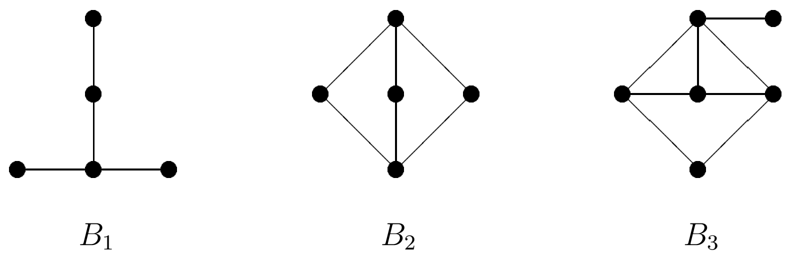

Figure 4. Graphs B_1, B_2 and B_3.

Proof. Necessity. It is easy to see that each B_i, $i = 1, 2, 3$, has a unique ι-set [consisting of two vertices], but at least two γ-sets. Since $\mathcal{UNDOM}$ is a hereditary class, G can not contain any of B_1, B_2 or B_3 as an induced subgraph.

Sufficiency. Suppose that G is a (B_1, B_2, B_3)-free graph that does not belong to $\mathcal{UNDOM}$. Moreover, let G be a minimal such a graph, i.e., $H \in \mathcal{UNDOM}$ for every proper induced subgraph H of G.

Claim 1. *G is a domination perfect graph.*

Proof. We use the characterization of domination perfect graphs [Theorem 3 and Figure 3]. Each of the graphs G_1, G_2, G_3 and G_5 contains B_1 as an induced subgraph. The graph G_4 has an induced subgraph isomorphic to B_2. Each of the graphs $G_6, G_7, \ldots, G_{17}$ contains B_3 as an induced subgraph. Since G is a (B_1, B_2, B_3)-free graph, G is also a $(G_1, G_2, \ldots, G_{17})$-free graph. By Theorem 3, G is a domination perfect graph. $\square$

Claim 2. *G has a unique ι-set, but at least two γ-sets.*

Proof. By Claim 1, $\iota(G) = \gamma(G)$. Therefore each ι-set in G is also a γ-set. It follows from $G \notin \mathcal{UNDOM}$ and minimality of G that G has a unique ι-set, but there are at least two γ-sets in G. □

Claim 2 implies that we can choose a dominating set $D \subseteq V(G)$ such that

(D1) D is a γ-set, but D is not an ι-set, and

(D2) the induced subgraph $G(D)$ has the minimum number of edges among all sets satisfying (D1).

Since D is not an independent domination set, there are adjacent vertices u and v in D. We denote by P the private neighborhood of $\{u, v\}$ with respect to D, i.e., $P = \{x \in V(G) : N[x] \cap D \subseteq \{u, v\}\}$. Let H be the subgraph induced by the set $P \cup \{u, v\}$.

Claim 3. $\gamma(H) = \iota(H) = 2$.

Proof. If H has a dominating vertex x, then the set $D' = (D \backslash \{u, v\}) \cup \{x\}$ is a dominating set in G with $|D'| < |D|$, a contradiction to minimality of D. Hence $\gamma(H) \geq 2$. But the definition of P implies that $\{u, v\}$ is a dominating set in H, therefore $\gamma(H) = 2$.

By Claim 1, G is a domination perfect graph. In particular, $\iota(H) = \gamma(H)$. □

Claim 4. $G = H$.

Proof. Suppose that H is a proper induced subgraph of G. By minimality of G, $H \in \mathcal{UNDOM}$. By Claim 3, H has at least two γ-sets, namely $\{u, v\}$ [that is not stable] and a minimum independent dominating set. It follows from the definition of $\mathcal{UNDOM}$ that there are at least two distinct ι-sets I_1 and I_2 in H.

The sets $D_i = (D \backslash \{u, v\}) \cup I_i$, $i = 1, 2$, are dominating sets in G. Since $|D| = |D_i|$, each D_i is a γ-set in G. Clearly, $G(D_i)$, $i = 1, 2$, has less edges than $G(D)$. The condition (D2) implies that each of D_1, D_2 must be an ι-set. Since $D_1 \neq D_2$, we have a contradiction to Claim 2. □

According to Claim 4, $|D| = \gamma(H) = \gamma(H) = 2$, i.e., $D = \{u, v\}$. Since D is a γ-set, there exist vertices $x \in PN(u, D)$ and $y \in PN(v, D)$. Clearly, u, v, x and y are pairwise distinct vertices.

At least one of the sets $\{u, y\}$ or $\{v, x\}$ is not an ι-set in G [by Claim 2]. By symmetry, we may assume that $\{u, y\}$ is not an ι-set. Therefore there exists a vertex $z \notin N[u] \cup N[y]$. Since $D = \{u, v\}$ is a dominating set and z is not adjacent to u, we must have the edge zv. Figure 5 shows all four possible variants for the subgraph induced by $\{u, v, x, y, z\}$. Two of them are isomorphic to B_1 and B_2 that contradicts to the choice of G. The two others variants, (a) and (b), are considered below.

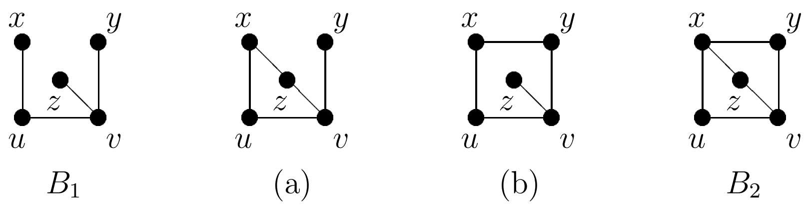

Figure 5. Variants for $G(u, v, x, y, z)$.

(a) Either $\{v, x\}$ or $\{x, y\}$ (Figure 5) is not an ι-set. Hence there exists a vertex a such that either $a \notin N[v] \cup N[x]$ or $a \notin N[x] \cup N[y]$. Since $D = \{u, v\}$ is a dominating set, a must be adjacent to at least one of the vertices u or v.

Suppose that a is non-adjacent to both v and x. Then a is adjacent to u. Now we specify the undefined edges ay and az:

- if a is non-adjacent to y, then the set $\{u, v, x, y, a\}$ induces B_1, a contradiction,
- if a is adjacent to z, then the set $\{u, v, x, z, a\}$ induces B_2, a contradiction.

Thus, a is adjacent to y, and a is non-adjacent to z, that is we obtain the graph A shown in Figure 6.

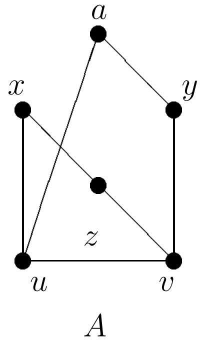

Figure 6. The graph A induced by $\{u, v, x, y, z, a\}$.

Now suppose that a is non-adjacent to both x and y. If a is adjacent to exactly one of u, v, then we delete z and obtains B_1, a contradiction. Let a be adjacent to both u and v:

- if a is non-adjacent to z, then the set $\{v, x, y, z, a\}$ induces B_1, a contradiction,
- if a is adjacent to z, then the set $\{u, v, x, y, z, a\}$ induces B_3, a contradiction.

Thus, it remains to consider the graph A in Figure 6. Either $\{a, z\}$ or $\{x, y\}$ is not an ι-set. By symmetry, we may assume that $\{x, y\}$ is not an ι-set. Hence there exists a vertex b such that $b \notin N[x] \cup N[y]$. Again, b must be adjacent to at least one of the vertices u or v, say u.

- If b is non-adjacent to v, then $G(u, v, x, y, b)$ is isomorphic to B_1, a contradiction. Therefore a is adjacent to v.
- If b is non-adjacent to z, then $G(v, x, y, z, b)$ is again isomorphic to B_1, a contradiction. Hence a is adjacent to v.

We delete the vertex a and obtain the forbidden subgraph B_3, a contradiction.

(b) Either $\{v, x\}$ or $\{x, z\}$ (Figure 5) is not an ι-set. As in (a), there exists a vertex a such that either $a \notin N[v] \cup N[x]$ or $a \notin N[x] \cup N[z]$. In the former case we obtain B_1 or B_2 or A. The latter case produces B_1 or B_2. It follows that (b) is also impossible. $\square$

Corollary 1. *Every graph in $\mathcal{UNDOM}$ is a domination perfect graph.*

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