**DIMACS Technical Report 2002-38**
**September 2002**

# Boundary classes of graphs for the dominating set problem

by

Vladimir E. Alekseev
University of Nizhny Novgorod
Gagarina 23, Nizhny Novgorod, Russia
ave@uic.nnov.ru

Dmitry V. Korobitsyn
University of Nizhny Novgorod
Gagarina 23, Nizhny Novgorod, Russia
dvk@nts.ru

Vadim V. Lozin[1]
RUTCOR, Rutgers University
640 Bartholomew Rd., Piscataway
New Jersey 08544-8003
lozin@rutcor.rutgers.edu

[1]Permanent Member

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, NEC Research Institute and Telcordia Technologies (formerly Bellcore).

DIMACS was founded as an NSF Science and Technology Center, and also receives support from the New Jersey Commission on Science and Technology.

**ABSTRACT**

The notion of a boundary class has been recently introduced as a tool for classification of hereditary classes of graphs according to the time complexity of NP-hard graph problems. In the present paper we concentrate on the dominating set problem and obtain three boundary classes for it.

# 1 Introduction

Typically, a problem, which is NP-hard in general graphs, becomes tractable (solvable in polynomial time) when restricted to some particular classes of graphs. A helpful tool for classification of graph classes according to the time complexity of a given NP-hard problem is the notion of a boundary class. Originally it has been introduced with respect to the independent set problem [2]. In the present paper we define this notion in its general form in Section 2 and apply it to the dominating set problem in Section 3. As a result, we discover three boundary classes for the problem in question.

All graph classes in this paper are *hereditary*, i.e. closed under deletion of vertices. A hereditary class is called *monotone* if it is closed under deletion of edges. If a graph $G$ does not contain any induced subgraph isomorphic to a graph in a set $\mathbf{Y}$, we say that $G$ is $\mathbf{Y}$-free. The set of all $\mathbf{Y}$-free graphs will be denoted $Free(\mathbf{Y})$. It is well known that for every hereditary class of graphs $\mathbf{X}$, there is a set $\mathbf{Y}$ such that $\mathbf{X} = Free(\mathbf{Y})$. The minimal set $\mathbf{Y}$ with this property is unique and will be denoted $Forb(\mathbf{X})$. If $Forb(\mathbf{X})$ is finite, we call $\mathbf{X}$ a *finitely defined* class.

Given a graph $G = (V, E)$ and a subset of vertices $U \subseteq V$, we denote by $G - U$ the subgraph of $G$ induced by $V - U$. The set of vertices adjacent to a vertex $v \in V$ is denoted $N(v)$ and is called the neighborhood of $v$. The degree of $v$ is $|N(v)|$. As usual, $C_n$ is the chordless cycle and $K_n$ is the complete graph on $n$ vertices. Also, $2K_2$ is the disjoint union of two copies of $K_2$, and $K_n - e$ is the graph obtained from $K_n$ by deleting a single edge. $K_{n,m}$ stands for the complete bipartite graph with parts of the size $n$ and $m$. By $H_i$ we denote the graph in Figure 1 (a).

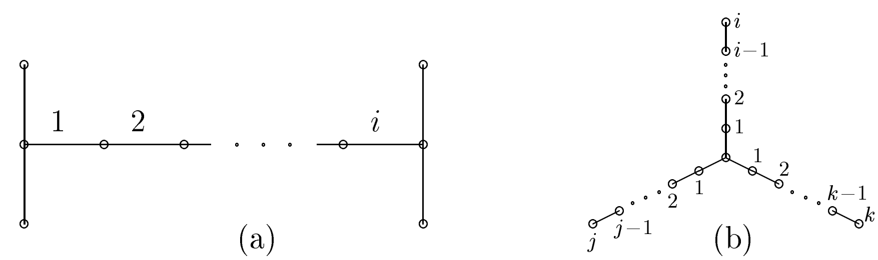


Figure 1: Graphs $H_i$ (a) and $T_{i,j,k}$ (b)

Throughout the paper we use special notations for two particular classes of graphs:

**$\mathbf{Z}_k$** is the class of $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free graphs with maximum degree three,

**$\mathbf{T}$** is the class of graphs every connected component of which is of the form $T_{i,j,k}$ with some values of $i, j, k \geq 0$ (Figure 1 (b)). Notice that if at least one of the indices $i, j, k$ equals 0, then $T_{i,j,k}$ is a path.

In a graph $G = (V, E)$, a subset of vertices $I \subseteq V$ is called *independent* if no two vertices in $I$ are linked by an edge. A *clique* is a subset of pairwise adjacent vertices. A graph is called *split* if its vertices can be partitioned into an independent set and a clique. It has been proven in [5] that the class of split graphs is exactly $Free(2K_2, C_4, C_5)$.

## 2 Fundamentals of boundary classes

Let $\Pi$ be a graph problem which is NP-hard in general graphs. A hereditary class of graphs $\mathbf{X}$ will be called $\Pi$-*hard* if the problem $\Pi$ remains NP-hard when restricted to graphs in $\mathbf{X}$. In case that $\Pi$ has a polynomial time solution for graphs in $\mathbf{X}$, we shall say that $\mathbf{X}$ is $\Pi$-*easy*.

**Definition 1** *A hereditary class of graphs* $\mathbf{X}$ *will be called a limit class for* $\Pi$ *if there exists a sequence* $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$ *of* $\Pi$*-hard classes such that* $\bigcap_{n \geq 1} \mathbf{X}_n = \mathbf{X}$.

Notice that every $\Pi$-hard class is limit for $\Pi$. The following example shows that the converse is not true in general.

Example 1. Let $\Pi$ be the independent set problem, i.e. the problem of finding in a graph an independent set of maximum cardinality, and $\mathbf{X}_k := Free(C_3, \ldots, C_k)$. It has been proven in [8] that the independent set problem is NP-hard in the class $\mathbf{X}_k$ for every fixed $k \geq 3$. It is not hard to see that $\mathbf{X}_k \supseteq \mathbf{X}_{k+1}$ for each $k$, and $\bigcap_n \mathbf{X}_n$ is the class of forests, i.e. graphs every connected component of which is a tree. Thus, the class of forests is a limit class for the independent set problem.

This example in conjunction with the well-known fact that the independent set problem has a polynomial time solution in the class of forests imply that a limit class for a problem $\Pi$ is not necessarily $\Pi$-hard. However, if we restrict ourselves to finitely defined classes, then the two notions – $\Pi$-hard and limit for $\Pi$ – become equivalent.

**Lemma 1** *A finitely defined class* $\mathbf{X}$ *is a limit class for* $\Pi$ *if and only if it is* $\Pi$*-hard.*

**Proof.** Let $Forb(\mathbf{X}) = \{G_1, \ldots, G_k\}$, and $\mathbf{X}$ be a limit class, i.e. $\mathbf{X} = \bigcap_n \mathbf{X}_n$ for a sequence $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$ of $\Pi$-hard classes. Clearly, there must exist a natural $n$ such that $\mathbf{X}_n$ does not contain $G_1, \ldots, G_k$. But then $\mathbf{X}_i = \mathbf{X}$ for each $i \geq n$, and hence $\mathbf{X}$ is $\Pi$-hard. ■

The following two lemmas establish some important properties of limit classes.

**Lemma 2** *If* $\mathbf{X}$ *is a limit class and* $\mathbf{Y} \supseteq \mathbf{X}$*, then* $\mathbf{Y}$ *also is a limit class.*

**Proof.** Let $\mathbf{X} = \bigcap_n \mathbf{X}_n$, where $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$ is a sequence of $\Pi$-hard graph classes. Then the class $\mathbf{Y}_n := \mathbf{X}_n \cup \mathbf{Y}$ is $\Pi$-hard for every $n$, $\mathbf{Y}_1 \supseteq \mathbf{Y}_2 \supseteq \ldots$, and $\mathbf{Y} = \bigcap_n \mathbf{Y}_n$. ■

**Lemma 3** *If* $\mathbf{X} = \bigcap_n \mathbf{X}_n$*, where* $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$*, and* $\mathbf{X}_n$ *is a limit class for each* $n$*, then* $\mathbf{X}$ *is a limit class.*

**Proof.** Let $Forb(\mathbf{X}) = \{G_1, G_2, \ldots\}$. For each natural $k$, define $\mathbf{X}^{(k)}$ to be the class $Free(G_1, \ldots, G_k)$. Obviously, for every $k$, there exists an $n$ such that $\mathbf{X}_n$ does not contain $G_1, \ldots G_k$, and hence $\mathbf{X}_n \subseteq \mathbf{X}^{(k)}$. By Lemma 2, $\mathbf{X}^{(k)}$ is a limit class, and by Lemma 1, it is $\Pi$-hard for each $k$. Obviously $\mathbf{X}^{(k)} \supseteq \mathbf{X}^{(k+1)}$ and $\bigcap_k \mathbf{X}^{(k)} = \mathbf{X}$. Therefore, $\mathbf{X}$ is a limit class. ∎

Example 1 (continued). An important observation concerning the class of forests is that it is not a minimal limit class for the independent set problem. In fact, the paper [8] shows that the problem is NP-hard even for graphs with maximum degree three in the class $Free(C_3, \ldots, C_k)$. Hence, the forests with vertex degree at most three constitute a limit class for the independent set problem. A stronger result has been obtained in [1]:

**Lemma 4** *For every natural $k \geq 3$, the independent set problem is NP-hard in the class $\mathbf{Z}_k$.*

It is not hard to verify that $\bigcap_k \mathbf{Z}_k$ coincides with the class $\mathbf{T}$ defined in the introduction. Therefore, $\mathbf{T}$ is a limit class for the independent set problem. Moreover, it has been recently proven in [2] that $\mathbf{T}$ is a minimal under inclusion limit class for this problem (assuming that $P \neq NP$). Minimal limit classes are of particular interest in our study. Below we introduce a special name for such classes and prove two results that explain our interest in them.

**Definition 2** *A minimal limit class for a problem $\Pi$ will be called a boundary class for $\Pi$.*

**Theorem 1** *For every $\Pi$-hard class $\mathbf{X}$, there is a boundary class $\mathbf{Y}$ for $\Pi$ such that $\mathbf{Y} \subseteq \mathbf{X}$.*

**Proof.** Consider a bijection between the class of all graphs and the natural numbers. This bijection defines a linear order, which will be called *standard*. Let us define a sequence of graph classes $\mathbf{X} = \mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \ldots$ as follows. Suppose that the class $\mathbf{X}_n$ has been defined. Then we find the first graph $G \in \mathbf{X}_n$ with respect to the standard order such that $\mathbf{X}_n \cap Free(G)$ is a limit class. If there is no such $G$, then set $\mathbf{X}_{n+1} := \mathbf{X}_n$, otherwise set $\mathbf{X}_{n+1} := \mathbf{X}_n \cap Free(G)$.

Now consider the class $\mathbf{Y} := \bigcap_n \mathbf{X}_n$. Clearly $\mathbf{Y} \subseteq \mathbf{X}$. By Lemma 3, $\mathbf{Y}$ is a limit class. In order to prove the minimality of $\mathbf{Y}$, assume there is a limit class $\mathbf{Z}$ such that $\mathbf{Z} \subset \mathbf{Y}$. Let $H \in \mathbf{Y} - \mathbf{Z}$. Then $\mathbf{Z} \subseteq \mathbf{Y} \cap Free(H) \subseteq \mathbf{X}_k \cap Free(H)$ for each $k$. Therefore, by Lemma 2, $\mathbf{X}_k \cap Free(H)$ is a limit class for each $k$. For some $k$, the graph $H$ becomes the first (under the standard order) graph with this property. But then $\mathbf{X}_{k+1} := \mathbf{X}_k \cap Free(H)$, and $H$ belongs to no class $\mathbf{X}_n$ with $n > k$, which contradicts $H \in \mathbf{Y}$. ∎

**Theorem 2** *A finitely defined class is $\Pi$-hard if and only if it includes some boundary class for $\Pi$.*

**Proof.** If $\mathbf{X}$ is a finitely defined class that includes a boundary class, then by Lemma 2, $\mathbf{X}$ is a limit class, and by Lemma 1, it is $\Pi$-hard. The converse statement follows from Theorem 1. ∎

# 3 Boundary classes for the dominating set problem

In a graph $G = (V, E)$, a subset of vertices $U \subseteq V$ is called *dominating* if every vertex of $G$ outside $U$ has a neighbor in $U$. The dominating set problem is that of finding in a graph a dominating set of minimum cardinality. It is a well known NP-hard graph problem [6]. In the present section we describe three boundary classes for this problem. Our arguments are based on the following result obtained in [3].

**Theorem 3** *If $\mathbf{X}$ is a monotone graph class and $\mathbf{T} \not\subseteq \mathbf{X}$, then the dominating set problem and the independent set problem can be solved for graphs in $\mathbf{X}$ in polynomial time.*

**Theorem 4** *If $P \neq NP$, then $\mathbf{T}$ is a boundary class for the dominating set problem.*

**Proof.** It has been proven in [7] that the dominating set problem is NP-hard in the class $\mathbf{Z}_k$ for any fixed $k \geq 3$. Since $\mathbf{T} = \bigcap_{k \geq 3} \mathbf{Z}_k$, we conclude that $\mathbf{T}$ is a limit class for the problem. Now let us prove its minimality.

Assume by contradiction that a limit class $\mathbf{X}$ is a proper subclass of $\mathbf{T}$, and let $G$ be a graph in $\mathbf{T} - \mathbf{X}$. Then $\mathbf{X} \subseteq \mathbf{T} \cap Free(G)$. If $G$ contains at least one connected component without vertices of degree three, we extend $G$ to an arbitrary graph $H \in \mathbf{T}$ in which every connected component has a vertex of degree three. Otherwise, set $H := G$. Since $G$ is an induced subgraph of $H$, we have $Free(G) \subseteq Free(H)$. Now let $\mathbf{M}$ denote the set of graphs containing $H$ as a spanning subgraph. In other words, every graph in $\mathbf{M}$, other than $H$, is obtained from $H$ by adding some edges. Obviously $H$ is the only graph in $\mathbf{T}$ belonging to $\mathbf{M}$, because addition of an edge to $H$ results in appearing either a cycle or a graph $H_i$ (Figure 1 (a)). We thus obtain the following inclusions:

$$\mathbf{X} \subseteq \mathbf{T} \cap Free(G) \subseteq \mathbf{T} \cap Free(H) \subseteq Free(\mathbf{M}).$$

Clearly $\mathbf{M}$ is a finite set, and therefore the dominating set problem is NP-hard in the class $Free(\mathbf{M})$ by Lemmas 2 and 1. On the other hand, $Free(\mathbf{M})$ is a monotone graph class and $\mathbf{T} \not\subseteq Free(\mathbf{M})$, since $H \in \mathbf{T} - Free(\mathbf{M})$. Consequently, the dominating set problem is polynomially solvable in the class $Free(\mathbf{M})$ by Theorem 3. We have a contradiction with the assumption $P \neq NP$, which completes the proof. ■

Denote by $\mathbf{T}^{\Delta}$ the class of graphs every connected component of which is of the form $T^{\Delta}_{i,j,k}$ with some values of $i, j, k \geq 0$ (Figure 2 (b)). Similarly to $T_{i,j,k}$, the graph $T^{\Delta}_{i,j,k}$ may not have vertices of degree three if at least one of the indices $i, j, k$ is 0.

**Theorem 5** *If $P \neq NP$, then $\mathbf{T}^{\Delta}$ is a boundary class for the dominating set problem.*

**Proof.** Denote by $\mathbf{Y}_k$ the class of $(K_{1,3}, K_4 - e, K_4, C_4, \ldots, C_k, \Phi_1, \ldots, \Phi_k)$-free graphs with vertex degree at most 3, where $\Phi_i$ is the graph depicted in Figure 2 (a).

It has been proven in [7] that the dominating set problem is NP-hard in the class $\mathbf{Y}_k$ for any particular value of $k \geq 4$. It is not hard to verify that $\mathbf{T}^{\Delta} = \bigcap_{k \geq 4} \mathbf{Y}_k$. Hence $\mathbf{T}^{\Delta}$ is a limit class for the problem. In what follows we prove that $\mathbf{T}^{\Delta}$ is a minimal limit class.

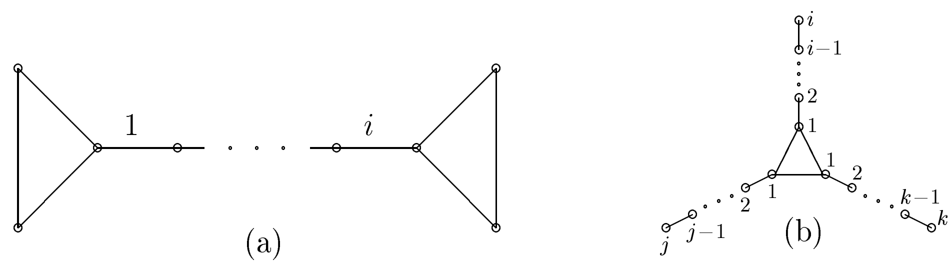


Figure 2: Graphs $\Phi_i$ (a) and $T^{\Delta}_{i,j,k}$ (b)

Assume to the contrary that a limit class $\mathbf{X}$ is properly contained in $\mathbf{T}^{\Delta}$, and let $G$ be a graph in $\mathbf{T}^{\Delta} - \mathbf{X}$. Then $\mathbf{X} \subseteq \mathbf{T}^{\Delta} \cap Free(G)$. Let $H$ be a graph in $\mathbf{T}^{\Delta}$ such that $G$ is an induced subgraph of $H$, and every connected component of $H$ is of the form $\mathbf{T}^{\Delta}_{j-1,j-1,j}$ with some value of $j > 1$ (obviously any graph $G \in \mathbf{T}^{\Delta}$ can be extended to such graph $H \in \mathbf{T}^{\Delta}$). Since $G$ is an induced subgraph of $H$, we have $Free(G) \subseteq Free(H)$. It is easy to see that one can delete an edge from each connected component of $H$ in such a way that in the resulting graph, denoted $H'$, every connected component has the form $\mathbf{T}_{j,j,j}$, i.e. $H'$ belongs to $\mathbf{T}$. Now let $\mathbf{M}$ be the set of all graphs containing $H'$ is a spanning subgraph. It is not hard to verify that $\mathbf{M} \cap \mathbf{T}^{\Delta} = \{H\}$. Hence, the following inclusions hold:

$$\mathbf{X} \subseteq \mathbf{T}^{\Delta} \cap Free(G) \subseteq \mathbf{T}^{\Delta} \cap Free(H) \subseteq Free(\mathbf{M}).$$

These inclusions in conjunction with Lemmas 2 and 1 imply NP-hardness of the dominating set problem in the class $Free(\mathbf{M})$, since $\mathbf{M}$ is a finite set. On the other hand, $Free(\mathbf{M})$ is a monotone graph class and $\mathbf{T} \not\subseteq Free(\mathbf{M})$, because $H' \in \mathbf{T} - Free(\mathbf{M})$. Therefore, the problem in question is polynomially solvable in the class $Free(\mathbf{M})$ by Theorem 3. This contradiction completes the proof. ■

To describe one more boundary class for the dominating set problem we introduce more notations and prove two auxiliary results.

For a graph $G = (V, E)$, denote by $Q(G)$ the split graph with the vertex set $V \cup E$ and the edge set

$$\{(x, y) \ : \ x, y \in V, \ x \neq y\} \cup \{(x, e) \ : \ x \in V, e \in E \text{ and } e \text{ is incident to } x \text{ in } G\}.$$

With each hereditary class of graphs $\mathbf{X}$ we associate a subclass of split graphs $\mathbf{Q}(\mathbf{X})$ defined by $\mathbf{Q}(\mathbf{X}) := \{Q(G) \ : \ G \in \mathbf{X}\}$. Notice that $\mathbf{Q}(\mathbf{X})$ is not hereditary in general. By $\mathbf{Q}^*(\mathbf{X})$ we denote the minimal hereditary class of graphs containing $\mathbf{Q}(\mathbf{X})$. In other words, $\mathbf{Q}^*(\mathbf{X})$ is obtained by adding to $\mathbf{Q}(\mathbf{X})$ all induced subgraphs of the graphs in $\mathbf{Q}(\mathbf{X})$. Obviously, $\mathbf{Q}^*(\mathbf{X})$ is again a subclass of split graphs.

Let $\mathbf{\Gamma}$ denote the class of all graphs. In the sequel, we shall use the following characterization of graphs in $\mathbf{Q}^*(\mathbf{\Gamma})$: a graph $G$ belongs to $\mathbf{Q}^*(\mathbf{\Gamma})$ if and only if the vertices of $G$ can be partitioned into a clique $C$ and an independent set $I$ in such a way that

(1) every vertex in $I$ has degree at most 2;

(2) any two different vertices of degree 2 in $I$ have distinct neighborhoods.

The necessity of this characterization follows directly from the definition. The sufficiency can be seen from the following observation: every graph $G$ that meets the characterization can be easily extended to a split graph $H$ satisfying (1) and (2) with the additional requirement that every vertex in $I$ has exactly two neighbors. Such graph $H$ can be obtained in different ways. For instance, one can add to $C$ a private neighbor for each vertex of degree one in $I$, and a couple of private neighbors for each isolated vertex in $I$. Clearly $H \in \mathbf{Q}(\mathbf{\Gamma})$ and hence $G \in \mathbf{Q}^*(\mathbf{\Gamma})$.

From the above characterization we can easily derive the description of $\mathbf{Q}^*(\mathbf{\Gamma})$ in terms of minimal forbidden induced subgraphs.

**Lemma 5** *$\mathbf{Q}^*(\mathbf{\Gamma}) = Free(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$, where $K_{2,3} + e$ denotes the graph obtained from a $K_{2,3}$ by linking its vertices of degree 3 by an edge.*

**Proof.** Let $G$ be a graph in $\mathbf{Q}^*(\mathbf{\Gamma})$. Then clearly $G$ is $(2K_2, C_4, C_5)$-free, since it is a split graph [5]. Furthermore, $G \in Free(K_5 - e, K_{2,3} + e)$ because neither $K_5 - e$ nor $K_{2,3} + e$ meet the above characterization. Thus, $\mathbf{Q}^*(\mathbf{\Gamma}) \subseteq Free(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$.

Conversely, let $G \in Free(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$. From $(2K_2, C_4, C_5)$-freeness it follows that the vertices of $G$ can be partitioned into a clique $C$ and an independent set $I$. Among all such partitions let us consider any one in which $C$ is a maximal under inclusion clique. Then the condition (1) holds. Indeed, assume that a vertex $v \in I$ has three neighbors $a, b, c \in C$. From the maximality of $C$ we know that there is a vertex $d \in C$ non-adjacent to $v$. But then $a, b, c, d, v$ induce a $K_5 - e$. To prove (2), assume there are two vertices $u$ and $v$ in $I$ such that $N(u) = N(v) = \{a, b\} \subseteq C$. By the maximality of $C$ we conclude that there must be one more vertex in $C$, say $c$. But now $a, b, c, u, v$ induce a $K_{2,3} + e$. ■

**Lemma 6** *The independent set problem in a hereditary class of graphs $\mathbf{X}$ is polynomially equivalent to the dominating set problem in the class $\mathbf{Q}^*(\mathbf{X})$.*

**Proof.** The proof of the lemma is based on the following simple observation:

(a) in a connected split graph $G$ whose vertices can be partitioned into a clique $C$ and an independent set $I$, there is a minimum dominating set which is a subset of $C$.

Indeed, if a minimum dominating set $D$ in $G$ contains a vertex $u \in I$, then any neighbor $v \in C$ of $u$ is not in $D$ (else $D - \{u\}$ is a smaller dominating set), and hence $(D - \{u\}) \cup \{v\}$ is a minimum dominating set without $u$.

Now let $G = (V, E)$ be a graph in $\mathbf{X}$, and $Q(G)$ the respective split graph in $\mathbf{Q}(\mathbf{X})$. Notice that $Q(G)$ is necessarily connected, and hence it contains a minimum dominating set $D \subseteq V$. It is not hard to verify that

(b) $D$ is dominating in $Q(G)$ if and only if $V - D$ is an independent set in $G$.

Hence the independent set problem in $\mathbf{X}$ polynomially reduces to the dominating set problem in $\mathbf{Q}(\mathbf{X})$, and vice versa.

To complete the proof we have to show how the dominating set problem for graphs in $\mathbf{Q}^*(\mathbf{X})-\mathbf{Q}(\mathbf{X})$ can be reduced to the independent set problem for graphs in $\mathbf{X}$. Without loss of generality we may consider only connected graphs, because any isolated vertex belongs to every dominating set, and any split graph without isolated vertices is connected.

Let $G$ be a connected graph in $\mathbf{Q}^*(\mathbf{X})-\mathbf{Q}(\mathbf{X})$ with a partition of its vertices into a clique $C$ and an independent set $I$ satisfying (1) and (2) (according to the arguments in the proof of Lemma 5 such partition can be found in polynomial time, because the number of maximal cliques in $C_4$-free graphs is bounded by a polynomial in the size of the graph [4], and hence the algorithm in [9] can generate all of them in polynomial time). Denote by $J \subseteq I$ the set of vertices of degree 1, and by $B \subseteq C$ the set of neighbors of the vertices in $J$. By (a), there is a minimum dominating set $D$ in $G$ such that $D \subseteq C$. Moreover, $B \subseteq D$, since otherwise some vertices in $J$ are not dominated by $D$. Now let us create the graph $G' = G - J \in \mathbf{Q}(\mathbf{X})$, and the graph $H = (C, I - J)$ such that $Q(H) = G'$. Finally, we find a maximum independent set $I'$ in the graph $H - B \in \mathbf{X}$. Then, by (b), $C - I'$ is a dominating set in $G'$. Moreover, it is a minimum dominating set containing $B$, since $I' \cap B = \emptyset$. Therefore, $C - I'$ is a minimum dominating set of the graph $G$, which completes the reduction.

Such a reduction can also be derived from the result in [3] stating that the dominating set problem in a graph $G$ polynomially reduces to the same problem for blocks in $G$. Obviously, any block in a connected graph $G \in \mathbf{Q}^*(\mathbf{X})$ is either a $K_2$ or a graph in $\mathbf{Q}(\mathbf{X})$. ■

**Theorem 6** *If $P \neq NP$, then $\mathbf{Q}^*(\mathbf{T})$ is a boundary class for the dominating set problem.*

**Proof.** It is not hard to verify that $\mathbf{Q}^*(\mathbf{T}) = \bigcap\limits_k \mathbf{Q}^*(\mathbf{Z}_k)$. Thus, from Lemmas 4 and 6 it follows that $\mathbf{Q}^*(\mathbf{T})$ is a limit class for the dominating set problem. To prove its minimality, assume there is a limit class $\mathbf{X}$ which is properly contained in $\mathbf{Q}^*(\mathbf{T})$. We consider a graph $F \in \mathbf{Q}^*(\mathbf{T}) - \mathbf{X}$, a graph $G \in \mathbf{Q}(\mathbf{T})$ such that $F$ is an induced subgraph of $G$, and a graph $H \in \mathbf{T}$ such that $G = Q(H)$. From the choice of $G$ and Lemma 5, we know that $\mathbf{X} \subseteq Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$. Therefore, by Lemmas 2 and 1 the dominating set problem is NP-hard in the class $Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$.

Denote by $\mathbf{M}$ the set of all graphs containing $H$ as a spanning subgraph. Clearly $Free(\mathbf{M})$ is a monotone class, and $\mathbf{T} \not\subseteq Free(\mathbf{M})$. Consequently, the independent set problem has a polynomial time solution for graphs in $Free(\mathbf{M})$ by Theorem 3, and hence the dominating set problem has a polynomial time solution for graphs in $\mathbf{Q}^*(Free(\mathbf{M}))$ by Lemma 6.

To provide a contradiction let us show that

$$\mathbf{Q}^*(Free(\mathbf{M})) = Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e).$$

Consider first a graph $H'$ in $\mathbf{Q}^*(Free(\mathbf{M}))$ and let $Q(H")$ denote a graph containing $H'$ as an induce subgraph and belonging to $\mathbf{Q}(Free(\mathbf{M}))$, i.e. $H" \in Free(\mathbf{M})$. By Lemma 5, $H' \in Free(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$. Moreover, $H'$ does not contain $G$ as an induced subgraph, since otherwise $H$" contains $H$ as a subgraph, which is forbidden. Thus, $\mathbf{Q}^*(Free(\mathbf{M})) \subseteq Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$.

To prove the converse inclusion, we observe that both $Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ and $\mathbf{Q}^*(Free(\mathbf{M}))$ are subsets of $\mathbf{Q}^*(\mathbf{\Gamma})$. Therefore, it suffices to show that $Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e) \cap \mathbf{Q}(\mathbf{\Gamma})$ is a subset of $\mathbf{Q}(Free(\mathbf{M}))$. Let $Q(H')$ be a graph in $Free(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e) \cap \mathbf{Q}(\mathbf{\Gamma})$. Then $H'$ does not contain $H$ as a subgraph, for otherwise $Q(H')$ would contain $G$ as an induced subgraph. Therefore, $H' \in Free(\mathbf{M})$, and hence $Q(H') \in \mathbf{Q}(Free(\mathbf{M}))$. ■

# References


[1] V.E. Alekseev, *On the local restrictions effect on the complexity of finding the graph independence number*, Combinatorial-algebraic methods in applied mathematics, Gorkiy Univ. Press, Gorky, (1983), 3-13 (in Russian).

[2] V.E. Alekseev, *On easy and hard hereditary classes of graphs with respect to the independent set problem*, Discrete Applied Mathematics, accepted.

[3] V.E. Alekseev and D.V. Korobitsyn, *On the complexity of some problems on hereditary classes of graphs*, Diskret. Mat. 4, N 4 (1992), 34-40 (in Russian).

[4] M. Farber, M. Hujter and Z. Tuza, *An upper bound on the number of cliques in a graph*, Networks, 23 (1993) 207-210.

[5] S. Foldes and P.L. Hammer, *Split graphs*, Congres. Numer. 19 (1977) 311–315.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208. Marcel Dekker, Inc., New York, 1998.

[7] D.V. Korobitsyn, *On the complexity of determining the domination number in monogenic classes of graphs*, Diskret. Mat. 2, N 3 (1990), 90-96 (in Russian, translation in Discrete Mathematics and Applications 2 (1992), no. 2, 191-199).

[8] O.J. Murphy, *Computing independent sets in graphs with large girth*, Discrete Applied Mathematics, 35 (1992) 167-170.

[9] S. Tsukiyama, M. Ide, H. Ariyoshi and I. Shirakawa, *A new algorithm for generating all the maximal independent sets*, SIAM J. Computing, 6 (1977) 505-517.