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Stable Matchings in Three-Sided Systems with Cyclic Preferences [1]

by

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ABSTRACT

We consider generalizations of the Gale-Shapley (1962) Stable Marriage Problem to three-sided families. Alkan (1988) gave an example which shows that in this case stable matchings do not always exist. Here we provide a simpler example demonstrating this fact. Danilov (2001) proved that stable matchings always exist for the special case of certain acyclic preferences and he raised the problem for another special case involving cyclic preferences. Here we show that the answer is still negative by constructing a three-sided system with lexicographically cyclic preferences for which no stable matching exists. Finally, we also consider possible generalizations to m-sided families, for $m > 3$.

Keywords: Gale-Shapley Theorem, stable matching, stable marriage, cyclic preferences.

1 Introduction

The *Stable Marriage* problem was considered and solved by Gale and Shapley (1962). Given two sets M and W (of men and women), $|M| = |W| = n$, a pair $(m, w) \in M \times W$ is called a *family*. For each individual there is also given a complete ranking of the individuals in the other group. More precisely, we write $w_j \succ_{m_i} w_k$ for $m_i \in M$, $w_j, w_k \in W$, if man m_i prefers woman w_j to w_k, and similarly, $m_j \succ_{w_i} m_k$, $w_i \in W$, $m_j, m_k \in M$, if woman w_i prefers m_j to m_k. It is assumed that for each individual the corresponding preference forms a complete ordering of the individuals in the other group. A *matching* is a set of n families such that each individual in $M \cup W$ belongs to exactly one of these families. Given a matching $\mathcal{M}$, a family F is said to be *breaking* (with respect to $\mathcal{M}$) if both members of F prefer the other more than their respective partners in the families they belong to in $\mathcal{M}$. The matching $\mathcal{M}$ is called *stable* if there exists no breaking family with respect to it. The *Stable Matching Theorem* of Gale and Shapley (1962) claims that such a stable matching always exists, and can be found in $O(n^2)$ time.

The above result of Gale and Shapley (1962) induced a significant amount of work, relating to algorithms, lattice structure of solutions, strategic behavior, etc. In this short paper, we are interested in generalizations of the Stable Marriage Theorem for "families" consisting of more than two individuals.

One such generalization involves three groups of individuals (say for instance, men, women and cats) and a "family" includes one of each type. In a most general setting the given individual preference for each individual is over all pairs of the other two types. Alkan (1988) gave an example which shows that in this case stable matchings do not always exist. For completeness, we provide in the next section a simpler example demonstrating this fact.

The natural next step was to consider restricted preference orders. Danilov (2001) proved that the Gale-Shapley Theorem generalizes to the case of three-sided systems with the following *acyclic preferences*: the men primarily care about the women (but coupled with different women, the same man may have different secondary preferences over the cats), and analogously, the women primarily care about the men, and finally the cats may have arbitrary preferences over all man-woman pairs. Using an extension of the Gale-Shapley algorithm Danilov showed that in such a case a stable matching always exists. Further, he raised the question if the same remains true for *lexicographically cyclic preferences*, in which the men primarily care about the women, the women primarily care about the cats, and the cats primarily care about the men. In section 3 we show that the answer is still negative by constructing a three-sided system with lexicographically cyclic preferences for which no stable matching exists.

As a further special case, we consider *purely cyclic* preference orders, in which the men only care about the women, the women only care about the cats, and the cats only care about the men. Let us note that such preferences are not complete (e.g., in case of a man, his preference order is not complete over all possible woman-cat pairs), so we should define more precisely the notion of stable matching. We do it in the weakest possible sense. Given a matching $\mathcal{M}$, a family $F \notin \mathcal{M}$ is *breaking* if every member of F *strictly* prefers his/her/its partners in F to his/her/its partners in the respective families they belong to in $\mathcal{M}$. (For instance, a family $F = (m, w, c)$ is not breaking with respect to a matching $\mathcal{M}$, if m and w belong to the same family of $\mathcal{M}$, since in this case the man m will be equally happy/unhappy with both families.) As before, the matching $\mathcal{M}$ is called stable if no breaking family exists with respect to it. The notion of purely cyclic preferences naturally extends to the case of m-sided systems. We prove in section 4 that the Gale-Shapley Theorem generalizes to the case of purely cyclic preferences provided $m \geq n$, where n is the number of individuals in each group, leaving the case $m < n$ open.

2 Three-Sided Families

Let us consider three sets M, W, C (say, men, women, and cats) having the same cardinality, $|M| = |W| = |C| = n$. A triple $F = (m, w, c) \in M \times W \times C$ is called a *family*. A set of n families $\mathcal{M}$ is called a *matching* if each member of $M \cup W \cup C$ belongs to exactly one family $F \in \mathcal{M}$.

Furthermore, we are given for every man $m \in M$ a complete preference order $\mathcal{P}_{(m)}$ over $W \times C$, i.e. of all woman-cat pairs. We write $(w_i, c_j) \succ_m (w_{i'}, c_{j'})$ if m prefers (w_i, c_j) to $(w_{i'}, c_{j'})$. Analogously, we are given complete preference orders $\mathcal{P}_{(w)}$ and $\mathcal{P}_{(c)}$ for the women and the cats, respectively over $C \times M$ and $M \times W$.

Given a matching $\mathcal{M}$, a family $F = (m, w, c) \notin \mathcal{M}$ is called *breaking* (with respect to $\mathcal{M}$) if each individual of F prefers the pair of his/her/its partners in F to the pair of his/her/its partners in the family he/she/it belongs to in $\mathcal{M}$. We call a matching $\mathcal{M}$ *stable* if no breaking family exists with respect to it.

Alkan (1988) showed that the Gale-Shapley (1962) Stable Marriage Theorem does not generalize to the case of three-sided families. He constructed a three-sided example with n=3 individuals in each group (and appropriate preferences for each individual over all pairs of possible partners) for which no stable matching exists. Here we provide a simpler example involving only two individuals in each group. Let $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$, $C = \{c_1, c_2\}$ and let the preferences of the individuals be given by the following table:

$$\begin{array}{rccccccc}
\mathcal{P}_{(m_1)}: & (w_2, c_1) & \succ_{m_1} & (w_1, c_2) & \succ_{m_1} & (w_2, c_2) & \succ_{m_1} & (w_1, c_1) \\
\mathcal{P}_{(m_2)}: & (w_2, c_1) & \succ_{m_2} & (w_1, c_1) & \succ_{m_2} & (w_2, c_2) & \succ_{m_2} & (w_1, c_2) \\
\mathcal{P}_{(w_1)}: & (c_2, m_2) & \succ_{w_1} & (c_1, m_2) & \succ_{w_1} & (c_1, m_1) & \succ_{w_1} & (c_2, m_1) \\
\mathcal{P}_{(w_2)}: & (c_2, m_2) & \succ_{w_2} & (c_1, m_1) & \succ_{w_2} & (c_1, m_2) & \succ_{w_2} & (c_2, m_1) \\
\mathcal{P}_{(c_1)}: & (m_1, w_2) & \succ_{c_1} & (m_2, w_1) & \succ_{c_1} & (m_1, w_1) & \succ_{c_1} & (m_2, w_2) \\
\mathcal{P}_{(c_2)}: & (m_1, w_1) & \succ_{c_2} & (m_1, w_2) & \succ_{c_2} & (m_2, w_2) & \succ_{c_2} & (m_2, w_1)
\end{array}$$

It is immediate to see that there are 8 possible families formed by these 6 individuals, which can be grouped into 4 possible matchings:

$$\begin{aligned}
\mathcal{M}_1 &= \{F_{1a} = (m_1, w_1, c_1), \quad F_{1b} = (m_2, w_2, c_2)\} \\
\mathcal{M}_2 &= \{F_{2a} = (m_2, w_1, c_1), \quad F_{2b} = (m_1, w_2, c_2)\} \\
\mathcal{M}_3 &= \{F_{3a} = (m_1, w_2, c_1), \quad F_{3b} = (m_2, w_1, c_2)\} \\
\mathcal{M}_4 &= \{F_{4a} = (m_1, w_1, c_2), \quad F_{4b} = (m_2, w_2, c_1)\}
\end{aligned}$$

With respect to the above preferences, each of these matchings have a breaking family, showing that no stable matching exists for this example. For instance, it is easy to check that family F_{2a} is breaking for $\mathcal{M}_1$, F_{3a} is breaking for both $\mathcal{M}_2$ and $\mathcal{M}_4$, and finally family F_{1b} is breaking for $\mathcal{M}_3$.

3 Lexicographically Cyclic Preferences

Let us consider again a three-sided system with $|M| = |W| = |C| = n$, and *lexicographically cyclic* preferences, as defined in the introduction, i.e., if $w \neq w'$ and $(w, c) \succ_m (w', c')$ then $(w, c'') \succ_m (w', c''')$ for every cat, and analogous implications for the women and cats hold, too. Let us say that a woman w is the *first choice* for a man m if $(w, c) \succ_m (w', c')$ holds for all women $w' \neq w$ and all cats c, c'. Analogously, we shall talk about first choices for the women and the cats, as well. Let us call a family *ideal* if each member of it is the first choice of the cyclically previous member. Let us further call a family *super-ideal* if each pair is the most preferred pair for the third member of the family, according to his/her/its preference order. Obviously, super-ideal families are also ideal.

Theorem 1 *In three-sided systems with lexicographically cyclic preferences stable matchings always exist if $n \leq 2$, and may fail to exist if $n \geq 3$.*

Before the proof of this theorem, let us first make a simple but useful observation.

Lemma 1 *Let F be an ideal family and let $\mathcal{M}$ be a matching. If $F \in \mathcal{M}$, then no breaking family (with respect to $\mathcal{M}$) can have a common member with F. Furthermore, if $F \notin \mathcal{M}$ and F is super-ideal, then $\mathcal{M}$ is not stable.*

Proof. This claim is quite obvious, since by definition, no member of an ideal family can "imagine to be happier" with somebody else from the cyclically next group. preferences). The second part is also immediate, since if $F \notin \mathcal{M}$ and F is super-ideal, then F is a breaking family for $\mathcal{M}$. □

Let us show next the easier part of the theorem.

Lemma 2 *A stable matching exists in a three-sided system with lexicographically cyclic preferences whenever $n \leq 2$.*

Proof. For $n = 1$ there is nothing to prove. Suppose $n = 2$. Let us first assume that two men, as well as two women, have no conflict in their first choices. In this case we just assign to each man and each woman his/her first choice and obtain two families forming a stable matching $\mathcal{M}$. Indeed, suppose $F = (m, w, c)$ is a breaking family for $\mathcal{M}$. Then m and w must have their first choices also in F, implying that $F \in \mathcal{M}$ which is impossible. Let us now assume that two men (or two women) have a conflict in their top preference, that is the same women (respectively same cat) is their first choice. In this case still there exists an ideal family, which together with the complementary family forms a stable matching. Let w^* be the first choice woman for both men, let $c^* \in C$ as the first choice of w^*, and let $m^* \in M$ be the first choice of c^*. Let $F = (m^*, w^*, c^*)$ and let F' be the complementary family formed by the rest of the individuals. Then the family F is ideal by its construction, and thus $\mathcal{M} = \{F, F'\}$ is a stable matching, since no breaking family can intersect F by Lemma 1, and since in the considered case of $n = 2$ all families but $F' \in \mathcal{M}$ intersect F. $\square$

Let us show next that the negative part of the Theorem can easily be reduced to the case of $n = 3$.

Lemma 3 *If a three-sided system $\mathcal{S}_n$, having n individuals in each group, and having lexicographically cyclic preferences, has no stable matching, then it can be extended to an analogous system $\mathcal{S}_{n+1}$ with $n + 1$ members in each group, which still does not have a stable matching.*

Proof. Let us denote the groups of $\mathcal{S}_n$ by $M = \{m_1, ..., m_n\}$, $W = \{w_1, ..., w_n\}$ and $C = \{c_1, ..., c_n\}$ and let $\succ$ denote the lexicographically cyclic preference order in S_n. Let us introduce three new members, m_{n+1}, w_{n+1}, and c_{n+1}, different from all others, and let $M' = M \cup \{m_{n+1}\}$, $W' = W \cup \{w_{n+1}\}$, and $C' = C \cup \{c_{n+1}\}$. Let us next define an extension $\succ'$ of $\succ$, accommodating the new members, so that the following relations

$$
\begin{array}{rcl}
(w_j, c_k) & \succ'_{m_i} & (w_{n+1}, c_{n+1}), \\
(c_j, m_k) & \succ'_{w_i} & (c_{n+1}, m_{n+1}), \\
(m_j, w_k) & \succ'_{c_i} & (m_{n+1}, w_{n+1}), \\
(w_{n+1}, c_{n+1}) & \succ'_{m_{n+1}} & (w_j, c_k), \\
(c_{n+1}, m_{n+1}) & \succ'_{w_{n+1}} & (c_k, m_i), \quad \text{and} \\
(m_{n+1}, w_{n+1}) & \succ'_{c_{n+1}} & (m_i, w_j)
\end{array}
$$

hold for all $i, j, k \leq n$. In addition, we assume that $\succ'$ agrees with $\succ$ for all other cases not involving the new members. Clearly, we can do this and ensure that the extended preference order $\succ'$ is still lexicographically cyclic over the $n + 1$ member groups M', W' and C'.

We claim that the obtained three-sided system $\mathcal{S}_{n+1}$ with the preference order $\succ'$ cannot have a stable matching, either. To see this let us observe first that the family $F = (m_{n+1}, w_{n+1}, c_{n+1})$ is super-ideal in $\mathcal{S}_{n+1}$. Thus, by Lemma 1, F is a breaking family with respect to each matching $\mathcal{M}$ which does not contain it. On the other hand, if $F \in \mathcal{M}$, then no breaking family with respect to $\mathcal{M}$ can intersect F, again by Lemma 1. Thus $\mathcal{M}$ is stable in $\mathcal{S}_{n+1}$ if and only if $\mathcal{M} \setminus \{F\}$ is stable in $\mathcal{S}_n$. Since by our assumption $\mathcal{S}_n$ has no stable matching, $\mathcal{M}$ cannot be stable, either. $\square$

Proof of Theorem 1: According to lemmas 2 and 3, it is enough to construct a three-sided system with lexicographically cyclical preference order and with $n = 3$ for which there is no stable matching. Let $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and $C = \{c_1, c_2, c_3\}$, and let us define a lexicographically cyclic preference order $\succ$ satisfying the following relations:

$$\begin{array}{llll}
(w_1, c_i) & \succ_m (w_2, c_j) & \succ_m (w_3, c_k) & \text{for } m \in \{m_1, m_3\} \\
(w_2, c_i) & \succ_m (w_3, c_j) & \succ_m (w_1, c_k) & \text{for } m = m_2 \\
(w_i, c_2) & \succ_m (w_i, c_1) & \succ_m (w_i, c_3) & \text{for } m \in \{m_1, m_2, m_3\}; \\[1ex]
(c_1, m_i) & \succ_w (c_2, m_j) & \succ_w (c_3, m_k) & \text{for } w \in \{w_1, w_3\} \\
(c_2, m_i) & \succ_w (c_3, m_j) & \succ_w (c_1, m_k) & \text{for } w = w_2 \\
(c_i, m_2) & \succ_w (c_i, m_1) & \succ_w (c_i, m_3) & \text{for } w \in \{w_1, w_2, w_3\}; \\[1ex]
(m_1, w_i) & \succ_c (m_2, w_j) & \succ_c (m_3, w_k) & \text{for } c \in \{c_2, c_3\} \\
(m_2, w_i) & \succ_c (m_3, w_j) & \succ_c (m_1, w_k) & \text{for } c = c_1 \\
(m_i, w_2) & \succ_c (m_i, w_1) & \succ_c (m_i, w_3) & \text{for } c \in \{c_1, c_2, c_3\},
\end{array}$$

for all $1 \leq i, j, k \leq 3$. Clearly, the preference order $\succ$ defined by the above relations is lexicographically cyclic. Let us also note the following property of $\succ$: for each individual, his/her/its secondary preference does not depend on the first partner, moreover, the corresponding sequence of indexes 2,1,3 is the same for all 9 individuals.

There are $(3!)^2 = 36$ possible matchings in this example. We show that for each one of these there exists a breaking family. For 9 of these 36 matchings there exists in fact a *unique* breaking family. We list these matchings together with the corresponding breaking families first.

$$\begin{array}{lll}
\mathcal{M}_1 & = \{(m_1, w_1, c_1), (m_2, w_2, c_2), (m_3, w_3, c_3)\} \longleftarrow & (m_3, w_3, c_1) = F_1 \\
\mathcal{M}_2 & = \{(m_1, w_1, c_2), (m_2, w_2, c_1), (m_3, w_3, c_3)\} \longleftarrow & (m_3, w_2, c_3) = F_2 \\
\mathcal{M}_3 & = \{(m_1, w_1, c_2), (m_2, w_2, c_3), (m_3, w_3, c_1)\} \longleftarrow & (m_3, w_1, c_1) = F_3 \\
\mathcal{M}_4 & = \{(m_1, w_1, c_2), (m_2, w_3, c_1), (m_3, w_2, c_3)\} \longleftarrow & (m_2, w_2, c_3) = F_4 \\
\mathcal{M}_5 & = \{(m_1, w_2, c_2), (m_2, w_1, c_1), (m_3, w_3, c_3)\} \longleftarrow & (m_2, w_3, c_3) = F_5 \\
\mathcal{M}_6 & = \{(m_1, w_2, c_2), (m_2, w_3, c_3), (m_3, w_1, c_1)\} \longleftarrow & (m_2, w_3, c_1) = F_6 \\
\mathcal{M}_7 & = \{(m_1, w_2, c_2), (m_2, w_3, c_1), (m_3, w_1, c_3)\} \longleftarrow & (m_1, w_1, c_3) = F_7 \\
\mathcal{M}_8 & = \{(m_1, w_3, c_3), (m_2, w_2, c_2), (m_3, w_1, c_1)\} \longleftarrow & (m_1, w_3, c_2) = F_8 \\
\mathcal{M}_9 & = \{(m_1, w_3, c_2), (m_2, w_2, c_3), (m_3, w_1, c_1)\} \longleftarrow & (m_1, w_2, c_2) = F_9
\end{array}$$

Let us next consider a matching $\mathcal{M}$ such that $(m_2, w_2, c_2) \notin \mathcal{M}$ and $(m_1, w_i, c_2) \notin \mathcal{M}$ for $i = 1, 2, 3$. Then it is easy to check that $(m_2, w_2, c_2) = F_{10}$ will be a breaking family for every such matching. This way we break twenty other matchings, leaving us with only 7 to check. Finally, we list these 7 matchings with their corresponding breaking families.

$$\begin{array}{lll}
\mathcal{M}_{10} & = \{(m_1, w_1, c_3), (m_2, w_2, c_2), (m_3, w_3, c_1)\} & \longleftarrow \quad (m_3, w_1, c_1) = F_3 \\
\mathcal{M}_{11} & = \{(m_1, w_1, c_2), (m_2, w_3, c_3), (m_3, w_2, c_1)\} & \longleftarrow \quad (m_2, w_3, c_1) = F_6 \\
\mathcal{M}_{12} & = \{(m_1, w_2, c_2), (m_2, w_1, c_3), (m_3, w_3, c_1)\} & \longleftarrow \quad (m_2, w_3, c_1) = F_6 \\
\mathcal{M}_{13} & = \{(m_1, w_3, c_2), (m_2, w_1, c_1), (m_3, w_2, c_3)\} & \longleftarrow \quad (m_1, w_2, c_2) = F_9 \\
\mathcal{M}_{14} & = \{(m_1, w_3, c_2), (m_2, w_1, c_3), (m_3, w_2, c_1)\} & \longleftarrow \quad (m_1, w_2, c_2) = F_9 \\
\mathcal{M}_{15} & = \{(m_1, w_3, c_2), (m_2, w_2, c_1), (m_3, w_1, c_3)\} & \longleftarrow \quad (m_1, w_2, c_2) = F_9 \\
\mathcal{M}_{16} & = \{(m_1, w_3, c_1), (m_2, w_2, c_2), (m_3, w_1, c_3)\} & \longleftarrow \quad (m_1, w_1, c_3) = F_7
\end{array}$$

Let us note that F_3, F_6, F_7, F_9 were the only breaking families for matchings $\mathcal{M}_3, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_9$ respectively. Thus the obtained set of 10 breaking families is a minimum cardinality set necessary to break every matching. $\square$

4 Purely Cyclic Preferences

Purely cyclic preferences, as defined in the introduction, constitute a further special case of lexicographically cyclic preferences. Let us formally extend this notion to m member families where $m \geq 2$ is an integer.

Let $\mathbb{Z}_m = \{0, 1, ..., m-1\}$ denote the additive group of modulo m addition of integers and let $\mathbb{I}_n = \{1, 2, ..., n\}$. Let us consider m groups $G_j = \{(i, j) \mid i \in \mathbb{I}_n\}$ for $j \in \mathbb{Z}_m$, with exactly n individuals in each group. Let us further denote the preference order of individual $(i, j) \in G_j$ over the cyclically next group G_{j+1} by $\succ_{i,j}$ for all $j \in \mathbb{Z}_m$ (additions within $\mathbb{Z}_m$ are taken *modulo* m). A *family* consists of exactly one individual from each of the groups. For simplicity, let us represent a family $F \in G_0 \times G_1 \times \cdots \times G_{m-1}$ by an associated vector $(f_0, f_1, ..., f_{m-1})$, where $F = \{(f_j, j) \mid j \in \mathbb{Z}_m\}$. A *matching* $\mathcal{M}$ is a collection of n pairwise disjoint families. We say that a family F is *breaking* with respect to a matching $\mathcal{M}$, if all individuals $(f_j, j) \in F$ prefer $(f_{j+1}, j+1)$ to $(i, j+1)$, where $(i, j+1)$ is their current partner in the family they belong to in $\mathcal{M}$. More formally, if $\mathcal{M} = \{F^i \mid i \in \mathbb{I}_n\}$, and $F^i = \{(f^i_j, j) \mid j \in \mathbb{Z}_m\}$, for $i \in \mathbb{I}_n$, then $F = \{(f_j, j) \mid j \in \mathbb{Z}_m\}$ is breaking with respect to $\mathcal{M}$, if

$$(f_{j+1}, j+1) \succ_{(f_j, j)} (f^i_{j+1}, j+1) \quad \text{where} \quad f_j = f^i_j,$$

for all $j \in \mathbb{Z}_m$. Finally, we call a matching *stable* if there is no breaking family with respect to it.

Let us note that for $m = 2$ all classical stable matching problems are purely cyclic by the above definitions, and thus a stable matching always exists for these purely cyclic problems when $m = 2$ by Gale and Shapley (1962). The same question is open already for $m = 3$, since the negative example for three-sided systems with lexicographically cyclic preferences in the previous section is not purely cyclic. In this section we prove that a stable matching always exists for purely cyclic preferences, whenever $n \leq m$, leaving the cases $n > m$ open.

Before proving our claim, let us introduce a few more notations, to simplify our analysis.

To an m-sided system with purely cyclic preferences let us associate a directed graph $G = (V, E)$, where $V = G_0 \cup G_1 \cup \cdots \cup G_{m-1}$, and where there is a directed edge from every individual of G_j to every individual of the cyclically next group G_{j+1} for $j \in \mathbb{Z}_m$. To a directed edge $e = [(i, j), (k, j+1)]$ let us associate its *rank*, denoted by $r(e)$, which is the rank of individual $(k, j+1)$ in the preference list of (i, j), i.e., $r(e) = 1 + |\{l \; : \; (l, j+1) \succ_{(i,j)} (k, j+1)\}|$. Given a subset $W \subseteq V$ of the vertices of G, let us denote by $G[W]$ the subgraph induced by W. We call a directed path $P = \{(i_j, j), (i_{j+1}, j+1), \cdots, (i_{j+s}, j+s)\}$ a *best choice path* in $G[W]$, if $P \subseteq W$ and the directed edges of this path $e_l = [(i_{j+l}, j+l), (i_{j+l+1}, j+l+1)]$ have the lowest possible rank among the outgoing edges from $(i_{j+l}, j+l)$ in $G[W]$ for $l = 0, 1, ..., s-1$.

Given a permutation $\pi = (a_0^1, a_0^2, ..., a_0^n)$ of $\mathbb{I}_n$, let us construct a corresponding matching $\mathcal{M}_\pi$ as follows: Let $W = V$ initially, and let us consider the (unique) best choice path

$$F^1 = \{(a_0^1, 0), (a_1^1, 1), \cdots, (a_{m-1}^1, m-1)\}$$

starting from $(a_0^1, 0)$ in $G[W]$. Let us then delete these vertices, set $W' = W \setminus F^1$, and consider the (unique) best choice path

$$F^2 = \{(a_0^2, 0), (a_1^2, 1), \cdots, (a_{m-1}^2, m-1)\}$$

starting from $(a_0^2, 0)$ in the induced subgraph $G[W']$, etc. Clearly, the paths F^1, F^2, ..., F^n are pairwise disjoint families, and thus

$$\mathcal{M}_\pi = \{F^1, F^2, ..., F^n\}$$

is a matching. Let us call a matching $\mathcal{M}$ a *best choice matching* if $\mathcal{M} = \mathcal{M}_\pi$ for some permutation π.

Theorem 2 *If $n \leq m$, a stable best choice matching always exists in an m-sided system with purely cyclic preferences.*

Before proving this claim, let us state a few helpful lemmas.

Note that we can view the sets $\{a_j^i \mid i \in \mathbb{I}_n\}$ for $j \in Z_m$ as permutations of $\mathbb{I}_n$, and a family as a selection of one element from each of these sets. Accordingly, we shall refer to a family F as $F = (a_j^{i(j)} \mid j \in \mathbb{Z}_m)$ in the sequel.

Lemma 4 *Let $\mathcal{M}_\pi$ be a best choice matching, and let $F = (a_j^{i(j)} \mid j \in \mathbb{Z}_m)$ be a breaking family with respect to $\mathcal{M}_\pi$. Then*

$$i(0) > i(1) > \cdots > i(m-1).$$

Proof. By the definition of a breaking family, we must have

$$(a_{j+1}^{i(j+1)}, j+1) \succ_{(a_j^{i(j)},j)} (a_{j+1}^{i(j)}, j+1)$$

for all $j \in \mathbb{Z}_m$, where $(a_j^{i(j)}, j)$ belongs to in the matching $\mathcal{M}_\pi$. By the definition of $\mathcal{M}_\pi$, $(a_{j+1}^{i(j)}, j+1)$ is the best choice of $(a_j^{i(j)}, j)$ among all those in G_{j+1} not appearing in the first $i(j)-1$ families $F^1, ..., F^{i(j)-1}$. Hence, the individual $(a_{j+1}^{i(j+1)}, j+1)$ must belong to one of these first $i(j)-1$ families, i.e. $i(j) > i(j+1)$. □

Corollary 1 *If $n < m$, all best choice matchings are stable in a m-sided system with purely cyclic preferences.*

Proof. Clearly, the condition in Lemma 4 cannot hold, unless the matching $\mathcal{M}_\pi$ has at least m families. Thus, if $n < m$, a best choice matching $\mathcal{M}_\pi$ cannot have a breaking family with respect to it. □

Proof of Theorem 2: In view of Corollary 1, it is enough to show that if $n = m$ then there always exists a stable best choice matching in an m-sided system with purely cyclic preferences.

To see this, let us observe first that in a best choice matching $\mathcal{M}_\pi$ the element a_{m-1}^1 depends only on the selection of a_0^1 (and not on the rest of the permutation π). This defines a mapping $\tau : \mathbb{I}_n \longrightarrow \mathbb{I}_n$ by setting $\tau(a_0^1) = a_{m-1}^1$ for all possible choices of $a_0^1 \in \mathbb{I}_n$.

Let us observe next that if $n = m$ then by Lemma 4 there is only one candidate breaking family, namely $F = (a_0^n, a_1^{n-1}, ..., a_{m-1}^1)$. For this unique family to be breaking with respect to $\mathcal{M}_\pi$ it is necessary that

$$(a_0^n, 0) \succ_{(a_{m-1}^1, m-1)} (a_0^1, 0), \qquad \text{and} \tag{1}$$

$$(a_1^{n-1}, 1) \succ_{(a_0^n,0)} (a_1^n, 1). \tag{2}$$

Case I: There exists an index $i \in \mathbb{I}_n$ such that $(i, 0) \in G_0$ is not the worst choice of the individual $(\tau(i), m-1) \in G_{m-1}$. In this case let us choose π such that $a_0^1 = i$, and a_0^n is the element for which $(a_0^n, 0) \in G_0$ is the worst choice of $(\tau(i), m-1) \in G_{m-1}$. Then the unique candidate family F cannot be breaking by the necessity of (1), and thus the matching $\mathcal{M}_\pi$ is stable.

Case II: For all $i \in \mathbb{I}_n$ the individual $(i, 0) \in G_0$ is the worst choice of $(\tau(i), m-1) \in G_{m-1}$. This implies that $\tau(i) \neq \tau(i')$ whenever $i \neq i'$ (since no individual can have two different worst choices). This fact further implies that all individuals in G_0 have different first choices (since otherwise two first choice paths would converge, implying the existence of indices $i \neq i'$ with $\tau(i) = \tau(i')$, i.e., *Case I* would apply). Thus, $(a_1^n, 1) \in G_1$ is the first choice of

$(a_0^n, 0)$, and hence it is preferred more than $(a_1^{n-1}, 1)$, implying that F cannot be breaking by the necessity of (2). Since this conclusion is reached regardless how the permutation π was chosen, all best choice matchings are stable in this case. $\square$

Let us remark finally that for $n > m$ we may not be able to find a stable matching among best choice ones. Namely, in a best choice matching at least $m - 1 > 0$ individuals realize their first choice (the first $m-1$ individuals in the first family F^1 surely do so). This however is not true already for $m = 2$. To see this, let us consider an example involving 4 men and 4 women, $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$, and the purely cyclic preferences

$$\begin{array}{c} w_2 \succ_{m_1} w_1 \succ_{m_1} w_3 \succ_{m_1} w_4 \\ w_3 \succ_{m_2} w_2 \succ_{m_2} w_1 \succ_{m_2} w_4 \\ w_2 \succ_{m_3} w_3 \succ_{m_3} w_1 \succ_{m_3} w_4 \\ w_3 \succ_{m_4} w_4 \succ_{m_4} w_2 \succ_{m_4} w_1 \\ \\ m_4 \succ_{w_1} m_1 \succ_{w_1} m_2 \succ_{w_1} m_3 \\ m_4 \succ_{w_2} m_2 \succ_{w_2} m_1 \succ_{w_2} m_3 \\ m_1 \succ_{w_3} m_3 \succ_{w_3} m_4 \succ_{w_3} m_2 \\ m_1 \succ_{w_4} m_4 \succ_{w_4} m_2 \succ_{w_4} m_3 \end{array}$$

It is easy to see that $\mathcal{M} = \{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$ is the only stable matching in this instance, and in this matcing no individual could realize his/her first choice. In fact, all individuals got their second best choices.

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