DIMACS Technical Report 2002-31
August 2002

Tournaments that omit N_5 are well-quasi-ordered

by

Brenda J. Latka[1]
Department of Mathematics, Lafayette College, Easton, PA 18042
latka@lafayette.edu

[1]Visitor

DIMACS is a partnership of Rutgers University, Princeton University, AT&T Labs-Research, Bell Labs, NEC Research Institute and Telcordia Technologies (formerly Bellcore).

DIMACS was founded as an NSF Science and Technology Center, and also receives support from the New Jersey Commission on Science and Technology.

ABSTRACT

The tournament N_5 can be obtained from the transitive tournament on $\{1, \ldots, 5\}$, with the natural order, by reversing the edges between successive vertices. Tournaments that do *not* have N_5 as a subtournament are said to *omit* N_5. We describe the structure of tournaments that omit N_5 and use this with Kruskal's Tree Theorem to prove that this class of tournaments is well-quasi-ordered. The proof involves an encoding of the indecomposable tournaments omitting N_5 by a finite alphabet.

The main technical problem is giving an explicit description of the indecomposable tournaments omitting N_5. The key to the proof that the explicit description is complete is the observation that for any indecomposable tournament T with $n > 1$ vertices, there is a proper indecomposable subtournament of T with $n-2$ or $n-1$ vertices. Thus the claim can be verified by a natural inductive procedure; it suffices to check that for any tournament T in the explicitly given list, any indecomposable extension of T by at most 2 vertices that omits N_5 will also be found in our list.

Keywords: alphabet, antichain, decidability, decomposition, embedding, encoding, forbidden, Higman, indecomposable, Kruskal, local order, obstruction, polynomial time, substructure, subtournament, tournament, well-quasi-order, words

1 Introduction

This paper gives a structural characterization of the tournaments that do not have a certain forbidden subtournament N_5. The tournament N_5 can be obtained from the transitive tournament on $\{1, \dots, 5\}$, with the natural order, by reversing the edges between successive vertices. N_5 is shown in Figure 1; all tournaments considered in this paper are finite.

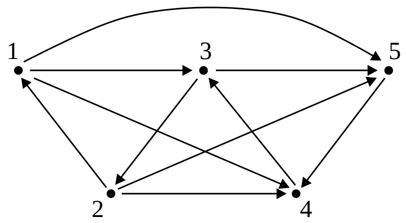

Figure 1: N_5

For tournaments T_1 and T_2, if T_1 is isomorphic to a subtournament of T_2 then we say that T_1 *embeds* in T_2 (in symbols: $T_1 \leq T_2$). Otherwise we say T_2 *omits* T_1. The structural characterization of the tournaments that omit N_5 is used with Kruskal's Tree Theorem to prove that:

Theorem 1.1 *If $\mathcal{Q}$ is an infinite set of tournaments each omitting N_5 then there exist non-isomorphic tournaments $T_1, T_2 \in \mathcal{Q}$ such that $T_1 \leq T_2$.*

We can view Theorem 1.1 from a Kuratowski point of view, thinking of N_5 as an obstruction or forbidden substructure that defines a class of tournaments. Any tournament defines a class in this manner.

Definition 1.2 *Let T be a tournament. We denote by $\mathcal{Q}_T$ the class of all tournaments that omit T. T is referred to as the* forbidden subtournament *in this context.*

One of the most striking examples is the class $\mathcal{Q}_{C_3}$ of transitive tournaments. This class is determined by the oriented 3-cycle and has a simple structure theory. If T_1, T_2 each omit C_3 then $T_1 \leq T_2$ or $T_2 \leq T_1$, that is, given two transitive tournaments, one embeds in the other.

We can also view Theorem 1.1 from a quasi-ordering point of view. A quasi-ordering is a class together with a reflexive and transitive binary relation, $\leq$, on that class. An antichain in a quasi-ordering is a subclass whose elements are pairwise incomparable. A well-quasi-ordering (wqo) is a quasi-ordering containing no infinite strictly descending sequence and no infinite antichain. We say a class $\mathcal{Q}$ of tournaments is well-quasi-ordered if $(\mathcal{Q}, \leq)$ is well-quasi-ordered, where $\leq$ is the embeddability relation.

In particular, a class $\mathcal{Q}_T$ will be well-quasi-ordered if and only if this class contains no infinite antichain, since there can be no infinite strictly descending sequence of finite tournaments. Equivalently, $\mathcal{Q}_T$ will be well-quasi-ordered if and only there is no infinite antichain of tournaments with T as an element. If a class of tournaments determined by a forbidden subtournament has a reasonably simple structure theory, Higman's Lemma or Kruskal's Tree Theorem can be used to show that the class is well-quasi-ordered under the embeddability relation.

We will call a tournament T *tight* if the corresponding class $\mathcal{Q}_T$ is well-quasi-ordered. T is thought of as a negative constraint in this context. As an example, we note that any transitive tournament L is tight since the class $\mathcal{Q}_L$ consists of tournaments of bounded order. This follows from a result of Erdös and Moser [12] or by the following simple argument: using induction on n, if a tournament T has at least 2^{n-1} vertices, then a transitive tournament on n vertices embeds in T. In the induction step, one considers any vertex a in a tournament T with at least 2^n vertices and partitions the vertices in $V(T) - \{a\}$ into the set of those that dominate a and the set of those that are dominated by a. One of these sets contains at least half the vertices and the induced tournament on this set embeds a transitive tournament on n vertices, L_n. The induced subtournament on $V(L_n) \cup \{a\}$ is a transitive tournament on $n + 1$ vertices.

We have obtained the precise classification of the tight tournaments:

Theorem 1.3 (Tightness Theorem) *A class of the form $\mathcal{Q}_T$ is wqo if and only if T is transitive, or T is isomorphic to a subtournament of N_5, $IC_3(I, I, L_3)$, or $C_3(I, I, L_3)I$.*

This has been discussed at length in [30], using the main theorems of [32] and the present paper. The tournament $IC_3(I, I, L_3)$ is shown in Figure 2. It has a vertex that dominates the remaining vertices. The subtournament on these dominated vertices is a 3-cycle in which one vertex has been replaced by a transitive tournament on 3 vertices. This substitution operation will be discussed in more detail below. The tournament $C_3(I, I, L_3)I$ is the dual of $IC_3(I, I, L_3)$, that is, the tournament obtained from $IC_3(I, I, L_3)$ by reversing the orientation of all the edges. N_5 is self-dual.

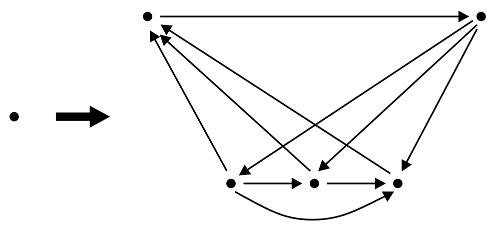

Figure 2: $IC_3(I, I, L_3)$

Our classification of the tight tournaments involves three quite separate steps. First we show [30] that any tight tournament other than a transitive one embeds in one of three tournaments, N_5, $IC_3(I, I, L_3)$, or $C_3(I, I, L_3)I$. At this point, it is already clear that the tight tournaments can be correctly identified by some algorithm, since there are only finitely many cases left open. The second and third steps are verifications that N_5 and $IC_3(I, I, L_3)$ are tight, noting that if a tournament is tight then its dual is tight. Each of these two results requires a rather long structural analysis. The present paper is devoted entirely to the structure theorem for the class $\mathcal{Q}_{N_5}$. These two structure theorems both depend on an application of Kruskal's Tree Theorem to a set of isomorphism types encoded by words in a finite alphabet, though the proofs have little in common beyond this basic combinatorial idea.

The idea of a tight tournament can be generalized to that of a tight set of tournaments.

Definition 1.4 *Let Γ be a set of tournaments.*

- *A tournament T is said to* omit Γ *if no subtournament of T is isomorphic to any element of Γ.*
- *We denote by $\mathcal{Q}_\Gamma$ the class of all tournaments T that omit Γ. Γ will be referred to as the set of* forbidden subtournaments *in this context.*

We will call a finite set Γ of tournaments *tight* if the corresponding class $\mathcal{Q}_\Gamma$ is well-quasi-ordered, in particular, if this class contains no infinite antichain.

The following question was raised in [5]:

> Is there an algorithm that determines, for every finite set Γ of tournaments, whether the corresponding class $\mathcal{Q}_\Gamma$ is tight?

The classification of tight sets Γ in general remains open. It seems considerably easier to show (non-constructively) that there is an algorithm that identifies tight tournaments than it is to actually produce an exact criterion for tightness. The main result of [27] is that any nontransitive tournament with at least 18 vertices is not tight. The proof uses the fact that the tournaments C_3L_2, L_2C_3, and $C_3(I, I, L_4)$ are not tight (Definitions are given in §3.1.) This fact is demonstrated by constucting infinite antichains in $\mathcal{Q}_{C_3L_2}$, $\mathcal{Q}_{L_2C_3}$, and $\mathcal{Q}_{C_3(I,I,L_4)}$. There is overlap here, only two antichains are required [34]. In fact, it is shown in [7] that given $k \geq 0$, there is a fixed finite set of infinite antichains that can be used to show that any k-set of tournaments is not tight. This provides some support for the view that the general question of whether or not a k-set of tournaments is tight is algorithmically decidable for fixed k. For more of what is known on this problem see [7, 8, 34, 35].

We will now sketch the proof of the main result of the present paper, which is a restatement of Theorem 1.1:

Theorem 1.5 *$\mathcal{Q}_{N_5}$ is well-quasi-ordered.*

We provide a structural characterization of the tournaments omitting N_5. This structural characterization makes use of the substitution operation for composing two tournaments.

Definition 1.6 *Let T_1 and T_2 be tournaments with disjoint vertex sets and let v be a vertex of T_1. We construct a tournament T* by substituting T_2 for v in T_1 *as follows:*

The vertex set of T is $V(T_1 - v) \cup V(T_2)$.
The restriction of T to $V(T_1 - v)$ is $T_1 - v$.
The restriction of T to $V(T_2)$ is T_2.
For $a \in V(T_1 - v)$ and $b \in V(T_2)$, (a, b) is an edge of T if and only if (a, v) is an edge of T_1.

A tournament is *indecomposable* (also called *simple* [16]) if it cannot be obtained from tournaments of smaller order by substitution. (A formal definition is given in §3.) Any tournament can be decomposed, in a tree-like way, into indecomposable pieces. Since N_5 is indecomposable, to describe the class of tournaments omitting N_5, we will show that it suffices to describe the class of indecomposable tournaments omitting N_5. This class is described in the following theorem.

Theorem 1.7 *T is an indecomposable tournament omitting N_5 if and only if T is isomorphic to one of the following tournaments: I, L_2, B_6, P_7, S_{2n+1}, or D_{2n+1}, for $n \geq 1$.*

The tournaments I and L_2 are the tournaments with one vertex and one edge, respectively. P_7 is the quadratic residue tournament, also known as the Paley tournament, of order 7. Since P_7 is vertex transitive, we can understand its structure by considering the subtournaments dominating and dominated by the vertex 0, as shown in Figure 3.

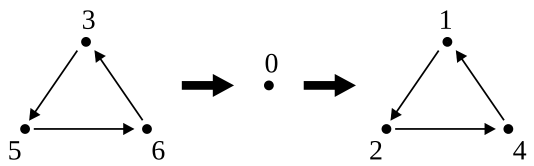

Figure 3: P_7 from the point of view of vertex 0

B_6 is the tournament obtained by deleting any vertex in P_7 and is shown in Figure 4.

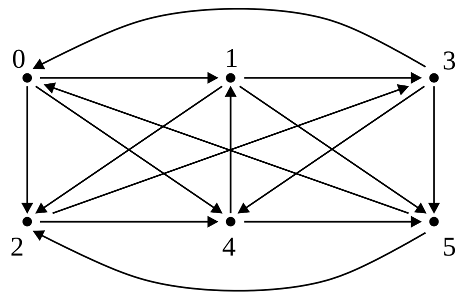

Figure 4: B_6

S_{2n+1} can be obtained from the directed cycle on $2n + 1$ vertices by adding edges from each vertex to all vertices at most n steps ahead. S_5, with $n = 2$, is shown in Figure 5.

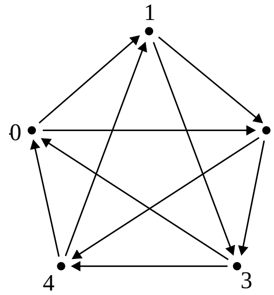

Figure 5: S_5

Equivalently, by Fact 4.2 and Lemma 4.4, S_{2n+1} can be obtained from the transitive tournament on $\{1, \ldots, 2n+1\}$ by partitioning the vertices into sets of odd and even vertices, and reversing the edges that connect vertices of differing parity. In the illustration shown, this description is realized by renaming the vertices $0, 1, 2, 3, 4$ to $1, 3, 5, 2, 4$, respectively.

D_{2n+1} is obtained from the same transitive tournament with the same partitioning of the vertices by reversing the edges that connect two odd vertices. D_5 is shown in Figure 6.

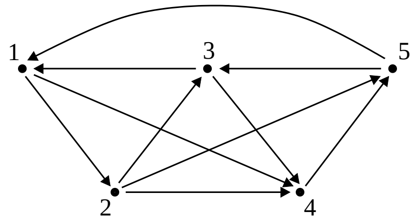

Figure 6: D_5

The tournament $S_3 \simeq D_3$ is the cycle on 3 vertices, which we also call C_3. Formal definitions of these tournaments are given in §3 and §4, where we show that they omit N_5.

The main technical problem is to prove Theorem 1.7. We will show that each of the tournaments I, L_2, B_6, P_7, S_{2n+1}, and D_{2n+1}, for $n \geq 1$, is indecomposable and omits N_5. We will then show that every indecomposable tournament in $\mathcal{Q}_{N_5}$ is isomorphic to one of the tournaments listed. The key to the proof that our list is complete is the observation that for any indecomposable tournament T with $n > 1$ vertices, we can find a proper indecomposable subtournament of T with $n-2$ or $n-1$ vertices. Thus our claim can be verified by a natural inductive procedure; it suffices to check that for any tournament T in our explicitly given list, any indecomposable extension of T by at most 2 additional vertices that omits N_5 will also be found in our list. We call this result, or family of results, the Extension Lemma. The detailed verification of the Extension Lemma is in the final sections of this paper.

Given the explicit description of the indecomposable tournaments in $\mathcal{Q}_{N_5}$, we encode the isomorphism types of these tournaments by words in a finite alphabet, of a type to which Kruskal's Tree Theorem applies. Theorem 1.5 then follows.

Something should also be said about the complexity of the arguments required to prove an explicit structure theorem for $\mathcal{Q}_{N_5}$ or $\mathcal{Q}_{IC3(I,I,L_3)}$. Both arguments are intricate, the latter considerably more so than the former. This may be contrasted with the straightforward structure theorem for the class $\mathcal{Q}_{C_3}$ of transitive tournaments. However the *only* tournaments T for which structure theorems of this particular type are true are the transitive tournaments on the one hand (and in this case the class $\mathcal{Q}_T$ contains only finitely many isomorphism types), and the subtournaments of N_5, $IC_3(I, I, L_3)$, and $C_3(I, I, L_3)I$ on the other. We sought the *precise* borderline between structure theorems on one side, and the failure of wqo on the other, and we found it. That the complexity of the proofs should increase noticeably near this boundary is not remarkable, particularly since the algorithmic problem under consideration remains open in general. For an example of a comparatively straightforward structure theorem intermediate between the case of transitive tournaments and $\mathcal{Q}_{N_5}$ we give the following characterization of $\mathcal{Q}_{C_3I}$: a tournament T omits C_3I, shown in Figure 7, if and only if (in the notation of §3.1 below) T is of the form LS with S a local order (cf. §4.1) and L transitive. This characterization is based on Definition 4.1 and Lemma 7.1.

Figure 7: IC_3 and C_3I

The structure of this paper is as follows. In §2 we provide some general background for the type of classification problem we are discussing here. In §3 we introduce some general notations and terminology, including the important concept of an indecomposable tournament, and we verify that $\mathcal{Q}_{N_5}$ is determined in a straightforward way by its subclass of indecomposable members, which we call $\mathcal{A}$. In §4 we exhibit various tournaments in the class $\mathcal{A}$ explicitly; by the end of the paper we will have shown that this list, which we call $\mathcal{A}_0$, is exhaustive, up to isomorphism. §5 contains some standard theorems of well-quasi-ordering theory and introduces the concept of a vertex encoding by a finite (or more generally, wqo) alphabet. In this section we show that the class of tournaments, $\mathcal{A}_0$, presented in §4 has a finite vertex encoding and how Kruskal's Tree Theorem can be applied to prove that $\mathcal{Q}_{N_5}$ is wqo, if $\mathcal{A}_0 = \mathcal{A}$. In §6 we prove the Reduction Lemma for indecomposable tournaments:

Lemma 6.1 (Reduction Lemma) *Let T be an indecomposable tournament with $|T| > 1$. Then there is a proper indecomposable subtournament T_0 of T such that $|T| - |T_0| \leq 2$.*

The Reduction Lemma is a special case of a result of Schmerl and Trotter [41] for binary relations. A similar result was obtained by Ille [22] and also by Ehrenfeucht and Rozenberg [11]. We will give a proof in §6 in the context of tournaments.

What remains to be proved at this point is just that the class $\mathcal{A}$ coincides with the closure under isomorphism of the class $\mathcal{A}_0$ of tournaments introduced in §4. Given the Reduction Lemma, to verify that $\mathcal{A}_0 = \mathcal{A}$ it suffices to show:

Lemma 6.2 (Extension Lemma) *Let T be an indecomposable tournament that omits N_5 and is an extension by at most two vertices of a tournament $T_0 \in \mathcal{A}_0$. Then $T \in \mathcal{A}_0$.*

The proof of the Extension Lemma requires a detailed consideration of the cases that correspond to various choices of T_0, with the smaller tournaments requiring the bulk of the analysis. The verification of the Extension Lemma is spread over §§7-10. In §11 we summarize the proof of the Extension Lemma and prove the main results: Theorem 1.5 and Theorem 1.7.

One can use results of Schmerl and Trotter [41] about indecomposable tournaments to simplify the proof of the Extension Lemma. Their paper states the results we need without proof. We do not use these results here, but instead give a complete and self-contained presentation. For a presentation using the Schmerl-Trotter results see [31].

2 Background

In this section we will present a fuller account of the contexts in which the type of classification problem considered here arises: homogeneous directed graphs, polynomial-time complexity, and the general analysis of structure theorems in a broad sense. None of this is directly relevant to the proof of the main theorem of this paper; it is intended to explain more fully the content of the Tightness Theorem (which of course provides only a partial answer to the recognition problem for tight finite sets of tournaments).

A directed graph G is said to be *homogeneous* if any isomorphism between two of its finite induced subgraphs extends to an automorphism of G. There are $2^{\aleph_0}$ isomorphism types of countable homogeneous graphs. These have been explicitly classified in [6], and with countably many exceptions they are associated with classes of finite tournaments. The most interesting examples are in the list of exceptions. It was pointed out in [5] that certain problems of a general character concerning homogeneous directed graphs are not settled by this classification, though of course many such problems are trivialized by the classification. One of the problems discussed in detail in [5] as an example of a problem that remains nontrivial after the classification theorem is the following: determine whether a class of directed graphs given by specifying finitely many forbidden finite directed subgraphs contains countably or uncountably many isomorphism types of countable homogeneous directed graphs. The classification theorem, combined with some additional argument, casts considerable light on this problem, and in fact shows that it is equivalent to the problem of determining whether a class $\mathcal{Q}_\Gamma$ of tournaments corresponding to a finite set Γ of forbidden finite subtournaments is wqo or not. Further discussion, with the details of the reduction, are presented in [5].

The connection with issues of polynomial time complexity comes about as follows. We consider recognition problems for classes of structures. One begins with a collection $\mathcal{Q}$ of finite structures (typically graphs, directed graphs, and similar combinatorial structures, possibly with vertex or edge colorings); see, for example, [2, 9, 10, 14, 21]. We assume that the class $\mathcal{Q}$ is given very explicitly, and specifically that the problem of determining whether a given structure lies in $\mathcal{Q}$ is decidable by an algorithm that runs in polynomial time (as a function of an appropriate measure of the complexity of the structures); this will certainly be the case if $\mathcal{Q}$ is determined by finitely many forbidden substructures. In such a context, we consider *recognition problems* for subclasses $\mathcal{X} \subseteq \mathcal{Q}$. The recognition problem for $\mathcal{X}$ is the problem of determining whether a structure belongs to $\mathcal{X}$, where in practice $\mathcal{X}$ is given by some combinatorially interesting property for which one knows no polynomial-time recognition algorithm in general. When the class $\mathcal{Q}$ is wqo with respect to the embeddability relation $\leq$ (or for that matter, with respect to any one of several natural variations on the embeddability relation), and when the class $\mathcal{X}$ is downward-closed with respect to this relation, in the sense that $S \leq T \in \mathcal{X} \Rightarrow S \in \mathcal{X}$, then a portion of the theory is trivialized by the following general principle:

Proposition 2.1 *Let $\mathcal{Q}$ be a wqo class of finite structures, with respect to the embeddability relation $\leq$. Assume that the recognition problem for $\mathcal{Q}$ is solvable in polynomial time. Then for any subclass $\mathcal{X}$ of $\mathcal{Q}$ that is downward-closed with respect to $\leq$, the recognition problem for $\mathcal{X}$ is also solvable in polynomial time.*

Proof: If Γ is a set of representatives for the $\leq$-minimal elements of $\mathcal{Q} - \mathcal{X}$, up to isomorphism, then Γ is an antichain with respect to $\leq$, and as $\mathcal{Q}$ is wqo it follows that Γ is finite. A structure $T \in \mathcal{Q}$ will belong to $\mathcal{X}$ if and only if none of the structures in Γ embeds in T. This is a property that can be checked in polynomial time. ■

There are useful generalizations of the principle, which would take us too far afield to explore here, but we remark that the embeddability relation $\leq$ can usefully be replaced here by a number of other relations, subject to some relatively minor constraints – notably, we require that for any fixed structure T_0, it should be possible to check in polynomial time whether $T_0 \leq T$ (T varies). For the usual sorts of embeddability relations, this can be checked in polynomial time, where the degree of the corresponding polynomial is $|T_0|$. For a different choice of $\leq$ see the extensive work of Robertson and Seymour, e.g. [38, 39, 40]. On the other hand, the claim that for downward-closed subclasses the theory of polynomial time complexity is trivial on wqo classes should be taken with a grain of salt since the argument just given is highly non-effective [17, 18]. In practice this proposition is used in conjunction with well-quasi-ordering theorems of Kruskal's type, and from such results one gets no way to extract effective bounds on the size of the set Γ occurring in the proof, or the degree of the corresponding polynomial, and thus one has very little information about the algorithms that are sought, apart from their existence.

Our third and final point of view is related to the second, but at a vastly greater level of generality, and a considerably less precise context. It is reasonable to view the wqo property, for classes determined by finitely many forbidden substructures, as offering one precise interpretation of what it means to say that a class of structures has a satisfactory structure theory. The line of thought involved in making this sort of identification runs as follows. A structure theory typically involves some sort of canonical notation, based on a finite alphabet, for the isomorphism types of elements of a class of structures. Such notations tend to be subject to Kruskal's Tree Theorem, and thus produce wqo theorems.

This particular scenario is very well illustrated by the whole of the present paper. To round out this argument in the context of tournaments, one would also like to say that whenever a class of the form $\mathcal{Q}_\Gamma$ is wqo, this fact can be proved by applying Kruskal's Tree Theorem to a suitable vertex encoding in the sense of §5 below. This is true when $|\Gamma| = 1$ by the results of [30, 32] and the present paper, and we conjecture that it is true in general.

3 The structure of $\mathcal{Q}_{N_5}$

After some preliminary definitions, we will define the notion of indecomposability and show that $\mathcal{Q}_{N_5}$ is generated in a straightforward manner by its indecomposable members.

3.1 Basic definitions

Let T be a tournament. We write $V(T)$ and $E(T)$ for the sets of vertices and edges of T respectively. For tournaments T_1, T_2 we write $T_1 \subseteq T_2$ if T_1 is a subtournament of T_2, $T_1 \leq T_2$ if T_1 is isomorphic to a subtournament of T_2, and $T_1 \simeq T_2$ if T_1 is isomorphic to T_2. If $T_1 \leq T_2$ we say that T_1 *embeds* in T_2. If T_1 does not embed in T_2 we say that T_2 *omits* T_1.

On occasion we write T for $V(T)$. If $X \subseteq V(T)$ we write $T \restriction X$ for the subtournament of T induced on X. If $T_0 \subseteq T$ and $X = V(T_0)$ we write $T - T_0$ or $T - X$ for $T \restriction V(T) - X$. $|T|$ means the cardinality of $V(T)$, which is called the *order* of T.

If x and y are vertices in the tournament T we write $x \to y$, read "x dominates y," to mean that $(x, y) \in E(T)$. For subsets X, Y of $V(T)$ we write $X \Longrightarrow Y$ if all vertices in X dominate all vertices in Y (in particular X and Y will be disjoint). The notations $x \Longrightarrow Y$ and $X \Longrightarrow y$ are abbreviations for $\{x\} \Longrightarrow Y$ and $X \Longrightarrow \{y\}$ respectively. When similar notations are applied to subtournaments of T, we are referring to the associated vertex sets.

For x a vertex of the tournament T, we write x' and $'x$ for the sets $\{y \in V(T) : x \to y\}$ and $\{y \in V(T) : y \to x\}$ respectively; the cardinalities of these sets are called the *outdegree* and the *indegree* of x in T, respectively.

The *dual* of T is the tournament on $V(T)$ obtained by reversing the orientation of every edge of T. T is *self-dual* if T is isomorphic to its dual.

It will be useful to have names for certain specific tournaments.

A tournament is *transitive* if its edge relation is transitive. L_n is the transitive tournament with vertex set $\{1, \ldots, n\}$ and edge relation $\{(i, j) : i < j\}$. L_1 will usually be denoted I.

C_3 is the tournament with vertex set $\mathbb{Z}/3\mathbb{Z}$ and edge relation $\{(i, i+1) : i \in \mathbb{Z}/3\mathbb{Z}\}$. Any tournament isomorphic to C_3 will be called a 3-cycle, and similarly any ordered triple of vertices (a, b, c) in a tournament will be called a 3-cycle if $a \to b \to c \to a$.

N_n is the tournament on $\{1, \ldots, n\}$ obtained from L_n by reversing the orientation of the edges that connect successive vertices. The edge relation on N_n is therefore

$$\{(i, j) : i + 1 < j \text{ or } i = j + 1\}.$$

We are specifically interested in N_5, and we will frequently have to verify that certain tournaments contain subtournaments isomorphic to N_5. For this purpose it is useful to note that vertex 1 in N_5 dominates a 3-cycle $C = (3, 5, 4)$ and that vertex 2 dominates 1 and dominates exactly two of the vertices in C.

$C_3(I, I, L_n)$ denotes the tournament obtained by substituting the tournament L_n for one of the vertices of C_3. Another construction we require is that of adjoining one additional vertex v to a tournament T, with v dominating or dominated by T. As examples we have the tournaments IC_3 and C_3I, shown in Figure 7, and $IC_3(I, I, L_3)$, shown in Figure 2 and occurring in the Tightness Theorem. More generally, T_1T_2 denotes the tournament obtained by substituting the tournaments T_1 and T_2 for the first and last vertices of L_2, respectively.

Some other tournaments introduced below in §4 will play a key role. These are the Paley tournament P_7 on seven vertices together with some of its subtournaments, and two infinite families, the local orders S_{2n+1} and the related partitioned-transitive tournaments D_{2n+1}.

3.2 Indecomposable tournaments

Definition 3.1 *Let T be a tournament, $T_0 \subseteq T$, $a \in V(T - T_0)$.*

1. *The* type *of a over T_0 is the function $\tau : V(T_0) \to \{1, -1\}$ that assigns to each vertex v of T_0 the orientation of the edge connecting a and v, with the convention that (a, v) is oriented positively and (v, a) is oriented negatively. If $X \subseteq V(T)$ and $a \in T - X$, we also talk about the type of a over X, meaning the type of a over $T \restriction X$.*

2. *The type of a over T_0 is* mixed *if edges of both possible orientations occur between a and $V(T_0)$, and is* unmixed *otherwise.*

Definition 3.2 *Let T be a tournament.*

1. *A nonempty set B of vertices of T is called a* block *if every vertex of B has the same type over $T - B$. A block is* trivial *if $B = V(T)$ or B consists of just one vertex. It is worth noting that B is a block in T if and only if there do not exist vertices $x, y \in B$ and a vertex $z \in V(T) - B$ such that $x \to z \to y$. We also refer to $T \restriction B$ as a block in T.*

2. *A* congruence *on T is an equivalence relation on $V(T)$ whose equivalence classes are blocks. The congruence is nontrivial if one of its equivalence classes is nontrivial.*

3. *If $\sim$ is a congruence on T, the* quotient *$T/\!\sim$ is the tournament whose vertices are the equivalence classes of $\sim$, with edge relation induced by $E(T)$:*

$$x/\!\sim\; \to y/\!\sim \text{ in } T/\!\sim \text{ if and only if } x \to y \text{ in } T$$

4. *T is* decomposable *if there is a nontrivial congruence on T (or, equivalently, T has a nontrivial block.) We say that T is indecomposable (also called simple [16]) if $V(T) \neq \emptyset$ and there is no nontrivial congruence on T.*

Definition 3.3 *A class $\mathcal{X}$ of tournaments is said to be* closed under composition *if for every tournament T and for every congruence $\sim$ on T, if $(T/{\sim})$ is in $\mathcal{X}$ and the induced subtournament on each $\sim$-class belongs to $\mathcal{X}$, then $T \in \mathcal{X}$.*

For $\mathcal{C}$ a class of tournaments, $\mathcal{X}(\mathcal{C})$ denotes the smallest class, under inclusion, of tournaments that contains $\mathcal{C}$ and is closed under composition.

We observe that if $\mathcal{C}$ is closed under isomorphism then $\mathcal{X}(\mathcal{C})$ is also closed under isomorphism.

The composition operation here involves replacing the vertices of one tournament by copies of various other tournaments, using the substitution operation give in Definition 1.6. This composition operation is a more general version of a wreath product or lexicographic product.

3.3 The structure of $\mathcal{Q}_{N_5}$

Lemma 3.4 *If $\mathcal{A}$ is the class of indecomposable tournaments in a class of downward closed (with respect to embedding) tournaments $\mathcal{Q}$, then $\mathcal{Q} \subseteq \mathcal{X}(\mathcal{A})$.*

Proof:

Suppose that $T \in \mathcal{Q}$. We show by induction on $|T|$ that $T \in \mathcal{X}(\mathcal{A})$. If T is indecomposable then $T \in \mathcal{A} \subseteq \mathcal{X}(\mathcal{A})$; so we suppose that $\sim$ is a nontrivial congruence on T. Then $T/\sim$ and the induced subtournaments on the $\sim$-classes are of smaller order, and again belong to $\mathcal{Q}$, and so by induction both $T/\sim$ and the induced subtournaments on the $\sim$-classes belong to $\mathcal{X}(\mathcal{A})$. As $\mathcal{X}(\mathcal{A})$ is closed under composition, $T \in \mathcal{X}(\mathcal{A})$, as claimed. ■

Lemma 3.5 *If Γ is a set of indecomposable tournaments, and $\mathcal{A}_\Gamma$ is the class of indecomposable tournaments that omit Γ, then $\mathcal{Q}_\Gamma = \mathcal{X}(\mathcal{A}_\Gamma)$.*

Proof: Since the members of Γ are indecomposable, it is easily seen that $\mathcal{Q}_\Gamma$ is closed under composition. Therefore $\mathcal{X}(\mathcal{A}_\Gamma) \subseteq \mathcal{Q}_\Gamma$.

As $\mathcal{Q}_\Gamma$ is downward closed, $\mathcal{Q}_\Gamma \subseteq \mathcal{X}(\mathcal{A}_\Gamma)$ by the previous lemma. ■

Lemma 3.6 *N_n is indecomposable.*

Proof: Let B be a block in N_n containing more than one vertex and let i and j be the minimum and maximum elements of B respectively. Then $i < j$ and thus if $i > 1$ then $i-1$ has a mixed type over B, a contradiction. Thus $i = 1$ and dually $j = n$. Each vertex in $\{2, \ldots, n-1\}$ has a mixed type over $\{1, n\}$, so each of these vertices belongs to B and $B = V(N_n)$. Thus N_n is indecomposable. ■

Definition 3.7 *Let $\mathcal{A}$ be the class of indecomposable tournaments that omit N_5.*

Lemma 3.8 $\mathcal{Q}_{N_5} = \mathcal{X}(\mathcal{A})$.

Proof: Since N_5 is indecomposable, this follows from Lemma 3.5. ■

In the remainder of this paper we will be concerned with two distinct issues: the explicit identification of the class $\mathcal{A}$, and the exploitation of this identification to prove the main theorem. The former involves a detailed inductive analysis of the class $\mathcal{A}$, while the latter is really a chapter in the general theory of well-quasi-orderings.

4 Some tournaments in $\mathcal{A}$

In the present section we will give an explicit description of all tournaments in the class $\mathcal{A}$ of indecomposable tournaments omitting N_5, up to isomorphism. We will prove that the tournaments described do indeed belong to the class $\mathcal{A}$, but we defer the proof that the list given is exhaustive until we have shown how this fact may be used to prove the main theorem, which we do in §5.

4.1 Local orders

Definition 4.1 *A tournament T is said to be a* local order *if for all vertices x of T, $T{\restriction}x'$ and $T{\restriction}'x$ are transitive; equivalently, T omits IC_3 and C_3I.*

Local orders are defined in [3]. They are also described in [26, 29, 36] with the names *vortex-free*, *locally transitive*, and *tournaments with property $\mathcal{L}$*, respectively. We observe that if T is a local order then the dual of T is also a local order. The following characterization is quite useful.

Fact 4.2 *Let T be a tournament. The following are equivalent:*

1. T is a local order.

2. There exists a transitive tournament L and a partitioning of $V(L)$ into two sets such that T is isomorphic to a tournament obtained from L by reversing the orientations of all edges between the two sets.

Proof: If T is obtained as described in condition 2, then we check directly that x' and $'x$ are transitive for x a vertex of T. Assume that $V(L)$ is partitioned into a set of red vertices and a set of blue vertices and that x is a red vertex. If r is a red vertex then $x \to r$ if and only if $r > x$. If b is a blue vertex then $x \to b$ if and only if $b < x$. If r and b are in x' then $b < x < r$ and $r \to b$. A dual argument holds for $'x$.

If we begin with a local order T then in order to obtain a representation of the type given in condition 2, we must partition the vertices of T into two sets in such a way that after reversing the orientations of all edges between the two sets, a transitive tournament results. We claim that for any vertex x of T, $\{x\} \cup x'$ and $'x$ gives such a partition. If not, there is a

3-cycle (a, b, c), after reversing the orientations of all edges between the sets, with two of the vertices in one set and one in the other, in which case $T\restriction\{x, a, b, c\} \simeq IC_3$, a contradiction. ■

Using this result one obtains a classification of the indecomposable local orders.

Definition 4.3 *For $n \geq 1$ the tournament S_{2n+1} has vertex set $\mathbb{Z}/(2n+1)\mathbb{Z}$ and edge relation*

$$\{(i, j) : j - i \in \{1, \ldots, n\}\}.$$

The tournaments S_{2n+1} are described by Kendall and Babington Smith [25] and given the name *highly-regular* tournaments by Moon [36].

Lemma 4.4 *The indecomposable local orders of order at least 3 are exactly the tournaments S_{2n+1} for $n \geq 1$.*

Proof: The shift automorphisms $x \mapsto x+i$ for $i \in \mathbb{Z}/(2n+1)\mathbb{Z}$ act transitively on $V(S_{2n+1})$, so in order to see that S_{2n+1} is a local order it suffices to check that $0'$ and $'0$ are transitive, as they clearly are.

To see that S_{2n+1} is indecomposable, observe that any edge (i, j) is contained in a 3-cycle $(i, j, j+n)$. Thus any nontrivial block B in S_{2n+1} contains a 3-cycle. But if B contains i and j with $i \to j$ then B contains all vertices between i and j (in cyclic order), and thus if B contains a 3-cycle it embeds S_{2n+1}. (Part of this argument holds more generally: a nontrivial congruence on a local order has transitive classes.)

Now let T be any finite indecomposable local order. It suffices to check that T is of odd order, and is uniquely determined by its order, as this forces $T \simeq S_{2n+1}$. We use the representation of T as the result of partitioning the vertices of a transitive tournament L into two sets, A and B, and reversing the orientation of the edges between A and B. If successive vertices of L belong to the same set, then they form a block in T, which is a contradiction. Thus the vertices of L lie alternately in A and B. We may assume the first vertex of L lies in the set A. If the last vertex of L lies in B then the first and last vertices of L form a block in T, and we have a contradiction.

It follows that L is of odd order, and that A consists of vertices occurring in odd positions, and B consists of vertices occurring in even positions. Thus T is determined by its order, as claimed. ■

Lemma 4.5 *Local orders omit N_5.*

Proof: Local orders omit N_5 since they omit IC_3. ■

Lemma 4.6 *For $n \geq 1$, S_{2n+1} is an indecomposable tournament omitting N_5.*

Proof: S_{2n+1} is indecomposable by Lemma 4.4. S_{2n+1} omits N_5 by the previous lemma. ■

Thus the tournaments S_{2n+1} lie in the class $\mathcal{A}$ of indecomposable tournaments omitting N_5. We will now examine a second closely related infinite family of tournaments in $\mathcal{A}$.

4.2 Partitioned-transitive tournaments: a variant of local orders

Definition 4.7 *A tournament T is said to be a* partitioned-transitive *tournament if there is a subtournament $T_0 \subseteq T$ such that the tournament obtained by replacing T_0 by its dual is transitive.*

We observe that if T is a partitioned-transitive tournament then so is the dual of T.

Definition 4.8 *For $n \geq 1$ the tournament D_{2n+1} has vertex set $\{1, \dots, 2n+1\}$ and edge relation*

$$\{(i,j) : i < j \text{ and } i,j \text{ are not both odd, or } i > j \text{ and } i,j \text{ are both odd } \}.$$

Recall that S_{2n+1} can be obtained from a transitive tournament of order $2n+1$ by partitioning its vertex set into sets of odd and even vertices, and reversing the edges that connect vertices of differing parity. In much the same spirit, D_{2n+1} is obtained from the same transitive tournament and the same partition by reversing the edges that connect two odd vertices.

Lemma 4.9 *The indecomposable partitioned-transitive tournaments of order at least 3 are exactly the tournaments D_{2n+1} for $n \geq 1$.*

Proof: Let T_0 be the induced subtournament on the odd vertices of D_{2n+1}. The tournament obtained by replacing T_0 by its dual is transitive and hence D_{2n+1} is a partitioned-transitive tournament.

We check that D_{2n+1} is indecomposable. Let B be a nontrivial block of D_{2n+1}. We show first that B contains both an even and an odd vertex. If this fails then B contains vertices $i < j$ of equal parity, and then B contains the interval $[i,j]$, hence also contains a vertex of opposite parity.

Now let T be any indecomposable partitioned-transitive tournament. It suffices to check that T is of odd order, and is uniquely determined by its order, as this forces $T \simeq D_{2n+1}$. We use the representation of T as the result of partitioning the vertices of a transitive tournament L into two sets, A and B, and reversing the orientation of edges between vertices in A. If successive vertices of L belong to the same set, then they form a block in T, which is a contradiction. Thus the vertices of L lie alternately in A and B. If the first vertex, v, of L is in B then $V(L) - \{v\}$ form a block in T, a contradiction. We may assume the first vertex of L lies in the set A. If the last vertex, u, of L lies in B then $V(L) - \{u\}$ form a block in T, and we have a contradiction.

It follows that L is of odd order, and that A consists of vertices occurring in odd positions, and B consists of vertices occurring in even positions. Thus T is determined by its order, as claimed. ■

Lemma 4.10 *Partitioned-transitive tournaments omit N_5.*

Proof: We consider a tournament T derived from a transitive tournament L by partitioning $V(L)$ into two sets A and B and replacing the induced subtournament on A by its dual. Any 3-cycle C in T will have exactly 2 vertices $i < j$ in A and the third vertex k will be such that $i < k < j$. Any vertex x that dominates C will lie in B and $x < i < k < j$, and if any vertex y dominates x then $y < x$. If $y \in B$ then y dominates C, and otherwise y dominates just one vertex of C. On the other hand, N_5 embeds a 3-cycle C and has two distinct vertices x, y such that $y \to x \Longrightarrow C$ and y dominates exactly two vertices of C . Thus N_5 does not embed in T. ∎

Lemma 4.11 *For $n \geq 1$, D_{2n+1} is an indecomposable tournament omitting N_5.*

Proof: D_{2n+1} is indecomposable by Lemma 4.9. D_{2n+1} omits N_5 by the previous lemma. ∎

Thus the tournaments D_{2n+1} lie in the class $\mathcal{A}$ of indecomposable tournaments omitting N_5. We will now examine some "sporadic" tournaments in $\mathcal{A}$.

4.3 The Paley tournament P_7

Definition 4.12 *The Paley tournament P_7 has vertex set $\mathbb{Z}/7\mathbb{Z}$ and edge relation*

$$\{(i, j) \colon j - i \in \{1, 2, 4\}\}.$$

P_7 is a member of the family of Paley tournaments P_p, where p is a prime number congruent to 3 modulo 4. P_p has vertex set $\mathbb{Z}/p\mathbb{Z}$ and edge relation

$$\{(i, j) \colon j - i \text{ is a square in } \mathbb{Z}/p\mathbb{Z}\}$$

The tournaments P_p are also called the quadratic residue tournaments. Fried observed that P_p is edge transitive [15].

Fact 4.13 (Fried) *The automorphism group of P_p acts transitively on the edges of P_p.*

Proof: The additive automorphisms $x \mapsto x + i$ act transitively on vertices and the multiplicative automorphisms $x \mapsto ax$, for a a nonzero square in $\mathbb{Z}/p\mathbb{Z}$, act transitively on the edges originating at 0. ∎

It follows from Fact 4.13 that P_p is vertex and edge transitive and therefore the isomorphism types of the tournaments obtained from P_p by deleting one or two vertices do not depend on the vertices deleted. The only case of interest here is P_7. We let B_6 and B_5 be the tournaments obtained from P_7 by deleting one or two vertices respectively. When we require concrete realizations of these isomorphism types we take the vertex or vertices deleted to be vertex 6 or vertices 5 and 6 respectively, unless some other choice is made explicitly.

Proposition 4.14 *The tournaments I, L_2, B_5, B_6, and P_7 are indecomposable tournaments omitting N_5.*

Proof: The tournaments I and L_2 are indecomposable and omit N_5 by inspection. By checking outdegrees, we see that $N_5 \not\preceq B_5$, and hence B_5 omits N_5. By Fact 4.13, P_7 also omits N_5, and hence so does B_6.

Now $B_5 \simeq D_5$, so, by Lemma 4.11, B_5 is indecomposable. Suppose B_6 is decomposable and let B be a nontrivial block in B_6. If $|B| \geq 3$ then choose a vertex $v \in B$, otherwise choose $v \in V(B_6) - B$. Then $B - \{v\}$ is a nontrivial block in $B_6 - \{v\}$. However, by the proof of Fact 4.13, $B_6 - \{v\} \simeq B_5$, which is indecomposable. This contradiction proves that B_6 is indecomposable. A similar argument proves that P_7 is indecomposable. ■

Definition 4.15 *Let $\mathcal{A}_0$ be the class of tournaments that are isomorphic to one of the following tournaments: I, L_2, B_6, P_7, S_{2n+1}, or D_{2n+1}. ($B_5 \simeq D_5$.)*

This section may be summarized as follows.

Corollary 4.16 *$\mathcal{A}_0 \subseteq \mathcal{A}$.*

We show in §§6-10 that $\mathcal{A}_0 = \mathcal{A}$ and hence, in view of Lemma 3.8, that $\mathcal{Q}_{N_5}$ is the closure of $\mathcal{A}_0$ under composition.

5 Vertex encodings and tree representations

We review the theorems of Higman and Kruskal on well-quasi-orderings and develop some additional machinery that allows us to apply these theorems directly in the context of classes of tournaments. This leads us to the following results:

1. The class $\mathcal{A}_0$ introduced in the previous section has a finite vertex encoding in the sense of Definition 5.4(2) below.

2. If the class $\mathcal{A}$ of indecomposable tournaments omitting N_5 has a finite vertex encoding, then $\mathcal{Q}_{N_5}$ is well-quasi-ordered with respect to embeddability.

5.1 Higman's Lemma

Higman's Lemma can be used to show that a quasi-ordering is a well-quasi-ordering if the quasi-ordering can be encoded by words in a well-quasi-ordered alphabet.

Definition 5.1 *1. An* alphabet *is simply any set.*

2. *If $\mathcal{E}$ is an alphabet, then $\mathcal{E}^*$, the set of* words *in $\mathcal{E}$, is the set of nonempty finite sequences of elements of $\mathcal{E}$.*

3. *If $\mathcal{E}$ carries a quasi-ordering $\preceq$ then we impose the following quasi-ordering, called* embeddability, *on $\mathcal{E}^*$. If $w_a = a_1 \dots a_m$ and $w_b = b_1 \dots b_n$ are words in $\mathcal{E}^*$, then w_a* embeds *in w_b ($w_a \leq w_b$) if there is a strictly increasing function α from $\{1, \dots, m\}$ to $\{1, \dots, n\}$ such that for all i with $1 \leq i \leq m$ we have $a_i \preceq b_{\alpha(i)}$.*

The proof of the following is given in [20] and [37]:

Fact 5.2 (Higman's Lemma) *If $(\mathcal{E}, \preceq)$ is wqo then $(\mathcal{E}^*, \leq)$ is wqo.*

This applies in particular if $\mathcal{E}$ is finite and if $\preceq$ is the relation of equality, in which case the relation $\leq$ is a natural embeddability relation between words in a finite alphabet. For example, cat $\leq$ crate but cat $\not\leq$ fact.

We now give several applications of Higman's result. Let $\mathcal{Q}$ be a class of tournaments and let $(\mathcal{E}, \preceq)$ be a quasi-ordering. We say that a function f from $\mathcal{E}^*$ to isomorphism types of tournaments gives an $\mathcal{E}^*$-*covering* of $\mathcal{Q}$ if:

- The isomorphism type of each tournament in $\mathcal{Q}$ is in the image of f.
- f preserves embeddings in the following sense:

  If $a, b \in \mathcal{E}^*$ with $a \leq b$, then any tournament with isomorphism type $f(a)$ embeds in any tournament with isomorphism type $f(b)$.

If $\mathcal{E}^*$ is a wqo and if some function f gives an $\mathcal{E}^*$-covering of $\mathcal{Q}$, then $\mathcal{Q}$ is wqo with respect to the embeddability relation, since any antichain in $\mathcal{Q}$ could be pulled back to an antichain in $\mathcal{E}^*$. In the case of interest for the present paper, $\mathcal{E}^*$ is the set of words in a finite alphabet $\mathcal{E}$, and the $\mathcal{E}^*$-covering is of a very natural type, a "vertex encoding", as defined below. The idea of a vertex encoding is that the letters of the alphabet $\mathcal{E}$ represent facts about individual vertices that are adequate to reconstruct the edge relation, once the vertices are ordered appropriately. We illustrate this idea by giving a succinct description of the class of local orders. We start with the alphabet $\mathcal{E} = \{a, b\}$. Consider a sequence $w \in \mathcal{E}^*$ of length n. We label the vertices of the transitive tournament L_n by the elements of w in the natural way. Construct a tournament T from L_n by reversing the orientation of those edges of L_n whose vertices have different labels from $\mathcal{E}$. Note that if w_0 is a subsequence of w and T_0 is the local order obtained from w_0 then $T_0 \leq T$. Thus, up to isomorphism, we have encoded local orders with elements of $\mathcal{E}^*$ in a way that preserves embeddability.

In [32] we also make use of vertex encodings with respect to an infinite but wqo alphabet, and so we give the general notion here. There are some significant differences between the finite case and the more general wqo case, which are of considerable practical significance, though not explored here. These issues are more visible in [32].

Definition 5.3 *Let $(\mathcal{E}, \preceq)$ be a quasi-ordering.*

1. *An* edge-rule *on $\mathcal{E}$ is a map ϵ from $\mathcal{E} \times \mathcal{E}$ to $\{1, -1\}$ (in other words, to the set of possible orientations of edges) that satisfies:*

$$\textit{If } a_1 \preceq b_1 \textit{ and } a_2 \preceq b_2 \textit{ then } \epsilon(a_1, a_2) = \epsilon(b_1, b_2).$$

2. *If ϵ is an edge-rule on $\mathcal{E}$, then ϵ^* denotes the function from $\mathcal{E}^*$ to the set of isomorphism types of tournaments given by the following rule. Let $w \in \mathcal{E}^*$, $w = a_1 \dots a_n$ with $a_i \in \mathcal{E}$, then $\epsilon^*(w)$ is the isomorphism type of the tournament with vertex set $\{1, \dots, n\}$ such that for $1 \le i < j \le n$ we have:*

$$i \to j \textit{ if and only if } \epsilon(a_i, a_j) = 1$$

Observe that for any quasi-ordering $(\mathcal{E}, \preceq)$ and any edge-rule ϵ on $\mathcal{E}$, the corresponding map ϵ^* preserves embeddings. Thus any map ϵ^* derived from an edge-rule on $\mathcal{E}$ provides an $\mathcal{E}^*$-covering of the class of tournaments whose isomorphism types are in the range of ϵ^*.

Definition 5.4 *1. Let $(\mathcal{E}, \preceq)$ be a quasi-ordering and ϵ be an edge-rule on $\mathcal{E}$. If the map ϵ^* gives an $\mathcal{E}^*$-covering of a class $\mathcal{Q}$ of tournaments, we say that $\mathcal{Q}$ has a* vertex encoding *in the alphabet $\mathcal{E}$ (given by the edge-rule ϵ).*

2. *If the class $\mathcal{Q}$ of tournaments has a vertex encoding in a wqo alphabet or a finite alphabet, we say that $\mathcal{Q}$ has a* wqo vertex encoding, *or a* finite vertex encoding, *respectively.*

In §4 we introduced the class $\mathcal{A}_0$ of tournaments consisting of the indecomposable subtournaments of the Paley tournament P_7 on seven vertices, the local orders S_{2n+1} for $n \ge 1$, and the partitioned-transitive tournaments D_{2n+1} for $n \ge 1$. (There is some overlap: C_3 is included in all three categories, and B_5 in two of them.) We now check that $\mathcal{A}_0$ has a finite vertex encoding. In §§6-10 we verify that $\mathcal{A}$ consists of all tournaments isomorphic with a tournament in $\mathcal{A}_0$, and thus $\mathcal{A}$ also has a finite vertex encoding (the same one as $\mathcal{A}_0$). This provides the basis for the proof of the main theorem (see Corollary 5.14).

We state two straightforward formal consequences of our definitions.

Proposition 5.5 *1. Let T be a tournament. Then $\{T\}$ has a finite vertex encoding.*

2. *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be disjoint alphabets that carry quasi-orderings, and suppose $\mathcal{Q}_1$, $\mathcal{Q}_2$ are classes of tournaments that have vertex encodings in the alphabets $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Then $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ has a vertex encoding in the alphabet $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. (Here the quasi-orderings of $\mathcal{E}_1$, $\mathcal{E}_2$ are extended to $\mathcal{E}$ by taking the elements of $\mathcal{E}_1$ and $\mathcal{E}_2$ to be incomparable.)*

Proof:

1. Use $V(T)$ as the alphabet; let $v_1, v_2, \dots, v_n$ be a list of the vertices in $V(T)$ in some order. Define the ordering $\preceq$ on $V(T)$ by $v_i \preceq v_j$ if and only if $i = j$. Let ϵ be an edge-rule on $V(T)$ such that for $i < j$, $\epsilon(v_i, v_j) = 1$ if and only if $v_i \to v_j$.

2. Let ϵ_i for $i = 1, 2$ be an edge-rule that gives rise to a covering of $\mathcal{Q}_i$ by $\mathcal{E}_i^*$. Let ϵ be any edge-rule on $\mathcal{E}$ that extends the given edge-rules. Then ϵ^* gives an $\mathcal{E}^*$-covering of $\mathcal{Q}$. ■

Corollary 5.6 *If $\mathcal{Q}$ is a class of tournaments that can be decomposed into finitely many classes that have wqo vertex encodings or finite vertex encodings respectively, then $\mathcal{Q}$ has the same type of vertex encoding.*

Proof: Obviously there is no loss of generality in taking the various alphabets involved to be pairwise disjoint, and so the previous result applies. ■

Proposition 5.7 *The class $\mathcal{A}_0$ has a finite vertex encoding.*

Proof: Since $\mathcal{A}_0$ consists of finitely many tournaments together with the tournaments S_{2n+1} and D_{2n+1}, it suffices by Proposition 5.5 to give finite vertex encodings of S_{2n+1} and D_{2n+1} respectively. We make use of the characterization of local orders contained in Fact 4.2, and the definition of D_{2n+1}. Thus each of S_{2n+1} and D_{2n+1} can be constructed in a canonical fashion from a transitive tournament with a suitable partition of the vertex set into two sets. We therefore use an alphabet $\mathcal{E} = \{a_1, a_2\}$ containing two letters, and the following two edge-rules ϵ_S and ϵ_D:

$$\epsilon_S(a_1,a_1) = \epsilon_S(a_2,a_2) = 1; \quad \epsilon_S(a_1,a_2) = \epsilon_S(a_2,a_1) = -1$$

$$\epsilon_D(a_1,a_2) = \epsilon_D(a_2,a_1) = \epsilon_D(a_2,a_2) = 1; \quad \epsilon_D(a_1,a_1) = -1$$

These rules of course mean that we are to view the letters of a word as labels on successive vertices of a transitive tournament with vertices of two colors, and reverse the orientations of all edges to which the edge-rule assigns orientation -1, namely: according to ϵ_S, we reverse all edges between vertices of distinct colors, while according to ϵ_D, we reverse only the edges between vertices with the same color as the first vertex. Thus ϵ_S^* covers all local orders and ϵ_D^* covers all partitioned-transitive tournaments. ■

5.2 Kruskal's Tree Theorem

We review Kruskal's Tree Theorem [28, 37], which is a generalization of Higman's Lemma, and show how it may be used to convert a finite vertex encoding of $\mathcal{A}$ into a well-quasi-ordering theorem for $\mathcal{Q}_{N_5}$. First we establish some notation.

For brevity , the word "tree" is used to mean a finite tree T, in the usual graph theoretic sense, with a distinguished vertex r_T called its *root*. We define a partial ordering $\leq$ on the set of vertices of any tree T by the rule that $v \leq w$ if v lies in the path from r_T to w. We say that w lies *above* v and that v lies *below* w if $v \leq w$. A vertex that is both adjacent to and above a vertex v is called a *successor* of v. If $v \neq r_T$, the unique vertex adjacent to and below v is called its *predecessor*. Vertices that have no successors are called *leaves*.

Definition 5.8 *If X is a set let $\mathcal{D}(X)$ be the class of finite trees Δ equipped with the following additional structure:*

1. *A vertex labeling: each vertex carries a label from X;*
2. *For each vertex v of Δ, the set of edges originating at v carries a linear ordering, or, equivalently, the set of successors of v carries a linear ordering $\leq_v$.*

Following [28], we call the structured trees in $\mathcal{D}(X)$ "trees".

Definition 5.9 *Let $(X, \preceq)$ be a quasi-ordering. We define a quasi-ordering $\leq$ on $\mathcal{D}(X)$, called embeddability, as follows.*

1. *If $\Delta_1, \Delta_2 \in \mathcal{D}(X)$ and $f : V(\Delta_1) \to V(\Delta_2)$ then f is called a* weak embedding *if: f is injective and order-preserving (that is, if $u \leq v$ in the partial ordering on Δ_1 then $f(u) \leq f(v)$ in the partial ordering on Δ_2).*

2. *If $f : V(\Delta_1) \to V(\Delta_2)$ is a weak embedding and u_1 is the root of Δ_1 we define $\hat{f} : V(\Delta_1) - \{u_1\} \to V(\Delta_2)$ as follows. If $v \in V(\Delta_1)$ has predecessor u then $\hat{f}(v)$ is the unique successor of $f(u)$ below $f(v)$.*

3. *A weak embedding f of Δ_1 into Δ_2 is called an* embedding *if the associated map $\hat{f}$ respects the order structure and labeling in the following sense. For each vertex v of Δ_1, $\hat{f}$ carries the successors of v into the successors of $f(v)$ in an order-preserving fashion, and for each vertex v other than the root, if a_v is the label on v in Δ_1 and $b_{\hat{f}(v)}$ is the label on $\hat{f}(v)$ in Δ_2, then $a_v \preceq b_{\hat{f}(v)}$. Note that an embedding maps distinct successors of a vertex v to two vertices above $f(v)$ that are not in the same path from the root to a leaf in Δ_2.*

4. *$\Delta_1 \leq \Delta_2$ if and only if there is an embedding of Δ_1 into Δ_2.*

Let $(X, \preceq)$ be a quasi-ordering, $\Delta \in \mathcal{D}(X)$, and $v \in V(\Delta)$. The *branch* $\Lambda(\Delta, v)$ *of* Δ *at* v is the subtree of Δ whose vertices are v and all vertices above v in Δ, regarded as an element of $\mathcal{D}(X)$ that has root v and inherits from Δ the labels on its vertices and the linear ordering on the edges originating at each of its vertices. If the successors of v are $x_1, \ldots, x_r$, where $x_1 \leq_v x_2 \leq_v \cdots \leq_v x_r$ and if $\Lambda_i = \Lambda(\Delta, x_i)$ and a_i is the label on x_i for $i = 1, \ldots, r$, then $w(v, \Delta)$ will denote the word $(a_1, \Lambda_1)(a_2, \Lambda_2) \ldots (a_r, \Lambda_r) \in (X \times \mathcal{D}(X))^*$.

Theorem 5.10 (Modified Kruskal's Tree Theorem) *If $(X, \preceq)$ is wqo then $(\mathcal{D}(X), \leq)$ is wqo.*

We have departed somewhat from Kruskal's original formulation. Kruskal requires that his tree embeddings map a vertex to a vertex with a greater label. To insure that a tree embedding implies the existence of a tournament embedding we have a slightly different requirement. We require that the label of the relevant successor of the image of the predecessor of a vertex be greater than the label of that vertex. This is a stricter requirement in the sense that there are infinite antichains of labeled trees under this tree embedding that would not be antichains using Kruskal's formulation. Our proof of Kruskal's Tree Theorem with our requirements is an adaptation of the proof in [37].

Given a quasi-ordering $(\mathcal{Q}, \leq)$, a sequence $(q_i)_{1 \leq i < \omega}$, $q_i \in \mathcal{Q}$, is called *bad* if $q_i \not\leq q_j$ for all $i < j$. It is easy to show that $(\mathcal{Q}, \leq)$ is wqo if and only if it contains no bad sequence. We also use the following result from [37].

Lemma 5.11 *If* $(\mathcal{Q}_1, \leq)$ *and* $(\mathcal{Q}_2, \leq)$ *are wqo, then* $(\mathcal{Q}_1 \times \mathcal{Q}_2, \leq)$ *is wqo, where* $(q_1, q_2) \leq (q_1^*, q_2^*)$ *if and only if* $q_1 \leq q_1^*$ *and* $q_2 \leq q_2^*$.

Proof of Modified Kruskal's Tree Theorem
Proof: Suppose $(\mathcal{D}(X), \leq)$ is not wqo. We construct a bad sequence $(\Delta_i)_{1\leq i<\omega}$, inductively, such that Δ_i is a tree with minimum number of vertices among the trees that are the i^{th} elements of bad sequences beginning with $\Delta_1, \ldots, \Delta_{i-1}$. We begin with the empty sequence. Suppose $\Delta_1, \ldots, \Delta_n$ have been defined such that there are bad sequences beginning with $\Delta_1, \ldots, \Delta_n$. Consider the set of trees that are the $(n+1)^{th}$ elements of these bad sequences. Let Δ_{n+1} be any tree with a minimum number of vertices in this set. Then $\Delta_1, \ldots, \Delta_{n+1}$ are the first $n+1$ trees in a bad sequence.

Let r_i be the root of Δ_i and $\mathcal{B}_i$ be the subset of $\mathcal{D}(X)$ whose members are the branches of Δ_i at the successors of r_i. Let $\mathcal{B} = \cup_{i=1}^{\omega} \mathcal{B}_i$. If there exists a bad sequence in $\mathcal{B}$ then this sequence must have a bad subsequence $(\Gamma_i)_{1\leq i<\omega}$ such that there is a strictly increasing function f from $\{1, 2, \ldots\}$ to $\{1, 2, \ldots\}$ with $f(j)$ defined to be the least i such that $\Gamma_j \in \mathcal{B}_i$. If $\Delta_i \leq \Gamma_j$ then $\Delta_i \leq \Gamma_j \leq \Delta_{f(j)}$. Since $(\Delta_i)_{1\leq i<\omega}$ is bad, we have that $i \geq f(j) \geq f(1)$. Then $(\Delta_1, \ldots, \Delta_{f(1)-1}, \Gamma_1, \Gamma_2 \ldots)$ is a bad sequence, contradicting the definition of $\Delta_{f(1)}$ since Γ_1 has strictly fewer vertices than $\Delta_{f(1)}$. This contradiction shows that $(\mathcal{B}, \leq)$ is wqo. Hence, by Fact 5.2 and Lemma 5.11, $(X \times \mathcal{B})^*$ is wqo, where the ordering on $(X \times \mathcal{B})^*$ is defined as in Definition 5.1.

The infinite sequence $\{w(r_i, \Delta_i)\}_{1\leq i<\omega}$ is in $(X \times \mathcal{B})^*$. There exists $i < j$ such that $w(r_i, \Delta_i) \leq w(r_j, \Delta_j)$. This induces an embedding of Δ_i into Δ_j with r_i mapping to r_j, which contradicts $(\Delta_i)_{1\leq i<\omega}$ being a bad sequence. ■

We now proceed with the application of this modification of Kruskal's Tree Theorem.

Lemma 5.12 *Let* Θ *be a downward closed (with respect to embedding) class of indecomposable tournaments. In particular,* Θ *is closed under isomorphism. If* $T \in \mathcal{X}(\Theta)$ *and* $\sim$ *is a congruence on* T *such that* $T/\sim$ *is indecomposable, then* $(T/\sim) \in \Theta$ *and* $T{\restriction}C \in \mathcal{X}(\Theta)$ *for each* $\sim$*-class* C *in* T.

Proof: Let Γ be a set of representatives of the isomorphism classes of the indecomposable tournaments not in Θ and let $\mathcal{A}_\Gamma$ be the class of indecomposable tournaments that omit Γ. We claim that $\Theta = \mathcal{A}_\Gamma$. If $S \in \mathcal{A}_\Gamma$ then S is not isomorphic to any member of Γ and therefore $S \in \Theta$. Conversely, if $S \in \Theta$ then, since Θ is downward closed, all indecomposable subtournaments of S are in Θ and hence not in Γ and therefore $S \in \mathcal{A}_\Gamma$.

Since $T \in \mathcal{X}(\Theta) = \mathcal{X}(\mathcal{A}_\Gamma) = \mathcal{Q}_\Gamma$ by Lemma 3.5, T omits every element of Γ. Therefore, for every $\sim$-class C, $T{\restriction}C$ omits every element of Γ and so $T{\restriction}C \in \mathcal{Q}_\Gamma = \mathcal{X}(\Theta)$. Moreover, $T/\sim$ is clearly isomorphic to at least one subtournament of T. Therefore $T/\sim$ omits every element of Γ and so, being indecomposable, belongs to $\mathcal{A}_\Gamma = \Theta$. ■

Lemma 5.13 *If* Θ *is a downward closed class of indecomposable tournaments with a wqo vertex encoding, then* $\mathcal{X}(\Theta)$ *is wqo.*

Proof: Let $\mathcal{E}$ be a wqo alphabet and ϵ be an edge-rule on $\mathcal{E}$ giving rise to an $\mathcal{E}^*$-covering of Θ. We define a covering of $\mathcal{X}(\Theta)$ by the wqo $\mathcal{D}(\mathcal{E})$.

Let $\Delta \in \mathcal{D}(\mathcal{E})$. We define an associated tournament T_Δ, and then let $f(\Delta)$ be the isomorphism type of T_Δ. We take as the vertex set for T_Δ the leaves of the tree Δ. The edges are defined as follows. If u, v are two distinct leaves of Δ, let $w = \inf(u, v)$, the maximal vertex below both u and v. Then u and v lie above distinct successors of w, which we call u_1 and v_1. The edges from w to u_1 and v_1 lie in some order. We may assume that the edge from w to u_1 precedes the edge from w to v_1, and that the labels on u_1 and v_1 are the letters a_1 and a_2 respectively. We assign to the edge joining u and v the orientation $\epsilon(a_1, a_2)$, in other words: $u \to v$ if $\epsilon(a_1, a_2) = 1$, and $v \to u$ if $\epsilon(a_1, a_2) = -1$. This completes the description of T_Δ and hence of $f(\Delta)$.

It remains to be seen that the map f determines a $\mathcal{D}(\mathcal{E})$-covering of $\mathcal{X}(\Theta)$, that is:

(1) $$\text{If } \Delta, \tilde{\Delta} \in \mathcal{D}(\mathcal{E}) \text{ and } \Delta \leq \tilde{\Delta} \text{ then } T_\Delta \leq T_{\tilde{\Delta}}$$

(2) Every tournament in $\mathcal{X}(\Theta)$ is isomorphic to a tournament T_Δ for some $\Delta \in \mathcal{D}(\mathcal{E})$

We deal first with (1). Assume therefore that $\Delta \leq \tilde{\Delta}$ and fix an embedding $\phi : \Delta \to \tilde{\Delta}$. We define a map $\psi : T_\Delta \to T_{\tilde{\Delta}}$ as follows. If u is a vertex of T_Δ, and thus is a leaf of Δ, let $\psi(u)$ be some leaf of $\tilde{\Delta}$ that lies above $\phi(u)$ in $\tilde{\Delta}$. We claim that ψ preserves edges. Let u, v be distinct leaves of Δ, and let $w = \inf(u, v)$. Let u_1, v_1 be the successors of w lying below u and v respectively, and suppose that the edge connecting w to u_1 precedes the edge connecting w to v_1, and that u_1 and v_1 bear the labels a_1 and a_2 respectively. Let $\tilde{u}_1$ and $\tilde{v}_1$ be the successors of $\phi(w)$ lying below $\phi(u_1)$ and $\phi(v_1)$ respectively. Then the edge from $\phi(w)$ to $\tilde{u}_1$ precedes the edge to $\tilde{v}_1$. If $\tilde{u}_1$ and $\tilde{v}_1$ bear the labels b_1 and b_2 then $a_1 \leq b_1$ and $a_2 \leq b_2$. As ϵ is an edge-rule, $\epsilon(a_1, a_2) = \epsilon(b_1, b_2)$. On the other hand, $\tilde{u}_1$ and $\tilde{v}_1$ are also the successors of $\phi(w)$ lying below $\psi(u)$ and $\psi(v)$ respectively, and so according to our construction of $T_{\tilde{\Delta}}$, the orientation of the edge connecting $\psi(u)$ and $\psi(v)$ is the same as the orientation of the edge connecting u and v, since the data used to determine this orientation are identical in both cases.

We turn to (2). We have a tournament $T \in \mathcal{X}(\Theta)$ and we wish to construct a labeled tree $\Delta \in \mathcal{D}(\mathcal{E})$ for which $T \simeq T_\Delta$. We proceed by induction on the order of T. When T has only one vertex, this tree consists of the root alone. Assume T has more than one vertex, and let $\sim$ be a congruence on T having more than one equivalence class such that the quotient $T/\sim$ is indecomposable. By the previous lemma, the isomorphism type of $T/\sim$ is encoded by a word $w \in \mathcal{E}^*$, that is, $T/\sim$ has the isomorphism type $\epsilon^*(w)$. In particular we may carry over the ordering of the letters of w to an ordering of the equivalence classes of $\sim$. Now each $\sim$-class C in T is the vertex set of an induced subtournament, which we also call C, and this subtournament also is a member of $\mathcal{X}(\Theta)$ by Lemma 5.12, and so by induction there is a tree labeled tree $\Delta_C \in \mathcal{D}(\mathcal{E})$ for which $C \simeq T_{\Delta_C}$. We form a labeled tree Δ as follows. The vertices of Δ consist of a root together with the disjoint union of the vertices of the trees Δ_C. The successors of the root of Δ are the roots of the trees Δ_C, and above the root of Δ_C, Δ coincides with Δ_C. The root of Δ is labeled with any letter from $\mathcal{E}$. The root of Δ_C is labeled by the letter of w that corresponds to the class C in $T/\sim$. Finally, the edges from the root of Δ are ordered by the ordering on the letters of w.

We still need to check that the tournament T_Δ corresponding to the tree Δ we have constructed is in fact isomorphic with T.

An isomorphism from T_{Δ_C} to C is given by a map from the leaves of Δ above the root of Δ_C to the vertices of C. Since the set of leaves of Δ is the disjoint union of the sets of leaves above the roots of the various Δ_C, this gives us a well-defined bijection from the leaves of Δ to the vertices of T. We claim that this map carries edges of T_Δ to edges of T. If u, v are two vertices of Δ lying above the root of Δ_C, for some fixed $\sim$-class C, then u and v correspond to vertices of C, under a map that carries T_{Δ_C} isomorphically into T. Certainly edges are preserved in this case. If, on the other hand, u and v are vertices of Δ lying above the roots of Δ_C and Δ_{C^*} for distinct $\sim$-classes C, C^*, then u and v correspond to elements of C and C^*, and the edge connecting u and v in T has the orientation of the edge connecting C and C^* in $T/\sim$, which is determined by the order of the edge from the root of Δ to the roots of Δ_C and Δ_{C^*} respectively, and the labels on the roots of Δ_C and Δ_{C^*}. Since this order and these labels are both determined by w in accordance with an isomorphism between $T/\sim$ and a representative of $\epsilon^*(w)$, it follows that edges are preserved in this case as well. ∎

Corollary 5.14 *Let Γ be a collection of indecomposable tournaments such that $\mathcal{A}_\Gamma$, the class of indecomposable tournaments that omit Γ, has a wqo vertex encoding. Then $\mathcal{Q}_\Gamma$ is wqo.*

Proof: Combine the preceding lemma with Lemma 3.5. ∎

More generally, we have:

Corollary 5.15 *Let $\mathcal{Q}$ be a downward closed class of tournaments (with respect to embedding) and $\mathcal{A}$ the class of indecomposable tournaments in $\mathcal{Q}$. If $\mathcal{A}$ has a wqo vertex encoding then $\mathcal{Q}$ is wqo.*

Proof: As $\mathcal{Q}$ is downward closed, so is $\mathcal{A}$. By Lemma 5.13, $\mathcal{X}(\mathcal{A})$ is wqo. By Lemma 3.4, $\mathcal{Q} \subseteq \mathcal{X}(\mathcal{A})$. ∎

6 The Reduction Lemma

We have observed that $\mathcal{A}_0 \subseteq \mathcal{A}$. Our objective in the remainder of this paper is to complete the proof that $\mathcal{A} = \mathcal{A}_0$ by showing $\mathcal{A} \subseteq \mathcal{A}_0$.

In light of Proposition 5.5, it would of course be sufficient to show that up to isomorphism there are only finitely many "sporadic" examples in $\mathcal{A}$ (that is, finitely many such tournaments not of the form S_{2n+1} or D_{2n+1}), and one might well look for a more qualitative approach that bypasses the need for an explicit classification. On the other hand there is a strong formal parallel between the sort of induction argument used here and the more complex inductive analysis of finite simple groups. The possibility of a qualitative classification in that context that would avoid some of the complexities of explicit classification has been raised repeatedly, with no meaningful result. In our case, as in that case, such an alternative approach would of course be welcome.

The proof that $\mathcal{A} \subseteq \mathcal{A}_0$ is inductive, and the inductive step depends on a detailed analysis of the extensions of the tournaments in $\mathcal{A}_0$ by at most two vertices. A tournament T is said to be an *extension* of a tournament T_0 by k vertices if $T_0 \leq T$ and $|V(T)| = |V(T_0)| + k$. We also refer to this as adjoining k vertices to T_0. This approach is justified by the following:

Lemma 6.1 (Reduction Lemma) *Let T be an indecomposable tournament with $|T| > 1$. Then there is a proper indecomposable subtournament, T_0 of T such that $|T| - |T_0| \leq 2$.*

Proof: We note that T does contain proper indecomposable subtournaments, such as the induced subtournament on a single vertex.

Let T_0 be a proper indecomposable subtournament of T of maximal order. Suppose toward a contradiction that $|T| - |T_0| \geq 3$. In particular, as T is indecomposable and of order at least 4, T contains a 3-cycle as a proper subtournament, and it follows that $|T_0| \geq 3$.

Since T is indecomposable, some vertex z of $T - T_0$ has mixed type over T_0. As $T_0 \cup \{z\}$ is decomposable, z is in a nontrivial block. Let i be a vertex of T_0 in this block. As T_0 is indecomposable, this block has only one vertex of T_0. Therefore, z and i have the same type over $T_0 - \{i\}$.

Let C be the set of all vertices in T that have the same type as i over $T_0 - \{i\}$. Note that $C \cap T_0 = \{i\}$. As T is indecomposable, C is not a block and there is a vertex $y \in T - C$ that has mixed type over C. In particular there is some vertex $x \in C - \{i\}$ such that y has mixed type over $\{x, i\}$. As T_0 is indecomposable and of order at least 3, i has mixed type over $T_0 - \{i\}$, and hence any element of C has mixed type over $T_0 - \{i\}$. Specifically, x has mixed type over $T_0 - \{i\}$, and hence has mixed type over T_0. .

Let $T_1 = T \restriction V(T_0) \cup \{x, y\}$. As T_1 is a proper subtournament of T, there is a nontrivial block A in T_1. If $V(T_0) \subseteq A$ then as x has mixed type over T_0, we have $x \in A$. As y has mixed type over $\{x, i\}$, we also find $y \in A$, and $A = V(T_1)$, a contradiction to A being nontrivial. Thus $V(T_0) \not\subseteq A$, and since T_0 is indecomposable we conclude that A contains at most one vertex of T_0. If A is disjoint from T_0 then $A = \{x, y\}$ and in particular x and y have the same type over $T_0 - \{i\}$, a contradiction to $y \notin C$. Thus A contains exactly one vertex j of T_0.

If $x \in A$ then x and j have the same type over $T_0 - \{j\}$, therefore they have the same type over $T_0 - \{i, j\}$. Since $x \in C$, x and i have the same type over $T_0 - \{i\}$, therefore they have the same type over $T_0 - \{i, j\}$. We conclude that i and j have have the same type over $T_0 - \{i, j\}$, that is, $\{i, j\}$ is a block of T_0 and $i = j$. As y has mixed type over $\{x, i\}$, $y \in A$, so again x and y have the same type over $T_0 - \{i\}$, a contradiction. We conclude that $x \notin A$ and thus $y \in A$. If $j = i$ then y and i have the same type over $T_0 - \{i\}$ and $y \in C$, a contradiction; thus $j \neq i$.

Now suppose for example $i \to j$. Then $x \to j$ and $i \to y$. As y has mixed type over $\{x, i\}$ we find that $y \to x$ and thus x has mixed type over $\{y, j\}$. Thus A is not a block. This contradiction completes the argument. ■

We note that this result is not quite correct if T_0 is merely taken to be a maximal proper indecomposable subtournament of T. Indeed, B_5 has a maximal indecomposable subtournament of order 2. Taking B_5 to be the tournament obtained by deleting vertex 5 from B_6, then $B_5 \restriction \{0, 4\}$ is maximal, it is not an edge of a 3-cycle. The proof given does show that if T_0 is a maximal proper indecomposable subtournament of T of order at least 3, then $|T| - |T_0| \leq 2$.

In view of this result, the proof that $\mathcal{A} = \mathcal{A}_0$ reduces to the following.

Lemma 6.2 (Extension Lemma) *Let T be an indecomposable tournament that omits N_5 and is an extension by at most two vertices of a tournament $T_0 \in \mathcal{A}_0$. Then $T \in \mathcal{A}_0$.*

This lemma is proved for S_{2n+1} in §7, for B_5, B_6, and P_7 in §§8-9, and for D_{2n+1} $(n \geq 3)$ in §10. Since $D_3 \simeq C_3$ and $D_5 \simeq B_5$, this covers all cases. We note that if a tournament T is an indecomposable extension by one vertex of I or L_2 then T is isomorphic to L_2 or C_3, respectively.

7 Extensions of S_{2n+1}

In this section we will classify the indecomposable tournaments omitting N_5 that can be obtained by adjoining at most two vertices to an indecomposable local order. Such a tournament will either be isomorphic to B_5, or will again be a local order.

The following result is from [33].

Lemma 7.1 *If the tournament T omits C_3I (IC_3 respectively) then $T = LS$ (SL respectively), where L is transitive and S is a local order.*

Proof: Define S to be the subtournament of T of maximum order such that every vertex of S is a vertex in a copy of C_3. Then $L = T - S$ is transitive. We claim that $L \Longrightarrow S$. Suppose not. Then there are vertices $a \in V(L)$ and $x \in V(S)$ such that $x \to a$. By the definition of S, there is a copy C of C_3 with the vertex x. As $C \not\Longrightarrow a$, we have that a has mixed type over C. Then a is a vertex in a copy of C_3, contradicting $a \in T - S$.

It remains to be shown that S is a local order. Suppose not. Then by Definition 4.1, $IC_3 \leq S$. Let a, b, c, x be vertices in $V(S)$ such that $a \to b \to c \to a$ and $x \Longrightarrow \{a, b, c\}$. Then there are vertices $y, z \in V(S)$ such that $x \to y \to z \to x$. Clearly z is distinct from a, b, c. Suppose $a \to z$. Since $\{x, a\} \Longrightarrow b$, we have that $b \to z$ and $S \restriction \{b, z, x\} \simeq C_3$. Now $\{x, b\} \Longrightarrow c$ and so $c \to z$, hence $\{a, b, c\} \Longrightarrow z$, a contradiction. We have $z \to a$. Now $\{z, x, c\} \Longrightarrow a$ and so $z \to c$. Similarly $\{z, x, b\} \Longrightarrow c$ and so $z \to b$. We have that the vertex y, which dominates z, is distinct from a, b, and c. Since $\{x, z\} \Longrightarrow \{a, b, c\}$, we have that $\{a, b, c\} \Longrightarrow y$, a contradiction.■

Lemma 7.2 *Let T be an indecomposable tournament of order at most 5. Then T is isomorphic to I, L_2, C_3, S_5, N_5, or B_5.*

Proof: If T is a local order then it is one of the first four tournaments listed. Assume that T is not a local order. Then as T is indecomposable, it follows from the previous lemma that both $IC_3 \leq T$ and $C_3I \leq T$. In particular T has exactly 5 vertices.

Fix vertices x, y in T such that x dominates some 3-cycle C in T, and y is dominated by a 3-cycle. If $y \notin V(C)$ then $V(T) = \{x, y\} \cup V(C)$ and evidently $C \implies y$, so T is decomposable, a contradiction. Thus $y \in V(C)$ and $'y$ must consist of x, one vertex of C, and a third vertex z. Let $a \in V(C)$ be dominated by y. Observe that the structure of T is known apart from the orientation of the edge connecting a and z. Since we already know of two indecomposable tournaments of order 5 which are not local orders, namely N_5 and B_5, T must be isomorphic to one of them.■

Lemma 7.3 *For $n \geq 1$, no indecomposable extension of S_{2n+1} by one vertex omits N_5.*

Proof: Let $S = S_{2n+1}$ and suppose an extension T of S by one vertex x omits N_5. Assume first that $T \restriction x'$ and $T \restriction {}'x$ are transitive. Since every transitive subtournament of S has at most $n+1$ vertices, x' and $'x$ have cardinalities n and $n+1$ (in some order). We may suppose that x' has cardinality $n+1$ and $'x$ has cardinality n. Let a be the first element of x'. Since $x' - \{a\} \subseteq a'$ and $|a'| = |x' - \{a\}| = n$, it follows that $x' - \{a\} = a'$, a and x have the same type over $T - \{a, x\}$, and T is decomposable.

Now assume $T \restriction x'$ or $T \restriction {}'x$ is not transitive. We may suppose that x dominates a 3-cycle in S. We will show that x dominates S, which implies that T is decomposable.

Recall that the vertices of S may be identified with $\mathbb{Z}/(2n+1)\mathbb{Z}$ so that the edges are the pairs (i, j) with $j - i \in \{1, \ldots, n\}$. We will argue that if x dominates a 3-cycle containing the vertex k, then x dominates a 3-cycle containing $k + 1$; this clearly suffices.

Assume that $x \implies \{i, j, k\}$ where $i \to j \to k \to i$. If $(k - n) \to x$ then $T \restriction (x, k - n, i, k, j) \simeq N_5$, a contradiction. Thus $x \implies \{i, k - n, k\}$. Similarly $x \implies \{k - 2n, k - n, k\}$. Since $k - 2n = k + 1$, our claim follows. ■

We also need to classify indecomposable extensions of indecomposable local orders by two vertices – not surprisingly, since this is one way of constructing indecomposable local orders. The main step can be phrased in a way that will also be useful in the discussion of B_6 and P_7 below. A slightly more complicated result of the same character will also be invoked in the analysis of extensions of B_5 and D_{2n+1}.

Lemma 7.4 *Let T_0 be an indecomposable tournament in which every vertex has indegree and outdegree at least 2. Let T be an indecomposable extension of T_0 by two vertices x and y. Then one of the following two possibilities occurs:*

1. There is an indecomposable tournament T_1 that is an extension of T_0 by one vertex, and T is an extension of T_1 by one vertex.

2. There are distinct vertices i, j_1, j_2 such that j_1 and j_2 have the same type over $T_0 - \{i, j_1, j_2\}$, $\{x, i\}$ is a block of $T - \{y\}$, $\{y, j_1\}$ is a block of $T - \{x\}$, and $\{y, j_2\}$ is a block of $T - \{i\}$. In this case, given j_1 and j_2, the vertex i is determined as the unique vertex of T_0 having mixed type over $\{j_1, j_2\}$, and T is uniquely determined once the orientation of the edge between x and i is known.

Proof: We may assume we are not in Case 1 and both $T-\{x\}$ and $T-\{y\}$ are decomposable. As T is indecomposable, T_0 is not a block in T and either x or y has mixed type over T_0. We assume x has a mixed type over T_0. Then T_0 is not a block in $T - \{y\}$ and hence there is a block $\{x, i\}$ in $T - \{y\}$ with i a vertex of T_0. Let $T_0^* = T_0 \cup \{x\} - \{i\}$. If $T - \{i\}$ is indecomposable then we may take $T_1 = T - \{i\}$, since $T_1 - \{y\} = T_0^* \simeq T_0$.

Assume now that $T - \{i\}$ is decomposable. Toward a contradiction, we suppose y has unmixed type over T_0. As T is indecomposable, y has mixed type over T_0^*. Let B be a nontrivial block in $T - \{i\}$. If $y \notin B$ then $x \notin B$ and $B \subseteq V(T_0 - \{i\})$. Then x, hence also i, has unmixed type over B and B is a nontrivial block in T_0, contradicting the indecomposability of T_0. So we may assume $B = \{y\} \cup B_0$. B_0 is a block in $T - \{y, i\} = T_0^* \simeq T_0$, which is indecomposable, and hence $B_0 = \{j\}$ for some vertex j in $T - \{y, i\}$. If $j = x$ then since y has unmixed type over T_0, x, hence also i, has unmixed type over $T_0 - \{i\}$, contradicting the indecomposability of T_0. Accordingly, $j \in V(T_0) - \{i\}$ and j has unmixed type over $T_0 - \{i, j\}$ and hence has indegree or outdegree at most 1 in T_0, a contradiction.

Now suppose y has unmixed type over T_0^*. We will again reach a contradiction using a similar argument in which the roles of x and i are interchanged. As T is indecomposable, y has mixed type over T_0. Let B be a nontrivial block in $T - \{x\}$. If $y \notin B$ then $i \notin B$ and $B \subseteq V(T_0^* - \{x\})$. Then i, hence also x, has unmixed type over B and B is a nontrivial block in T_0^*, contradicting the indecomposability of T_0^*. So we may assume $B = \{y\} \cup B_0$. B_0 is a block in $T - \{y, x\} = T_0$, which is indecomposable, and hence $B_0 = \{j\}$ for some vertex j in $T - \{y, x\}$. If $j = i$ then since y has unmixed type over T_0^*, i, hence also x, has unmixed type over $T_0^* - \{x\}$, contradicting the indecomposability of T_0^*. Accordingly, $j \in V(T_0^*) - \{x\}$ and j has unmixed type over $T_0^* - \{x, j\}$ and hence has indegree or outdegree at most 1 in $T_0^* \simeq T_0$, a contradiction.

Assume now that y has mixed type over both T_0 and T_0^*. Then T_0 is not a block in $T - \{x\}$ and hence there is a block $\{y, j_x\}$ in $T - \{x\}$ with j_x a vertex of T_0. Similarly, T_0^* is not a block in $T - \{i\}$ and hence there is a block $\{y, j_i\}$ in $T - \{i\}$ with j_i a vertex of T_0^*. If $j_i = x$ then $\{i, j_x\}$ is a block in T_0, which implies that $i = j_x$, and hence $\{i, x, y\}$ is a block in T, a contradiction. We conclude that j_i is a vertex of $T_0 - \{i\}$.

If $j_i = j_x = j$ then $\{y, j\}$ is a block in T, a contradiction. Thus j_i, and j_x are distinct vertices.

Let z be any vertex in $T_0 - \{i, j_i, j_x\}$. Then z has unmixed type over $\{j_i, y, j_x\}$, and hence has unmixed type over $\{j_i, j_x\}$. As T_0 is indecomposable, there is a vertex in T_0 with mixed type over $\{j_i, j_x\}$ and that vertex is i, necessarily distinct from j_i and j_x.

Up to the orientation of the edge between x and i, T is now determined as follows:

$y \to z$ if and only if $j_x \to z$ for $z \in T - \{x, y, j_x\}$

$y \to z$ if and only if $j_i \to z$ for $z \in T - \{i, y, j_i\}$

$x \to z$ if and only if $i \to z$ for $z \in T - \{y, x, i\}$

$j_x \to y$ if and only if $j_x \to j_i$

We now let $j_1 = j_x$ and $j_2 = j_i$. ■

Lemma 7.5 *For $n \geq 2$, if T is an indecomposable extensions of S_{2n+1} by two vertices and $N_5 \not\leq T$, then $T \simeq S_{2n+3}$.*

Proof: Let $S = S_{2n+1}$ and suppose T is an indecomposable extension of S by two vertices $\{x, y\}$ and $N_5 \not\leq T$.

All vertices of S have indegree and outdegree at least 2. Since there is no indecomposable tournament T_1 omitting N_5 that can be obtained as an extension of S by a single vertex, T must fall under the second case of Lemma 7.4. The vertices j_1, j_2 occurring in that case have the property that there is only one vertex of S with a mixed type over j_1, j_2; this means that they are successive vertices in S (using the definition of S_{2n+1} as having vertex set $\mathbb{Z}/(2n+1)\mathbb{Z}$). For each such pair j_1, j_2, one can obtain a local order by adjoining two vertices, x and y to S; y is inserted between j_1 and j_2 and the block $\{x, i\}$ will consist of successive vertices. This local order will fall under case 2 of Lemma 7.4, with respect to the given vertices j_1, j_2, and i. Furthermore, the block $\{x, i\}$ consists of successive vertices in this local order. Since T is determined up to the orientation of the edge between x and i, it follows that T is either an indecomposable local order or is obtained from an indecomposable local order by reversing the orientation on one edge between successive vertices i and $i+1$. However, in the latter case, N_5 embeds in T, specifically, $T \restriction \{i{+}1, i{-}1, i{+}n{+}1, i{+}2, i\} \simeq N_5$. ■

8 Extensions of B_5, B_6, and P_7 by one vertex

In a sense the main case in the inductive analysis of indecomposable tournaments omitting N_5 comes in the consideration of extensions of B_5, which leads on the one hand to the "sporadic" examples of B_6 and P_7, as well as to the infinite family D_{2n+1}. The key fact is that the only indecomposable extension of B_5 that omits N_5 obtainable by adjoining a single vertex is B_6.

Lemma 8.1 *If T is an indecomposable tournament of order 6 and $N_5 \not\leq T$, then T has no vertex of outdegree 4.*

Proof: Suppose on the contrary that $x \in V(T)$ has outdegree 4. We simply inspect the various possibilities for the structure of $T_0 = T \restriction x'$. This may be organized as follows. Let a be the vertex of T that dominates x.

If $T_0 \simeq L_4$ then $T - \{a\} \simeq L_5$. Since $a \to x$, either a has an unmixed type over two successive vertices of $T - \{a\}$ or a dominates the last vertex of $T - \{a\}$, in each case contradicting the indecomposability of T.

If $T_0 \simeq IC_3$ then let y be the vertex in x' that dominates all the others, and let A be the 3-cycle $T \restriction x' \cap y'$. Note that $y \to a$ since otherwise $\{x, y\}$ would be a block of T. As T is indecomposable, a has mixed type over A. Accordingly we can find $b, c \in V(A)$ with $a \to b \to c \to a$, and then $T \restriction \{y, x, a, c, b\} \simeq N_5$, a contradiction.

If $T_0 \simeq C_3I$ then let y be the vertex in x' that is dominated by all the others, and let A be the 3-cycle $T \restriction x' \cap {}'y$. Since T is indecomposable, $y \to a$ and a has mixed type over A. If a dominates two vertices of A then $T \restriction \{x, a\} \cup V(A) \simeq N_5$, while if a dominates just one vertex of A, then $T \restriction V(A) \cup \{a, y\} \simeq N_5$, a contradiction in any case.

There remains the possibility that $T_0 \simeq S_4$, the unique nontransitive (and decomposable) local order of order 4. This has a block A of order 2 and two other vertices y_1, y_2; (b, y_1, y_2) is a 3-cycle for each $b \in A$. As T is indecomposable, a has a mixed type over A. If a dominates exactly two vertices of (b, y_1, y_2) for some $b \in A$ then $T \restriction \{x, a, b, y_2, y_1\} \simeq N_5$. Since a has mixed type over A, we reach a contradiction unless $\{y_1, y_2\} \Longrightarrow a$. Now $A = \{b_1, b_2\}$ with $b_1 \to b_2$, and if $b_1 \to a \to b_2$, then $T \restriction \{b_1, y_2, y_1, b_2, a\} \simeq N_5$, while if $b_2 \to a \to b_1$, then $T \restriction \{y_2, x, a, b_2, b_1\} \simeq N_5$. ■

Lemma 8.2 *If T is an indecomposable tournament of order 6 and $N_5 \not\leq T$, then $T \simeq B_6$.*

Proof: Since T is indecomposable, it has no vertex with outdegree 0 or 5. By the preceding lemma, applied to T and its dual, T has no vertex with outdegree 1 or 4. All outdegrees are 2 or 3, and thus there are three vertices of outdegree 3, and three vertices of outdegree 2.

Let T_0 and T_1 be the tournaments induced on the vertices of outdegree 3 and 2 respectively. In the dual tournament the roles of these subtournaments are reversed. If $T_0 \simeq L_3$ then label the vertices of T_0 v_1, v_2, v_3 in order, and observe that $v_3 \Longrightarrow T_1$. As v_2 dominates two vertices of T_1, if $T_1 \simeq C_3$ we find that $T \restriction \{v_3, v_2\} \cup V(T_1) \simeq N_5$. Thus if $T_0 \simeq L_3$ then also $T_1 \simeq L_3$. Since the vertices of T_0 and T_1 have outdegrees 3 and 2 in T respectively, the last vertex of T_0 and the first vertex of T_1 will form a block.

We conclude that T_0 and T_1 are both isomorphic to C_3. Now the relation $\leftarrow$ gives a bijection between $V(T_0)$ and $V(T_1)$, which is either an isomorphism or an anti-isomorphism. In the former case, T would have three blocks of cardinality 2, and in the latter case $T \simeq B_6$. ■

Corollary 8.3 *If the tournament T is an extension of B_5 by a single vertex and $N_5 \not\leq T$, then either $T \simeq IB_5$, B_5I, or B_6, or else T has a congruence relation with quotient isomorphic to B_5.*

Proof: Let T be an extension of B_5 by the vertex x. If T is decomposable then by Proposition 4.14 either B_5 is a block in T, or else x belongs to a block of order 2. These two cases correspond respectively to: $T \simeq IB_5$ or B_5I; T has a congruence relation with quotient isomorphic to B_5.

If T is indecomposable then the preceding lemma applies. ■

The following remark is quite useful.

Lemma 8.4 *If T is a tournament obtained from B_6 by reversing exactly one edge, then $N_5 \le T$.*

Proof: We may identify B_6 with $P_7 \restriction (\mathbb{Z}/7\mathbb{Z} - \{0\})$. Multiplication by nonzero squares acts as a group of automorphisms on B_6, and has five orbits on edges. On the other hand, B_6 is also self-dual and modulo duality these five orbits coalesce into three. Since N_5 is also self-dual it suffices to check representatives of the three classes of edges, for example $(1,2)$, $(6,1)$, and $(1,5)$. If one of these edges is reversed, then $T \restriction \{4,2,5,1,6\}$, $T \restriction \{1,4,3,6,5\}$, or $T \restriction \{5,4,2,1,6\}$ respectively, is isomorphic to N_5. ■

Of course the same statement applies *a fortiori* to P_7.

Lemma 8.5 *If the indecomposable tournament T is an extension of B_6 by a single vertex and $N_5 \not\le T$, then $T \simeq P_7$.*

Proof: We take the vertex set of P_7 to be $\mathbb{Z}/7\mathbb{Z}$, with the usual edge relation, and we take B_5 and B_6 to be $P_7 \restriction (\mathbb{Z}/7\mathbb{Z} - \{0,6\})$ and $P_7 \restriction (\mathbb{Z}/7\mathbb{Z} - \{0\})$ respectively.

Let T be an extension of B_6 by the vertex y. Suppose first that $T \restriction \{y\} \cup V(B_5)$ is indecomposable, hence isomorphic to B_6. One should observe that there are only two ways to extend B_5 to B_6: by adding a vertex of outdegree 2 or outdegree 3 (and there is only one possible type for the additional vertex in each case). As T is indecomposable, the type of y over B_5 is to distinct from the type of 6 over B_5, so y has the type of the vertex 0 in P_7 over B_5. Thus T coincides with P_7 up to the orientation of the edge between 6 and y; by the preceding lemma, the orientation of this edge must be as in P_7.

Now suppose toward a contradiction that $T \restriction \{y\} \cup B_5$ is decomposable. If $y \Longrightarrow B_5$ then $6 \to y$ since T is indecomposable and hence $T \restriction \{y,6,4,3,1\} \simeq N_5$, a contradiction. A similar contradiction involving $T \restriction \{4,2,1,6,y\}$ arises if $B_5 \Longrightarrow y$. Thus there must be a block of the form $\{y,i\}$ in $B_5 \cup \{y\}$, with i a vertex of B_5. But then 6 must have mixed type over $\{y,i\}$ and thus $T - \{i\}$ differs from B_6 by the orientation of exactly one edge, the edge connecting y and 6. By the previous lemma this is a contradiction. ■

Lemma 8.6 *If the tournament T is an extension of P_7 by a single vertex and $N_5 \not\le T$, then T is decomposable.*

Proof: Let T be an extension of P_7 by the vertex x. Let $B = P_7 - \{0\}$, $B^* = P_7 - \{6\}$, $T_1 = T - \{0\}$ and $T_1^* = T - \{6\}$. Since $B \simeq B_6$, which is indecomposable, it follows that either

(a) T_1 is indecomposable and hence (by Lemma 8.5) isomorphic to P_7 or

(b) x has unmixed type over B or

(c) T_1 has a block $\{x, i\}$ where $i \in V(B)$.

Similarly, since $B^* \simeq B_6$, either

(a*) $T_1^* \simeq P_7$ or

(b*) x has unmixed type over B^* or

(c*) T_1^* has a block $\{x, j\}$ where $j \in V(B^*)$.

If (a) is true then, since all vertices of P_7 have indegree and outdegree 3, the types of x and 0 over B are the same and so T is decomposable. Similarly, T is decomposable if (a*) is true. If (b) and (b*) are both true then T is clearly decomposable. Since (c) implies that the indegree and outdegree of x in T are at least the indegree and outdegree respectively of i in B, and hence are both at least 2, and since (c*) has similar consequences, neither (c) nor (c*) is compatible with (b) or (b*). We may therefore assume that (c) and (c*) are both true.

If at least one of i or j does not belong to $V(B) \cap V(B^*)$ then x has unmixed type over some 3-cycle C in $T \upharpoonright V(B) \cap V(B^*)$. We may suppose that $x \Longrightarrow C$ and $i \Longrightarrow C$. Then j dominates at least two vertices of C; so $j \notin C$ and thus $j \Longrightarrow C$. This forces $i = j$ and $\{x, i\}$ is a block in T.

The other possibility is that $i, j \in V(B) \cap V(B^*)$, in which case $\{i, j\}$ is a block in a tournament isomorphic to B_5, which again forces $i = j$. ■

9 Extensions of B_5, B_6, and P_7 by two vertices

We apply the classification of extensions of B_5, B_6 and P_7 by one vertex to determine the indecomposable extensions of these tournaments by two vertices.

Lemma 9.1 *If the tournament T is an extension of B_6 or P_7 by two vertices and $N_5 \not\leq T$, then T is decomposable.*

Proof: We apply Lemma 7.4 with $T_0 = B_6$ or P_7. The second possibility listed in that Lemma involves a pair of vertices j_1, j_2 of T_0 such that only one vertex of $T_0 - \{j_1, j_2\}$ has mixed type over $\{j_1, j_2\}$. However in P_7 for any two vertices j_1, j_2 there are at least three vertices with mixed type over $\{j_1, j_2\}$, and hence in B_6 there are at least two such vertices. Thus if T is indecomposable, the first alternative given in Lemma 7.4 must apply: T contains an indecomposable tournament T_1 that is an extension of B_6 or P_7 by a single vertex, and omits N_5. The only possibility is that $T_1 \simeq P_7$, but as T is an extension of T_1 by a single vertex, this is also impossible. ■

We will now need to reformulate Lemma 7.4 more precisely.

Lemma 9.2 *Let T_0 be an indecomposable tournament and let T be an indecomposable extension of T_0 by two vertices x and y. Then one of the following three possibilities occurs, after relabeling some of the vertices if necessary.*

1. *There is an indecomposable tournament T_1 that is an extension of T_0 by one vertex, and T is an extension of T_1 by one vertex.*
2. *There is a block $\{x, i\}$ of $T - \{y\}$ with i a vertex of T_0, and y has unmixed type over $T_0 - \{i\}$. If $y \Longrightarrow T_0$ and $N_5 \not\leq T$, then $i \to x \to y$, there is no 3-cycle C in $T_0 - \{i\}$ such that exactly two vertices of C are dominated by i, and there is no 3-cycle containing i that is dominated by a vertex of T_0.*
3. *There are distinct vertices i, j_1, j_2 of T_0 such that j_1 and j_2 have the same type over $T_0 - \{i, j_1, j_2\}$, $\{x, i\}$ is a block of $T - \{y\}$, $\{y, j_1\}$ is a block of $T - \{x\}$, and $\{y, j_2\}$ is a block of $T - \{i\}$. In this case, given j_1 and j_2, the vertex i is determined as the unique vertex of T_0 having mixed type over $\{j_1, j_2\}$, and T is uniquely determined once the orientation of the edge between x and i is known.*

Proof: We review the proof of Lemma 7.4, where we made use of an additional assumption on the indegrees and outdegrees at two stages, which differ by the interchange of the roles of two vertices. Thus the case that requires examination here is the following: $\{x, i\}$ is a block in $T - \{y\}$, with i a vertex of T_0, and y has unmixed type over T_0 or ${T_0}^*$, which is case 2 above. It remains to examine the situation when $y \Longrightarrow T_0$ and $N_5 \not\leq T$.

Since $y \to i$ and y has a mixed type over $\{x, i\}$, we have $x \to y$. As T_0 is strongly connected, there is a 3-cycle of the form (i, j, k) in T_0. Then (x, j, k) is also a 3-cycle, and since $T \restriction \{y, x, k, j, i\} \not\simeq N_5$, we find that $i \to x$. If exactly two vertices of any 3-cycle C in $T - \{i\}$ are dominated by i, then the same applies to x and $T \restriction \{y, x\} \cup C \simeq N_5$, a contradiction. Finally, if (i, j, k) is a 3-cycle containing i that is dominated by a vertex l of T_0, then $T \restriction \{l, y, x, k, j\} \simeq N_5$. ■

Lemma 9.3 *If the tournament T is an indecomposable extension of B_5 by two vertices and $N_5 \not\leq T$, then $T \simeq P_7$ or D_7.*

Proof: Let T be an extension of B_5 by the two distinct vertices x and y. We apply the preceding lemma with $T_0 = B_5$. If T is an extension of an indecomposable extension T_1 of B_5, then $T_1 \simeq B_6$ and $T \simeq P_7$. Otherwise, we are in case 2 or 3 of Lemma 9.2 and we claim that $T \simeq D_7$. Notice that we are free to replace T by its dual.

If we are in case 2 of the preceding lemma we may suppose that y dominates $B_5 - \{i\}$. As y must have a mixed type over $\{x, i\}$ and the vertices x and i are interchangeable, we may suppose $y \Longrightarrow B_5$. We may identify the vertices of B_5 with the vertices $1, 2, 3, 4, 5$ of P_7. Now i cannot dominate two vertices of a 3-cycle in B_5, so $i = 4$ or 5; but vertex 5 belongs to a 3-cycle dominated by 1, so finally $i = 4$. Since D_7 is in fact an extension of B_5 by two vertices, one of which dominates B_5, the unique configuration we have found must indeed be D_7.

If we are in case 3 of the preceding lemma then i is the only vertex of $T_0 - \{j_1, j_2\}$ having mixed type over $\{j_1, j_2\}$. To analyze this situation it is most convenient to think of B_5 as $D_5 = L_O \cup L_E$. Then j_1 and j_2 are successive elements either of L_O or of L_E, while i is the element coming "between" j_1 and j_2 in the other component. Thus, up to duality, there are only two possibilities: $\{j_1, j_2\} = \{1, 3\}$ and $i = 2$, or $\{j_1, j_2\} = \{2, 4\}$ and $i = 3$. Furthermore, interchanging x and i will result in j_1 and j_2 being switched; so we may suppose $(j_1, i, j_2) = (1, 2, 3)$ or $(2, 3, 4)$. In the former case, as $T \restriction \{x, 2, y, 4, 5\} \not\simeq N_5$, we must have $x \to 2$. In the latter case, as $T \restriction \{y, x, 5, 4, 3\} \not\simeq N_5$, we must have $3 \to x$. Thus there are two such configurations. On the other hand, if we consider pairs of points in D_7 that can be removed so as to leave a copy of D_5 over which the two points removed had mixed types, there are up to duality only two distinct possibilities: $\{2, 3\}$ and $\{3, 4\}$; so these must be the two possibilities we have just identified. ■

10 Extensions of D_{2n+1}

We show that no indecomposable tournament omitting N_5 can be obtained by adjoining one or two vertices to D_{2n+1}.

Lemma 10.1 *If the tournament T is an extension of D_{2n+1} by a single vertex, with $n \geq 3$, and $N_5 \not\leq T$, then T is decomposable.*

Proof: Let T be an extension of D_{2n+1} by the vertex x. Let B_5 denote the first copy of $B_5 \simeq D_5$ inside D_{2n+1}, in other words, $T \restriction \{1, \ldots, 5\}$. We examine the various possibilities for the type of x over B_5. The extension by x must, by Corollary 8.3, be either decomposable or isomorphic to B_6. Let $T_1 = T \restriction V(B_5) \cup \{x\}$.

First we assume that $T_1 \simeq B_6$. B_6 has three vertices of outdegree 3 and three of outdegree 2. This outdegree requirement implies that (using the vertex numbering of D_5) one of the following must hold in T_1:

(i) $\{3, 4, 5\} \Longrightarrow x \Longrightarrow \{1, 2\}$ or

(ii) $\{1, 4\} \Longrightarrow x \Longrightarrow \{2, 3, 5\}$ or

(iii) $\{1, 3, 4\} \Longrightarrow x \Longrightarrow \{2, 5\}$ or

(iv) $\{4, 5\} \Longrightarrow x \Longrightarrow \{1, 2, 3\}$ or

(v) $\{1, 4, 5\} \Longrightarrow x \Longrightarrow \{2, 3\}$ or

(vi) $\{3,4\} \Longrightarrow x \Longrightarrow \{1,2,5\}$.

If one of the first four cases holds then $T_1\restriction\{3,5,4,1,x\}$, $T_1\restriction\{2,x,4,3,5\}$, $T_1\restriction\{1,3,2,x,4\}$, or $T_1\restriction\{x,5,2,1,3\}$ respectively, is isomorphic to N_5. In the fifth case, as $T\restriction\{5,4,3,x,6\} \not\simeq N_5$, we must have $x \to 6$ and, as $T\restriction\{x,5,7,6,3\} \not\simeq N_5$, we must have $7 \to x$. Consequently, $T\restriction\{2,x,7,6,3\} \simeq N_5$, a contradiction. In the sixth case, as, $T\restriction\{4,3,5,x,6\} \not\simeq N_5$, we must have$x \to 6$ and, as $T\restriction\{4,3,7,6,x\} \not\simeq N_5$, we must have $x \to 7$. Consequently, $T\restriction\{2,x,3,7,6\} \simeq N_5$, a contradiction.

Thus T_1 is decomposable. If $x \Longrightarrow B_5$ then for $3 \le i \le n$, as $T\restriction\{x,2i+1,4,3,5\} \not\simeq N_5$, we must have $x \to 2i+1$, and, as $T\restriction\{2,x,2i,3,2i+1\} \not\simeq N_5$, we must have $x \to 2i$. Consequently $x \Longrightarrow D_{2n+1}$.

On the other hand, suppose $B_5 \Longrightarrow x$. If $D_{2n+1} \Longrightarrow x$, we are done. Otherwise, let v be the smallest vertex in D_{2n+1} such that $x \to v$. Then $v = 2i$ or $v = 2i+1$ for some $i \ge 3$. We will show that $\{x,2i\}$ is a block in T.

If $v = 2i+1$ then $\{1,2,\ldots,2i\} \Longrightarrow x \to 2i+1$ by the definition of v, and for $i < j \le n$, as $T\restriction\{2i,2i-1,2i+1,x,2j\} \not\simeq N_5$, we must have $x \to 2j$ and, as $T\restriction\{2j+1,2i,2i-1,2i+1,x\} \not\simeq N_5$, we must have $x \to 2j+1$.

If $v = 2i$ then $\{1,2,\ldots,2i-1\} \Longrightarrow x \to 2i$ by the definition of v, and for $i \le j \le n$, as $T\restriction\{2i-2,2i-3,2j+1,2i,x\} \not\simeq N_5$, we must have $x \to 2j+1$. In particular, $x \to 2i+1$ and so, for $i < j \le n$, as $T\restriction\{2i-2,2i-3,2i+1,x,2j\} \not\simeq N_5$, we must have $x \to 2j$. We have now seen that $\{1,2,\ldots,2i-1\} \Longrightarrow x \Longrightarrow \{2i+1,2i+2,\ldots,2n+1\}$. Therefore, $\{x,2i\}$ is a block in T.

Now suppose that x has a mixed type over B_5. Then, since T_1 is decomposable and B_5 is indecomposable, B_5 has a vertex j such that $\{x,j\}$ is a block in T_1. If $j = 1,2,3$, or 4 we examine, for $3 \le i \le n$:

$T\restriction\{2,x,3,2i+1,4\}$ and $T\restriction\{3,5,4,x,2i\}$;

$T\restriction\{x,2i+1,4,3,5\}$ and $T\restriction\{1,3,x,2i,4\}$;

$T\restriction\{x,2,1,2i+1,4\}$ and $T\restriction\{2,1,x,2i,4\}$; and

$T\restriction\{2,1,5,x,2i\}$ and $T\restriction\{2,1,2i+1,2i,x\}$, respectively.

In each case, as these subtournaments must not be isomorphic to N_5, we find that the type of x over $\{2i,2i+1\}$ coincides with the type of j over $\{2i,2i+1\}$ and conclude that $\{x,j\}$ is a block in T.

If $j = 5$ then $\{2,4\} \Longrightarrow x \Longrightarrow \{1,3\}$ because $\{x,5\}$ is a block in T_1. Define i_0 to be the greatest integer such that $2 \le i_0 \le n$ and $2i_0 \to x$. Then for $2 \le i < i_0$, as $T\restriction\{2i+1,2,1,x,2i_0\} \not\simeq N_5$, we must have $x \to 2i+1$ and so $x \to 2i_0 - 1$ and hence for $2 \le i < i_0$, as $T\restriction\{x,2,1,2i_0-1,2i\} \not\simeq N_5$, we must have $2i \to x$. If $i_0 = n$ we have established that $\{x,2n+1\}$ is a block in T. Otherwise, using the maximality of i_0, we examine $T\restriction\{x,2,1,2i+1,2i_0+2\}$ for $i_0 < i \le n$ and conclude that $\{x,2i_0+1\}$ is a block in T. ■

Lemma 10.2 *If the tournament T is an indecomposable extension of D_{2n+1} by two vertices, with $n \geq 3$, and $N_5 \not\leq T$, then $T \simeq D_{2n+3}$.*

Proof: Let T be an indecomposable extension of D_{2n+1} by the two distinct vertices x and y. We apply Lemma 9.2. As there is no indecomposable extension of D_{2n+1} by a single vertex, we may suppose we are in either the second or third situation listed there, that is:

(Case 2) There is a block $\{x, i\}$ of $T - \{y\}$ with i a vertex of D_{2n+1}, and y has unmixed type over $D_{2n+1} - \{i\}$; or

(Case 3) There are distinct vertices i, j_1, j_2 of D_{2n+1} such that j_1 and j_2 have the same type over $D_{2n+1} - \{i, j_1, j_2\}$, $\{x, i\}$ is a block of $T - \{y\}$, $\{y, j_1\}$ is a block of $T - \{x\}$, and $\{y, j_2\}$ is a block of $T - \{i\}$.

We use the standard model of D_{2n+1} consisting of a transitive tournament L whose vertex set is partitioned into the sets L_O and L_E of odd and even vertices, with induced subtournament on L_O replaced by its dual.

We deal first with Case 2. We may suppose that y dominates $D_{2n+1} - \{i\}$, and since y has mixed type over $\{x, i\}$ and the roles of x and i may be interchanged, we may suppose that y dominates D_{2n+1}. By Lemma 9.2, since $y \Longrightarrow D_{2n+1}$ and $N_5 \not\leq T$, we have:

1. $i \to x \to y$,
2. there is no 3-cycle C in $D_{2n+1} - \{i\}$ such that exactly two vertices of C are dominated by i,
3. and there is no 3-cycle containing i that is dominated by a vertex of D_{2n+1}.

We claim that these conditions on i imply $i = 1$. If $i = 2n$ then $2n-2$ dominates the 3-cycle $(2n, 2n+1, 2n-1)$, contradicting 3. If $i \neq 2n$ is even then i dominates exactly two vertices in the 3-cycle $(i-1, i+2, i+3)$, contradicting 2. We may assume that i is odd. If $i \geq 3$ then 2 dominates the 3 cycle $(i, i+1, i+2)$, contradicting 3. Since $i = 1$ it follows that $T \simeq D_{2n+3}$, via the map $x \mapsto 1$, $y \mapsto 2$, and $i \mapsto i+2$ for i a vertex of D_{2n+1}.

Now we deal with Case 3. The vertices j_1 and j_2 have the property that exactly one vertex of D_{2n+1} has mixed type over $D_{2n+1} - \{j_1, j_2\}$, and thus j_1, j_2 are a pair of successive vertices of equal parity in L, and i is the vertex of opposite parity that separates them. Switching x and i will result in interchanging j_1 and j_2, so writing j for j_1, we may suppose that $(j_1, i, j_2) = (j, j+1, j+2)$. Note that one way to arrive at this situation is to embed D_{2n+1} into D_{2n+3} by $k \mapsto k$ for $k \leq j+1$, and $k \mapsto k+2$ for $k \geq j+2$, taking $y = j+2$ and $x = j+3$. Since T is determined up to the orientation of the edge between x and i, it suffices to check that if T is obtained by reversing the orientation of one edge between successive vertices $k, k+2$ in L_O or L_E in D_{2n+3} then T embeds N_5. We need this for $n \geq 3$, and it is true for $n \geq 2$. Specifically, if $k = 2$ then $T \restriction \{2, 4, 3, 7, 6\} \simeq N_5$. If $k \neq 2$ is even then $T \restriction \{2,1,k+1,k,k+2\} \simeq N_5$. If $k=1$ then $T \restriction \{2,1,5,4,3\} \simeq N_5$. If $k \neq 1$ is odd then $T \restriction \{k,k-1,k-2,k+2,k+1\} \simeq N_5$.

It is noteworthy that this fails for $n=1$: reversing the edge $(2,4)$ in D_5 gives a tournament isomorphic to S_5. ■

With this, all cases of the Extension Lemma have been verified, and the proof of the main result can be completed.

11 Proof of the main result

Recall that $\mathcal{A}_0$ is the class of tournaments that are isomorphic to one of the following: I, L_2, B_6, P_7, S_{2n+1}, or D_{2n+1}, $n \geq 1$. We can now prove the Extension Lemma for $\mathcal{A}_0$.

Lemma 6.2 (Extension Lemma) *Let T be an indecomposable tournament that omits N_5 and is an extension by at most two vertices of a tournament $T_0 \in \mathcal{A}_0$. Then $T \in \mathcal{A}_0$.*

Proof: The cases $T_0 \simeq I$ or L_2 are trivial.

If $T_0 \simeq S_{2n+1}$ for some $n \geq 1$ then Lemma 7.2, 7.3, or 7.5 applies. Note that $D_3 \simeq S_3$.

If $T_0 \simeq B_5$, B_6, or P_7 then Lemma 8.2, 8.5, 8.6, 9.1, or 9.3 applies. Note that $D_5 \simeq B_5$.

Finally, if $T_0 \simeq D_{2n+1}$ for some $n \geq 3$ then Lemma 10.1 or 10.2 applies. ■

We are now able to prove Theorem 1.7.

Theorem 1.6 *T is an indecomposable tournament omitting N_5 if and only if T is isomorphic to one of the following tournaments: I, L_2, B_6, P_7, S_{2n+1}, or D_{2n+1}, for $n \geq 1$.*

Recalling that $\mathcal{A}$ is the class of tournaments in $\mathcal{Q}_{N_5}$ that are indecomposable, this can be restated as:

$\mathcal{A} = \mathcal{A}_0$.

Proof: By Corollary 4.16, $\mathcal{A}_0 \subseteq \mathcal{A}$. The reverse is proved by induction on the order of $T \in \mathcal{A}$. If $|T| = 1$ then $T \in \mathcal{A}_0$. Suppose $T \in \mathcal{A}$ and $|T| > 1$. By the Reduction Lemma, T is an extension by one or two vertices of an indecomposable tournament T_0. By induction, $T_0 \in \mathcal{A}_0$. By the Extension Lemma, $T \in \mathcal{A}_0$. ■

We have now completed all the steps needed to prove the main result.

Theorem 1.5 *$\mathcal{Q}_{N_5}$ is well-quasi-ordered.*

Proof: : By Proposition 5.7, the class $\mathcal{A}_0 = \mathcal{A}$ has a finite vertex encoding. By Lemma 5.13, $\mathcal{X}(\mathcal{A})$ is wqo. By Lemma 3.8, $\mathcal{Q}_{N_5} = \mathcal{X}(\mathcal{A})$ is wqo. ■

References

[1] V. E. Alekseev and D. V. Korobitsyn. Complexity of some problems on hereditary classes of graphs. *Diskret. Mat*, 4:34-40, 1992.

[2] M. Burlet and J. Fonlupt. Polynomial algorithm to recognize a Meyniel graph. In **Topics on Perfect Graphs**, eds. C. Berge and V. Chvátal, North Holland, Amsterdam, 225–252, 1984. MR 86g #05078.

[3] P. Cameron. Orbits of permutation groups on unordered sets II. *J. London Math. Soc.*, 23:249–264, 1981. MR 58 #11136.

[4] G. Cherlin. The classification of homogeneous directed graphs and countable homogeneous n-tournaments. *Mem. Amer. Math. Soc*, 131(No. 621), 1998.

[5] G. Cherlin. Combinatorial problems connected with finite homogeneity. In *Contemporary Mathematics*, Vol 131, 3-30, 1992.

[6] G. Cherlin. Homogeneous directed graphs. In *Finite and Infinite Combinatorics in Sets and Logic*, N. Sauer, R. Woodrow, and B. Sands, eds., NATO ASI Series C: Mathematical and Physical Sciences, vol. 411, Kluwer Academic Publishers, Norwell, MA, 81–95, 1993.

[7] G. Cherlin and B. J. Latka. A decision problem involving tournaments. *DIMACS Technical Report 96-11*, 1996.

[8] G. Cherlin and B. J. Latka. Minimal antichains in well-founded quasi-orders with an application to tournaments. *J. Combinatorial Theory Ser. B*, 80:258-276, 2000.

[9] V. Chvátal and N. Sbihi. Recognizing perfect graphs. *J. Combinatorial Theory Ser. B*, 44:154-176, 1988.

[10] D. G. Corneil and J. Fonlupt. Stable set bonding in perfect graphs and parity graphs. *J. Combinatorial Theory Ser. B*, 59:1-14, 1993.

[11] A. Ehrenfeucht and G. Rozenberg. Primitivity is hereditary for 2-structures. *Theor. Comp. Sci.* **70** 343–358, 1990.

[12] P. Erdös and Moser, L. A problem on tournaments. *Can Math Bull*, 7:351-356, 1964.

[13] M. Fellows and M. Langston. On well-partial-order theory and its application to combinatorial problems of VLSI design. *SIAM J. Discrete Math*, 5:117-126, 1992.

[14] J. Fonlupt and A. Zemirline. A polynomial algorithm for recognizing $K_4 \setminus e$-free perfect graphs. preprint.

[15] E. Fried. On homogeneous tournaments. In **Combinatorial Theory and its Applications Vol II**, P. Erdös, et. al. eds, North-Holland, Amsterdam, 467-476, 1970.

[16] E. Fried and H. Laskar, Simple tournaments. *Notices Amer. Math. Soc.* **18**,395, 1971.

[17] H. Friedman, N. Robertson, and P. Seymour. The metamathematics of the graph minor theorem. in **Logic and Combinatorics**, Arcata, California, 1985, 229-261, Contemporary Mathematics **65**, American Mathematical Society, Providence, R.I., 1987.

[18] J. Gallier. What's so special about Kruskal's Theorem and the ordinal Γ_0? A survey of some results in proof theory. *Annals of Pure and Applied Logic* **53** (1991), 199-260.

[19] C. Ward Henson. Countable homogeneous relational systems and categorical theories. *J. Symb. Logic*, 37:494–500, 1972. MR 48 #94.

[20] G. Higman. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.*, 2:326–336, 1952. MR 14-238.

[21] W. L. Hsu. Recognizing planar perfect graphs. *J. Assoc. Computing Machinery* 34:25-288, 1987.

[22] P. Ille. Recognition problem in reconstruction for decomposable relations In **Finite and Infinite Combinatorics in Sets and Logic**, N. Sauer, R. Woodrow, and B. Sands, eds., NATO ASI Series C: Mathematical and Physical Sciences, vol. 411, Kluwer Academic Publishers, Norwell, MA, 189–198, 1993.

[23] T. A. Jenkyns and C. St. J. A. Nash-Williams. Counterexamples in the theory of well-quasi-ordered sets. *Proc. 2nd Ann Arbor Graph Theory Conf.*, 87–91, 1968.

[24] D. S. Johnson. The NP-completeness column, an ongoing guide. *J. of Algorithms*, 8:285-303, 1987.

[25] M. G. Kendall and B. Babington Smith. On the method of paired comparisons. *Biometrika*, 21:324-345, 1939.

[26] D. Knuth. *Axioms and Hulls, Lecture Notes in Computer Science 606*. Springer-Verlag, Berlin, 1992.

[27] J. Komsa, B. J. Latka, E. Murray, and E. Weiss. An Improved Result on Infinite Antichains of Tournaments. *Bulletin of the Institute of Combinatorics and its Applications*, 21:22-26, 1997.

[28] J. B. Kruskal. Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture. *Trans. American Math. Soc.*, 95(2):210–225, May 1960. MR 22 #2566.

[29] A. H. Lachlan. Countable homogeneous tournaments. *Trans. Amer. Math. Soc.*, 284:431–461, 1984. MR 85i:05118.

[30] B. J. Latka. Finitely constrained classes of homogeneous directed graphs. *J. Symb. Logic*, 59:124–139, 1994. MR 95d:03065.

[31] B. J. Latka. A structure theorem for tournaments omitting N_5. *J. Graph Theory*, to appear.

[32] B. J. Latka. A structure theorem for tournaments omitting $IC_3(I, I, L_3)$. Submitted to *J. Graph Theory*.

[33] B. J. Latka. Some classes of tournaments defined by obstructions. *Congr. Num.*, 104:81–87, 1994.

[34] B. J. Latka. Tournaments, embeddings, and well-quasi-orderings. In **Graph Theory, Combinatorics, and Applications, Proc. 7th Quadrennial International Conference on the Theory and Application of Graphs**, vol. 2, 685–696, 1995.

[35] B. J. Latka. Well-quasi-ordered classes of tournaments. In **Proc. 8th Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms, and Applications**, vol. 2, 597–606, 1999.

[36] J. W. Moon. Tournaments whose subtournaments are irreducible or transitive. *Can. Math. Bull.*,21:75–79, 1979.

[37] C. St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proc. Cambridge Phil. Soc.*, 59:833–835, 1963. MR 27 #3564.

[38] N. Robertson and P. Seymour. Graph minors II Algorithmic aspects of tree-width *J. Algorithms* **7** (1986), 309-322.

[39] N. Robertson and P. Seymour. Graph Minors IV Tree-width and well-quasi-ordering. *J. Combinatorial Theory Ser. B* **48** (1990),227-254.

[40] N. Robertson and P. Seymour. Graph Minors X. Obstructions to tree-decomposition. *J. Combinatorial Theory Ser. B* **52** (1991),153-190.

[41] J. Schmerl and W. Trotter. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures. *Discrete Math* **113** (1993), 191-205.

[42] K. Truemper and Y. Soun. Minimal forbidden subgraphs of unimodular multicommodity networks. *Math. Oper. Res*, 4: 379-389, 1979.