DIMACS Technical Report 2002-33
July 2002

ON CONVEX POLYTOPES IN THE PLANE "CONTAINING" AND "AVOIDING" ZERO

by

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ABSTRACT

Let S be a finite set of points in the plane, $X \subseteq S$, and z a point not in S. We say that X is a z–*containing set* if z is in the interior of the convex hull of X. Also X is a z–*avoiding set* if z is not contained in the interior of the convex hull of X. Let $\mathcal{C}(S)$ and $\mathcal{A}(S)$ denote the sets of minimal z–containing and maximal z–avoiding subsets of S.

E. Boros and V. Gurvich raised the following question:

Is it true that $|\mathcal{A}(S)| \leq 2d|\mathcal{C}(S)|$ *where* d *is the dimension of the space*?

This inequality is obviously true for $d = 1$.

In this paper we verify the inequality for $d = 2$ by proving the following stronger result:

Theorem *Let S be a finite set of points in the plane and z a point not in S. Suppose that z is in the interior of the convex hull of S. Then $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$. Moreover, the equality holds if and only if $|S| = 4$, the convex hull of S is a quadrilateral Q, and z is the point of intersection of the two diagonals of Q.*

1 Introduction

The goal of this paper is to prove a certain inequality concerning some subsets of a finite set of points in the plane.

The main motivation for considering our problem is the study of monotone boolean functions. It is known and easy to show that a monotone boolean function f is uniquely defined by the family $\mathcal{T}_f$ of so-called *minimal true sets* as well as by the family $\mathcal{F}_f$ of so-called *maximal false sets* of variables of the function.

One of the natural problems here is to generate "efficiently" all minimal true sets (i.e. the elements of $\mathcal{T}_f$) or all maximal false sets (i.e. the elements of $\mathcal{F}_f$), or both (i.e. the elements of $\mathcal{F}_f \cup \mathcal{T}_f$).

In 1995 M. Fredman, V. Gurvich and L. Khachiyan [1, 2] found an incremental quasi-polynomial time algorithm to generate all minimal true and maximal false sets of a monotone boolean function, i.e. the elements of $\mathcal{F}_f \cup \mathcal{T}_f$. Unfortunately, it is not known whether or not there exists an incremental polynomial or quasi-polynomial time algorithm to generate the elements $\mathcal{T}_f$ (the same is true for $\mathcal{F}_f$).

Suppose that a monotone boolean function satisfies the condition:

(C) *there exists a polynomial function p such that $p(|\mathcal{T}_f|) \geq |\mathcal{F}_f|$, and so the number $|\mathcal{T}_f|$ of minimal true sets grows at least as a polynomial function of the number $|\mathcal{F}_f|$ of maximal false sets.*

Then, fortunately, the Fredman–Gurvich–Khachiyan algorithm allows also to generate all minimal true sets (the elements $\mathcal{T}_f$) incrementally in quasi-polynomial time.

For that reason, it is interesting to describe some natural classes of monotone boolean functions satisfying condition **(C)**. The consideration of convex polytopes "containing" and "avoiding" a special point, allows to describe various classes of monotone boolean functions. We will show that some of them satisfy condition **(C)**.

Now we will give necessary definitions, describe the corresponding boolean function in terms of its minimal true sets, and formulate the main result.

Let S be a finite set of points in the plane, $X \subseteq S$, and z be a point not in S.

X is a *z-containing set* if z is in the interior of the convex hull of X.

A z-containing set X is *minimal in S* if X has no proper z-containing subset.

X is a *z-avoiding set* if z is not contained in the interior of the convex hull of X.

A z-avoiding set X is *maximal in S* if there is no z-avoiding set containing X properly.

Let $\mathcal{C}(S)$ and $\mathcal{A}(S)$ denote the sets of minimal z-containing and maximal z-avoiding subsets of S, respectively.

The families $\mathcal{C}(S)$ and $\mathcal{A}(S)$ can be interpreted as the families of minimal true sets and maximal false sets, respectively, of some monotone boolean function f_S.

E. Boros and V. Gurvich raised the following interesting question:

Is it true that

$$|\mathcal{A}(S)| \le 2d|\mathcal{C}(S)| \tag{1.1}$$

where d is the dimension of the space?

This inequality is obviously true for $d = 1$. In this paper we verify the inequality for $d = 2$ by proving the following stronger theorem.

1.1 *Let S be a finite set of points in the plane and z a point not in S. Suppose that z is in the interior of the convex hull of S. Then $|\mathcal{A}(S)| \le 3|\mathcal{C}(S)| + 1$. Moreover, the equality holds if and only if $|S| = 4$, the convex hull of S is a quadrilateral Q, and z is the point of intersection of the two diagonals of Q.*

Using a different approach, A. Kelmans and A. Rubinov [3] proved the following more general result.

1.2 (A. Kelmans, A. Rubinov). *Let S be a finite set of points in the d–dimensional space and z be a point that belongs to no hyperplane containing at least two points from S. Suppose that z is in the interior of the convex hull of S. Then $|\mathcal{A}(S)| \le d|\mathcal{C}(S)| + 1$ and the equality holds if and only if $|S| = d + 1$ and the convex hull of S is a $(d + 1)$–dimensional simplex.*

Theorem **1.2** implies $|\mathcal{A}(S)| \le 2d|\mathcal{C}(S)|$ giving a positive answer to the above question for points in general position. They also proved that inequality (1.1) is true for $d \le 3$.

2 Proof of Theorem 1.1

It is known and not hard to prove that:

2.1 *Let z be a point and X be a finite set of points in the d–dimensional space, $d \geq 1$. If X is minimal z–containing then $d+1 \leq |X| \leq 2d$.*

We first prove:

2.2 *Let S be a finite set of points in the plane and z a point not in S. Suppose that z is in the interior of the convex hull of S. Then $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$.*

Proof We prove our claim by induction on $|S|$.

Suppose that S is a minimal z-containing set. Then $\mathcal{C}(S) = \{S\}$ and $S - s$ is a maximal avoiding set for every $s \in S$. Therefore $\mathcal{A}(S) = \{S - s : s \in S\}$. Thus $|\mathcal{C}(S)| = 1$ and $|\mathcal{A}(S)| = |S|$. Since S is a minimal z-containing set, by **2.1**, $|S| \leq 4$. Therefore $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$.

Suppose that $|S| = 3$. By the assumption of the theorem, z is an interior point of the convex hull of S. Since S is a set of three points in the plane, the convex hull of S is a triangle. Therefore $S - x$ is a z–avoiding set for every $x \in S$. Hence S is a minimal z–avoiding set, and so by the above arguments our claim holds.

Now suppose that $|S| \geq 4$ and S is not a minimal z–containing set. Then there exists $s \in S$ such that $S - s$ is a z–containing set. By the induction hypothesis, $|\mathcal{A}(S - s)| \leq 3|\mathcal{C}(S - s)| + 1$.

Let $\mathcal{A}^s(S)$ denote the set of all subsets A of S such that A is maximal z–avoiding in S and $A - s$ is maximal z–avoiding in $S - s$, i.e. $A \in \mathcal{A}(S)$ and $A - s \in \mathcal{A}(S - s)$. Let $\mathcal{A}_s(S) = \mathcal{A}(S) \setminus \mathcal{A}^s(S)$, and so a subset X of S is an element of $\mathcal{A}_s(S)$ if and only if $s \in X$, $X \in \mathcal{A}(S)$, and $X - s \notin \mathcal{A}(S - s)$.

Claim 2.2.1 *Given $X \in \mathcal{A}^s(S)$, let $\alpha(X) = X$ if $s \notin X$ and $\alpha(X) = X - s$ if $s \in X$. Then $\alpha : \mathcal{A}^s(S) \to \mathcal{A}(S - s)$ is a bijection, and so*

$$|\mathcal{A}^s(S)| = |\mathcal{A}(S - s)|. \tag{2.1}$$

Proof of Claim 2.2.1. Clearly α is an injective function. Let $Y \in \mathcal{A}(S-s)$. By definition of $\mathcal{A}(S-s)$, $Y+y$ is z–containing for every $y \in (S-s)\setminus Y$. If $Y+s$ is z–containing in S then Y is maximal z–avoiding in S. Hence $\alpha(Y)=Y$. Now if $Y+s$ is not z–containing in S, then $Y+s$ is maximal z–avoiding in S and $\alpha(Y+s)=Y$. Therefore α is a bijection, and our claim is proved. $\diamondsuit$

Let $\mathcal{C}_s(S)$ denote the set of all minimal z–containing subsets X of S such that $s \in X$, i.e. $\mathcal{C}_s(S)=\{X \in \mathcal{C}(S) : s \in X\}$.

Claim 2.2.2

$$|\mathcal{A}_s(S)| \leq 3|\mathcal{C}_s(S)|. \tag{2.2}$$

Proof of Claim 2.2.2. Let $A \in \mathcal{A}_s(S)$ and $A' = A - s$. Consider a line containing z and a point from A' such that A' belongs to one of the half–spaces defined by this line. Obviously there are at most two such lines $L_1 = L_1(A')$ and $L_2 = L_2(A')$. By definition of $\mathcal{A}_s(S)$, $A \in \mathcal{A}_s(S)$ implies $A' \notin \mathcal{A}(S-s)$. Therefore there exists $v \in S \setminus A$ such that $A'+v$ is a z–avoiding set, and so there exists $C \in \mathcal{C}_s(S)$ such that $v \in C \subseteq A+v$. Since $A \in \mathcal{A}(S)$, we have $A+v$ is a z–containing set.

Let G be an $(\mathcal{A},\mathcal{C})$–bipartite graph with the following properties:

$(c1)$ $\mathcal{A}=\mathcal{A}_s(S)$ and $\mathcal{C}=\mathcal{C}_s(S)$ and

$(c2)$ if $A \in \mathcal{A}$ and $C \in \mathcal{C}$ then (A,C) is an edge of G if and only if there exists $c \in C$ such that $C \subseteq A+c$ and $A'+c$ is z–avoiding.

Since $A \in \mathcal{A} = \mathcal{A}_s(S)$, there exists $C \in \mathcal{C}$ such that (A,C) is an edge. Therefore every vertex in $\mathcal{A}$ has degree at least one in G. By **2.1**, $|C| \in \{3,4\}$.

Let Δ denote the maximum degree of vertices in $\mathcal{C}$. Then $|\mathcal{A}| \leq \Delta \cdot |\mathcal{C}|$. Our goal is to show that $\Delta \leq 3$ which would imply the required inequality.

Let $i,j \in \{1,2\}$. Let H_1^i and H_2^i denote the half-planes defined by L_i and $K_{ij} = H_i^1 \cap H_j^2$, and so each K_{ij} is a cone in the plane. We can assume that $A' \subset K_{11}$. By definition of L_i's, $A' = (S-s)\cap K_{11}$ and $L_i \cap A' \neq \emptyset$ for each $i \in \{1,2\}$. Let $l_i \in L_i \cap A'$. Since the z–avoiding set A is maximal, $A = S \cap H_1^2$.

(p1) Suppose that $L_1 \neq L_2$. Then $l_1 \neq l_2$. If $\{s,v\} \subseteq H_1^i$ for some $i \in \{1,2\}$, then $A+v$ is z–avoiding, a contradiction. If $x \in \{s,v\}$ and x is in the interior of K_{22}, then the set $\{l_1,l_2,x\}$ is

z–containing. Since $\{l_1, l_2\} \subseteq A'$, we have $A' + x$ is z–containing, a contradiction. Therefore $\{s, v\} \cap K_{ij} \neq \emptyset$ where $\{i, j\} = \{1, 2\}$. We can assume that $s \in K_{21}$ and $v \in K_{12}$. Since the z–avoiding set A is maximal, it follows from the definition of L_1, L_2 that $(K_{21} \setminus L_1) \cap S = \{s\}$.

(p1.1) Suppose that $s \notin L_2$. Let P be the line containing s and z. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two half–planes defined by P such that $K_{ii} \subseteq \mathcal{P}_j$ where $\{i, j\} = \{1, 2\}$. Let $D_i = \mathcal{P}_i \cap H_i^i$. If $v \in D_2$, then $A + v$ is z–avoiding, a contradiction. Therefore $v \in D_1 \setminus P$. Since the z–avoiding set A is maximal, $D_2 \cap S = \emptyset$. As above let C be a minimal z–containing set such that $C \subseteq A + v$, and so $v \in C$. Clearly $|C| = 3$, say $C = \{s, v, a\}$ where $a \in A'$.

(p1.1.1) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, a\}$, and so B is adjacent to C in G. Let $B' = B - s$. Since the z–avoiding set B is maximal, $B \nsubseteq A$. We know that $A = S \cap H_1^2$. Therefore there exists $b \in B \cap (H_2^2 \setminus L_2)$. Since $D_2 \cap S = \emptyset$, clearly $b \in K_2^2 \cup (D_1 \setminus P)$.

If b is an interior point of D_1, then $\{s, a, b\}$ is z–containing, and so B is also z–containing, a contradiction. Therefore $b \in K_{22} \setminus L_2$.

Let R be the line containing a and z. If $b \notin R$, then either $\{a, b, s\}$ or $\{a, b, v\}$ is z–containing, and so either B or $B' + v$, respectively, is z–containing, a contradiction. Therefore $b \in R$ and $B \cap K_{22} = B \cap R$. $Q \in \mathcal{A}$, such that $Q \cap C = \{s, a\}$, and therefore Q is either A or $(R \cap S) + s$.

(p1.1.2) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, v\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal in S, clearly $B \not\subset A$.

(p1.1.2.1) Suppose that there exists $b \in B \cap K_{11}$. If $b \notin L_1$, then $\{b, s, v\}$ is z–containing, and therefore B is z–containing, a contradiction. If $b \in L_1$ and $v \notin L_1$, then again $\{b, s, v\}$ is z–containing, and therefore B is z–containing, a contradiction.

Therefore $b, v \in L_1$. If there exists $b' \in B \cap (D_1 \setminus L_1)$, then $\{s, b, b'\}$ is z–containing, and therefore B is z–containing, a contradiction. Hence $B \subseteq H_2^1$. Since the z–avoiding set B is maximal in S, clearly $B = S \cap H_2^1$. It follows that there exists only one set $Q \in \mathcal{A}(S)$, such that $Q \cap C = \{s, v\}$, and $Q = S \cap H_2^1$.

(p1.1.2.2) Now suppose that $B \cap K_{11} = \emptyset$. Then $B \subseteq \mathcal{P}_1$. Since the z–avoiding set B is maximal in S, clearly $B = S \cap \mathcal{P}_1$. Hence again there exists only one set $Q \in \mathcal{A}$, such that $Q \cap C = \{s, v\}$, and $Q = S \cap \mathcal{P}_1$.

(p1.2) Suppose $s \in L_2$. As above let $C \in \mathcal{C}_1(S)$ such that $v \in C \subseteq A + v$. We know that by **2.1**, $|C| \in \{3, 4\}$.

(p1.2.1) Suppose that $|C| = 3$, say $C = \{s, v, a\}$. We will show that in this case there are at most two different sets in $\mathcal{A}$, adjacent to C in G, and each one avoids exactly one point in $\{v, a\}$ (see **(p1.2.1.1)** and **(p1.2.1.2)** below). Since C is z–containing, $v \in L_1$ implies $a \notin L_1 \cup L_2$.

(p1.2.1.1) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$, $B \cap C = \{s, a\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal , $B \not\subset A$. Since the z–avoiding set A is maximal, $A = S \cap H_1^2$. Therefore there exists $b \in B \cap H_2^2 \setminus L_2$. Let R be the line containing a and z. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be the two half–planes defined R and let $s \in \mathcal{R}_1$. Then $v \in \mathcal{R}_2$.

If $b \in \mathcal{R}_1 \setminus R$, then $\{a, b, v\}$ is z–containing. Since $\{a, b, v\} \subseteq B$, clearly B is z–containing, a contradiction. If $b \in \mathcal{R}_2 \setminus R$, then $\{a, b, s\}$ is z–containing. Since $\{a, b, s\} \subseteq B$, clearly B is z–containing, a contradiction. It follows that $B - s = S \cap R$, and so there exists only one set $Q \in \mathcal{A}$ such that $Q \cap C = \{s, a\}$, and therefore $Q = A$.

(p1.2.1.2) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, v\}$, and so B is adjacent to C in G. As before let $B' = B - s$.

Suppose that there exists $b \in B$ is in the interior of K_{11}. Then $\{b, s, v\}$ is z–containing. Since $\{b, s, v\} \subseteq B$, clearly B is z–containing, a contradiction. Hence B has no points in the interior of K_{11}.

Suppose that there exists $b \in B$ is in the interior of $K_{22} \cap \mathcal{R}_1$. Then $\{b, a, v\}$ is z–containing. Since $\{b, a, v\} \subseteq B' + a$, clearly $B' + a$ is also z–containing, a contradiction.

Suppose that there exists $b \in B \cap H_2^2 \cap \mathcal{R}_2$. Since B is maximal z–avoiding in S, clearly $S \cap H_2^2 \cap \mathcal{R}_2 \subseteq B$. Suppose that there exists $b' \in B \setminus (H_2^2 \cap \mathcal{R}_2)$. Since $(K_{21} \setminus L_1) \cap S = s$, we have: b' belongs to $L_1 \cap H_1^2$.

Since $l_2 \in H_2^2 \cap \mathcal{R}_2 \cap S$, clearly $l_2 \in B$. Then $\{s, v, l_2, b'\}$ is z–containing. Since $\{s, v, l_2, b'\} \subseteq B$, B is z–containing, a contradiction.

Now suppose that $B \cap H_2^2 \cap \mathcal{R}_2 = \emptyset$. Then $B = (S \cap H_1^2 \cap L_1) + s$, and so there exists only one set $Q \in \mathcal{A}$, such that $Q \cap C = \{s, a\}$, and therefore $Q = (S \cap H_1^2 \cap L_1) + s$.

(p1.2.2) Suppose that $|C| = 4$, say $C = \{s, v, a_1, a_2\}$. Since the z–containing set C is minimal, $\{a_1, a_2\} \cap L_i \neq \emptyset$ for each $i \in \{1, 2\}$, say $a_i \in L_i$, and so $v \in L_1$. We will show that in this case there are at most three different sets in $\mathcal{A}$, adjacent to C in G, and each of them avoids exactly one point from $\{v, a_1, a_2\}$ (see **(p1.2.2.1)**, **(p1.2.2.2)**, and **(p1.2.2.3)** below).

(p1.2.2.1) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, a_1, a_2\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal, $B \not\subset A$. We know that $A = S \cap H_1^2$. Therefore there exists $b \in B \cap (H_2^2 \setminus L_2)$. Then $\{s, b, a_1, a_2\}$ is z–containing. Since $\{s, b, a_1, a_2\} \subseteq B$, clearly B is z–containing, a contradiction. Therefore there exists only one set $Q \in \mathcal{A}$, such that $Q \cap C = \{s, a_1, a_2\}$, and therefore $Q = A$.

(p1.2.2.2) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, a_1, v\}$. Since the z–avoiding set B is maximal and $s \in H_2^1 \setminus L_1$, clearly $S \cap H_2^1 \subseteq B$. If there exists $b \in B \cap K_{12} \setminus L_1$, then $\{s, v, b, a_1\}$ is a z–containing set, and since $\{s, v, b, a_1\} \subseteq B$, clearly B is also z–containing, a contradiction. Therefore $B = S \cap H_2^1$, and so there exists only one set $Q \in \mathcal{A}(S)$, such that $Q \cap C = \{s, a_1, v\}$, and therefore $Q = S \cap H_2^1$.

(p1.2.2.3) Now suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$ and $B \cap C = \{s, a_2, v\}$. Since B is a maximal z–avoiding set, $S \cap H_2^2 \subseteq B$. If there exists $b \in B \cap H_1^2 \setminus L_2$, then $\{s, v, b, a_2\}$ is a z–containing set, and since $\{s, v, b, a_2\} \subseteq B$, clearly B is also z–containing, a contradiction. Therefore $B = S \cap H_2^2$, and so there exists only one set $Q \in \mathcal{A}$, such that $Q \cap C = \{s, a_2, v\}$, and therefore $Q = S \cap H_2^2$. Hence we have the desired inequality.

(p2) Suppose that $L_1 = L_2 = L$. Let H_1 and H_2 be the two half–planes defined by L. By definition of L, clearly $A' = (S - s) \cap H_i$ for some $i \in \{1, 2\}$, say $A' = (S - s) \cap H_1$.

If $v \in H_1$, then $v \in A'$, a contradiction. Therefore we can assume that v is an interior point of H_2, i.e. $v \in H_2 \setminus L$. As above let C be a minimal z–containing set such that $v \in C \subseteq A + v$ and let P be the line containing s and z. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the two half–planes defined by P.

(p2.1) Suppose that $A' \subseteq L$. If $s \in H_2$ then $A' + s + v$ is z–avoiding because $v \in H_2 \setminus L$, a contradiction. Therefore $s \in H_1 \setminus L$. Since the z–avoiding set A is maximal, $A' = S \cap L$. Since $A' = (S - s) \cap H_1$ and $A' \subseteq L$, clearly $S \cap H_1 \setminus L = s$.

(p2.1.1) Suppose that $|C| = 3$, say $C = \{s, a, v\}$. We can assume that $a \in \mathcal{P}_1$. Since C

is z–containing, $v \notin \mathcal{P}_1$. We will show that in this case there are at most two different sets in $\mathcal{A}$, adjacent to C in G, each one avoids exactly one point in $\{a, v\}$ (see **(p2.1.1.1)** and **(p2.1.1.2)** below).

(p2.1.1.1) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$, $B \cap C = \{s, a\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal, $B \not\subset A$. Since $S \cap H_1 \setminus L = \{s\}$, clearly $B' \subseteq H_2$. If there exists $b \in B \cap \mathcal{P}_2 \setminus (L \cup P)$, then $\{a, b, s\}$ is z–containing, and so B is z–containing, a contradiction. Therefore $B \cap \mathcal{P}_2 \setminus (L \cup P) = \emptyset$.

If there exists $b \in B \cap H_2 \cap \mathcal{P}_1 \setminus L$ and $b' \in B \cap L \cap \mathcal{P}_2$, then $\{s, b, b', a\}$ is z–containing. Since $\{s, b, b', a\} \subseteq B$, clearly B is z–containing, a contradiction.

If $B \cap H_2 \cap \mathcal{P}_1 \setminus L = \emptyset$ then $B' \subseteq L$, implying that $B \subseteq A$, a contradiction. Therefore $B \cap H_2 \cap \mathcal{P}_1 \setminus L \neq \emptyset$. Since the z–avoiding set B is maximal in S, clearly $B = S \cap H_2 \cap \mathcal{P}_1 + s$ It follows that there exists exactly two sets $Q \in \mathcal{A}$ such that $Q \cap C = \{s, a\}$, and Q is either A or $S \cap H_2 \cap \mathcal{P}_1 + s$.

(p2.1.1.2) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$, $B \cap C = \{s, v\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal , $B \not\subset A$. Therefore $B' \subseteq H_2$. If there exists $b \in B \cap \mathcal{P}_1 \setminus P$, then $\{s, b, v\}$ is z–containing because v is an interior point of $\mathcal{P}_2 \cap H_2$, and so B is z–containing, a contradiction. Therefore $B' \subseteq \mathcal{P}_2 \cap H_2$. Since the z–avoiding set B is maximal, $B' = S \cap H_2 \cap \mathcal{P}_2$. It follows that there exists exactly one set $Q \in \mathcal{A}$ such that $Q \cap C = \{s, v\}$, and $Q = S \cap H_2 \cap \mathcal{P}_2$.

(p2.1.2) Now suppose that $|C| = 4$, say $C = \{s, a_1, a_2, v\}$. We can assume that $a_1 \in \mathcal{P}_1$. Since C is minimal z–containing, $a_2 \in \mathcal{P}_2$. Clearly $z \notin \{a_1, a_2\}$. Therefore each a_i is an interior point of $\mathcal{P}_i$. If $v \notin P$, then $\{a_i, s, v\}$ is z–containing for some i, and so the z–containing set C is not minimal, a contradiction. Therefore $v \in P$.

(p2.1.2.1) Suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$, $B \cap C = \{s, a_1, a_2\}$, and so B is adjacent to C in G. Since the z–avoiding set B is maximal, $B \not\subset A$. Since $S \cap H_1 \setminus L = \{s\}$, clearly $B' \subseteq H_2$. If there exists $b \in B \cap H_2 \cap \mathcal{P}_2 \setminus (L \cup P)$, then $\{a_1, s, b\}$ is z–containing, and so B is z–containing, a contradiction. If there exists $b \in B \cap H_1 \cap \mathcal{P}_2 \setminus (L \cup P)$, then $\{a_1, b, v\}$ is z–containing. Since $B \in \mathcal{A}_s(S)$, clearly $B' + v$ is z–avoiding. Since $\{a_1, b, v\} \subseteq B'$, clearly $B' + v$ is z–containing, a contradiction. If there exists $b \in B' \cap P$, then $\{a_1, a_2, s, b\}$ is z–containing, and so B is z–containing, a contradiction. Therefore $B' \subseteq L$.

Since $A' = S \cap L$, we have: $B \subseteq A$, a contradiction.

(p2.1.2.2) Now suppose that there exists $B \in \mathcal{A}_s(S)$ such that $B \neq A$, $B \cap C = \{s, a_i, v\}$ for some $i \in \{1, 2\}$, and so B is adjacent to C in G. Because of symmetry we can assume that $i = 1$. Since the z–avoiding set B is maximal, $B \not\subset A$. Since $S \cap H_1 \setminus L = s$, clearly $B' \subseteq H_2$. If there exists $b \in B' \cap H_2 \cap \mathcal{P}_2 \setminus P$, then $\{a_1, s, b, v\}$ is z–containing, and so B is z–containing, a contradiction. Therefore $B' \subseteq H_2 \cap \mathcal{P}_1$. Since the z–avoiding set B is maximal, $B' = S \cap H_2 \cap \mathcal{P}_1$. It follows that there exists exactly one set $Q \in \mathcal{A}(S)$ such that $Q \cap C = \{s, a_1, v\}$, and $Q = S \cap H_2 \cap \mathcal{P}_1$.

(p2.2) Now suppose that $A' \not\subseteq L$. Then there exists $a \in A' \cap H_1 \setminus L$, $l_1 \neq l_2$, and z is an interior point of the segment $[l_1, l_2]$. Since $A \in \mathcal{A}_s(S)$, the z–avoiding set A' is not maximal in $S - s$, and so there exists $v \in S - s$ such that $A' + v$ is z–avoiding. Since $A' = (S - s) \cap H_1$, clearly v is an interior point of H_2. Then $\{a, l_1, l_2, v\}$ is z–containing. Since $\{a, l_1, l_2, v\} \subseteq A'$, we have $A' + v$ is z–containing, a contradiction. Hence we have the desired inequality and this concludes the proof of Claim 2.2.2. $\diamondsuit$

Clearly $|\mathcal{A}(S)| = |\mathcal{A}_s(S)| + |\mathcal{A}^s(S)|$ and $|\mathcal{C}(S)| = |\mathcal{C}_s(S)| + |\mathcal{C}(S-s)|$. By Claim 1 and the induction hypothesis $|\mathcal{A}^s(S)| \leq 3|\mathcal{C}(S-s)| + 1$, and by Claim 2 we have $|\mathcal{A}_s(S)| \leq 3|\mathcal{C}_s(S)|$. Hence, $|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1$, and the theorem is proved. $\square$

2.3 *Let S be a finite set of points in the plane and z a point not in S. Suppose that z is in the interior of the convex hull of S. Then $|\mathcal{A}(S)| = 3|\mathcal{C}(S)| + 1$ if and only if $|S| = 4$, the convex hull of S is a quadrilateral Q, and z is the point of intersection of the two diagonals of Q.*

Proof By **2.2**,

$$|\mathcal{A}(S)| \leq 3|\mathcal{C}(S)| + 1. \tag{2.3}$$

It is easy to see that if $|S| = 3$ then we have strict inequality in (2.3) and if $|S| = 4$ then equality holds if and only if the convex hull of S is a quadrilateral Q and z is the point of intersection of the two diagonals of Q.

Suppose, on the contrary, that our claim is not true. Let S be a set of minimum number of elements for which our claim is not true. Let $s \in S$ and $S' = S - s$. Let $\mathcal{A}_1(S') = \mathcal{A}(S) \cap \mathcal{A}(S')$

and $\mathcal{A}_2(S') = \mathcal{A}(S') \setminus \mathcal{A}_1(S')$. Then $\mathcal{A}(S) = \mathcal{A}_s(S) \cup \mathcal{A}^s(S) \cup \mathcal{A}_1(S')$. Let $X \in \mathcal{A}^s(S)$ and $\alpha(X) = X - s$. Obviously $\alpha : \mathcal{A}^s(S) \to \mathcal{A}_2(S')$ is a bijection, and so $|\mathcal{A}^s(S)| = |\mathcal{A}_2(S')|$. Therefore

$$|\mathcal{A}(S)| = |\mathcal{A}_s(S)| + |\mathcal{A}(S')|. \tag{2.4}$$

By Claim 2.2.1, $|\mathcal{A}_s(S)| \leq 3|\mathcal{C}_s(S)|$. Since $|S'| = |S| - 1$, we have, by minimality of S, $|\mathcal{A}(S')| < 3|\mathcal{C}(S')| + 1$. Since $|\mathcal{C}_s(S)| + |\mathcal{C}(S')| = |\mathcal{C}(S)|$, we have from (2.4): $|\mathcal{A}(S)| < 3|\mathcal{C}_s(S)| + 3|\mathcal{C}(S')| + 1 = 3|\mathcal{C}(S)| + 1$. □

The results of this paper were presented at DIMACS Connect Institute (DCI): Graph Theory and its Applications to Problems of Society, DIMACS Center, Rutgers University, July 2001, and at Prague Midsummer Combinatorial Workshop VIII, DIMATIA, Charles University, Prague, August 2001.

Acknowledgement. We are thankful to E. Boros and V. Gurvich for useful and stimulating discussions.

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