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On the Edge-Forwarding Indices of Frobenius Graphs

by

Yan Wang[1,2] Xin Gui Fang[3,4] D. Frank Hsu[5]

[1]Supported by the Natural Sciece Foundation and RFDP of China, and by Com^{2}MaC-KOSEF
[2]LMAM & Department of Mathematics, Peking University, Beijing 100871, China
Department of Mathematics, Qufu Normal University, Qufu 273165, China
Combinatorial and Computational Mathematics Center(Com^{2}Mac), Pohang University of Science and Technology, San 31 Hyoja Dong, Nam-Gu, Pohang, 790-784, The Republic of Korea
Email: wangy@com2mac.postech.ac.kr or wangy42@yahoo.com.cn
[3]Supported by the Natural Sciece Foundation and RFDP of China.
[4]LMAM & Department of Mathematics, Peking University, Beijing 100871, China
[5]Department of Computer & Information Sciences, Fordham University, New York, New York 10023, USA
DIMACS Center, Rutgers University, 96 Frelinghuysen Road, Piscataway, New Jersey 08854, USA

ABSTRACT

A G-Frobenius graph, as defined recently by Fang, Li, and Praeger, is a connected orbital graph of a Frobenius group $G = K{:}H$ with Frobenius kernel K and Frobenius complement H. Γ is also shown to be a Cayley graph, $\Gamma = Cay(K, S)$ for K and some subset S of the group G. On the other hand, a network N with a routing function R, written as (N, R), is an undirected graph N together with a routing R which consists of a collection of simple paths connecting every pair of vertices in the graph. The edge-forwarding index $\pi(\Gamma)$ of a network (N, R), defined by Heydemann, Meyer, and Sotteau, is a parameter to describe the maximum load of edges of N. In this paper, we study the edge-forwarding index of Frobenius graphs. In particular, we obtain edge-forwarding index of a G-Frobenius graph Γ with $\mathrm{rank}(G) \leq 50$ and those of Γ which has $type-(n_1, n_2, ..., n_d)$ where $d = n, (1, 2, 3, ..., n); d = 2n-1, (1, 2, ..., n-1, n, n-1, ..., 2, 1); d = 2n, (1, 2, ..., n-1, n, n, n-1, ...2, 1)$, respectively.

1 Introduction

In[2], Chung et al introduced the concept of forwarding index of communication networks. Then in [6], Heydemann et al defined that edge-forwarding index $\Pi(\Gamma)$ of a finite graph Γ as a measure of the maximal load carried by an edge of Γ. In general, we use a graph to model an interconnection network which consists of hardware and/or software entities that are interconnected to facilitate efficient computation and communications (see [5]).

For a given graph(or a communication network) Γ, it is difficult to determine $\Pi(\Gamma)$ explicitly in general cases. However, a method was given in [6] to compute $\Pi(\Gamma)$ for a certain class of graphs Γ which involved taking the average of the distances between pairs of distinct vertices of Γ. The method can be used to graphs which have certain "routing" property. In [7], Sol*é* constructed a class of graphs called orbital regular graphs, showed that they possess the required routing property, and computed the edge-forwarding index explicitly for two families of orbital regular graphs.

Fang et al in [3] introduced the G-Frobenius graph Γ as a connectd orbital graph of a Frobenius group $G = K{:}H$ with Frobenius kernel K and Frobenius complement H. They showed that almost all finite orbital regular graphs are Frobenius graphs and identified a G-Frobenius graph Γ, $G = K{:}H$, as a Cayley graph $\Gamma = Cay(K, S)$ for K and for some Cayley subset S of the form $S = x^H$ or $x^H \bigcup (x^{-1})^H$. They also studied the problem of determining the edge-forwarding index for Frobenius graphs. In particular, they obtained $\Pi(\Gamma)$ for all G-Frobenius graphs Γ with G of rank at most 5.

The purpose of this paper is to continue (based on the work by Fang et al) studying the edge-forwarding indices of Frobenius graphs. In Section 2, we will give definitions and discuss previous results. In Sections 3,4 and 5, we obtain computationally the edge-forwarding indices of all G-Frobenius graphs Γ with $\gamma(G) \leq 50$, rank of 6. The possibilities for the type and $\Pi(\Gamma)$ are explicitly listed in Tables 3.1-3.6 and 4.1-4.5. In [3], Fang et al conjectured that there are no Frobenius graphs of type-(1,1,d). We show in Section 5 that this is true under certain conditions. In Section 6, three classes of G-Frobenius graphs arising from the Frobenius group G, $G = K{:}H$ where $K = Z_t$ (or $Z_{t_1} \times Z_{t_2}$) and $H = Z_4$ are investigated. These are valency-4 Frobenius graphs with type (1,2,...,n), (1,2,...,n-1,n,n-1,...,2,1), and (1,2,...,n,n,...,2,1) respectively. In particular, we successfully calculate the edge-forwarding index of these Frobenius graphs for $t = 2n^2+2n+1, 4n^2+1$, and $t_1 = t_2 = 2n+1$ respectively.

2 Preliminaries

Let G be a permutation group on a set V. G would induce a natural action on $V \times V$ by $g(x, y) = (g(x), g(y))$ for $(x, y) \in V \times V$ and $g \in G$. If G is transitive on V, then the orbits of G in the action on $V \times V$ are called orbitals. The set $\{(x, x)|x \in V\}$ is one of the

G-orbitals and is called the trivial G-orbital. Let $(V)_2 = \{(x,y)|x,y \in V, x \neq y\}$. $(V)_2$ is a G-invariant set and each G-orbit in $(V)_2$ is called a nontrivial orbital. A permutation group G on V is said to be regular if G is transitive on V and the only element of G which fixes a point of V is the identity element of G.

Definition 2.1 Let $\Gamma = (V,E)$ be a simple undirected connected graph and G be a subgroup of the automorphism group $\mathrm{Aut}\Gamma$ of Γ. Then Γ is said to be a G-orbital regular graph if G is regular on each of its orbits in $(V)_2$, and there is a G-orbital O in $(V)_2$ such that $E = \{\{x,y\}|\ (x,y) \in O\}$.

A graph Γ is said to be an orbital regular graph if it is G-orbital regular for some $G \leq \mathrm{Aut}\Gamma$.

A Frobenius group is a transitive permutation group on a set V which is not regular on V, but has the property that the only element of G which has more than one fixed point is the identity. It was shown by Thompson(see[4] and [8]) that a finite Frobenius group G has a nilpotent normal subgroup K, called the Frobenius kernel, which is regular on V. Hence G is the semidirect product $K{:}H$, where H is the stabilizer of a point of V; each such subgroup H is called a Frobenius complement for K in G.

The following definition of a G-Frobenius graph was given by Fang et al.

Definition 2.2 Let G be a Frobenius group on a set V. A G-Frobenius graph is defined to be a connected graph Γ with vertex set V such that for some G-orbital O in $(V)_2$, the edge set of Γ is $E = \{\{x,y\}|(x,y) \in O\}$. If the Frobenius kernel of G is elementary abelian, then the Frobenius graph Γ is said to be affine.

The classes of Frobenius graphs and orbital regular graphs are very similar to each other. In fact, it is easy to show, following the above definitions, that every Frobenius graph is an orbital regular graph. In [3], Fang et al showed, in the following, that almost all orbital regular graphs are Frobenius graphs.

Lemma 2.3 Let Γ be a graph with n vertices and let $G \leq \mathrm{Aut}\Gamma$. Then Γ is a G-orbital regular graph if and only if one of the following holds:

1. Γ is a G-Frobenius graph or,
2. $\Gamma = C_n$ (a cycle with n vertices) and $G = Z_n$ where $n \geq 3$ or,
3. $\Gamma = K_{1,n-1}$ (a bipartite graph).

Given a group G, a subset S of G satisfying $S^{-1} = \{x^{-1}|x \in S\} = S$ and $1 \notin S$ is called a Cayley subset of G. A Cayley graph $\mathrm{Cay}(G,S)$ for G relative to a Cayley subset S is defined to be the graph with vertex set G and edges the pairs $\{x,y\}$ from G such that $xy^{-1} \in S$. $\mathrm{Cay}(G,S)$ is connected if and only if S is a generating set for G. Let $G = K{:}H$

be a Frobenius group with Frobenius kernel K and let Γ be a G-Frobenius graph. Fang et al [3] showed that Γ is a connected Cayley graph for K.

Lemma 2.4 Let $G = K{:}H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. Then a G-Frobenius graph is a Cayley graph $\Gamma = Cay(K, S)$ for K, for some Cayley subset S of the form

$$S = \begin{cases} x^H, & \text{if } |H| \text{ is even or } |x| = 2, \\ x^H \cup (x^{-1})^H, & \text{if } |H| \text{ is odd and } |x| \neq 2, \end{cases}$$

for some $x \in K$ with the property $< x^H >= K$. Conversely, if $x \in K$ is such that $< x^H >= K$, then $\Gamma = Cay(K, S)$ is a G-Frobenius graph where S is defined above and for $x \in K$, x^H denotes the set of conjugates $x^h = h^{-1}xh$, for $h \in H$.

For a finite group G, $\Phi(G)$ denotes the Frattini subgroup of G, which is the intersection of all maximal subgroups of G. If G is a p-group, then $G/\Phi(G)$ is an elementary abelian p-group. The following lemma links the action of Frobenius complement on the Frobenius kernel K with its action on the sylow subgroup of K.

Lemma 2.5 Let $G = K{:}H$ be a finite Frobenius group with Frobenius kernal K. Let $p_1, p_2, ..., p_t$ be the distinct prime divisors of $|K|$ and let P_i be the Sylow p_i-subgroup of K, for $i = 1, 2, ..., t$, so that $K = P_1 \times P_2 \times \cdots \times P_t$. Let $x = x_1...x_t \in K$ with $x_i \in P_i$ for $i = 1, 2, ..., t$, then

1. $< x^H >= K$ if and only if $< x_i^H >= P_i$ for each $i = 1, 2, .., t$ and
2. for each $i = 1, 2, ..., t, < x_i^H >= P_i$ if and only if $< \Phi(P_i)x_i^H >= P_i/\Phi(P_i)$.

Now we turn to the problem of computing the edge-forwarding index for orbital regular graphs and Frobenius graphs. The load of a vertex v(or edge e) in a given routing R of a graph Γ is the number of paths of R going through v(or e), where v is not an end vertex. Forwarding index was introduced by Chung et al [2] as a parameter to describe the maximum load of vertices of a graph Γ. Similarly, Heyemann et al [6] defined the edge-forwarding index of (Γ, R) as

$$\Pi(\Gamma, R) = max\Pi(\Gamma, R, e), \text{ for } e \in E(\Gamma)$$

and the edge-forwarding index $\Pi(\Gamma)$ of Γ as

$$\Pi(\Gamma) = min\Pi(\Gamma, R) \text{ on R.}$$

In the study of computer and communication network, one wishes to find networks (as graphs) Γ with $\Pi(\Gamma)$ as small as possible so that congestion on the links is optimized. In general, it is difficult to calculate $\Pi(\Gamma)$ for a given graph Γ. For an orbital regular graph $\Gamma = (V, E)$, Solé [7] gave the edge-forwarding index $\Pi(\Gamma)$ as follows:

$$\Pi(\Gamma) = \frac{1}{|E|}\sum d(u,v), \text{ for } (u,v) \in V \times V,$$

where $d(u,v)$ is the distance in Γ between vertices u and v. As a restatement of this result for G-Frobenius graphs, Fang et al gave the following expression for $\Pi(\Gamma)$ in terms of the type of Γ and used this to determine $\Pi(\Gamma)$ for all G-Frobenius graphs with G of rank at most 5.

Lemma 2.6 Let $G = K{:}H$ be a Frobenius group and let Γ be a G-Frobenius graph of type-$(n_1, n_2, ..., n_d)$, where d is the diameter of Γ and for each $i = 1, 2, ..., d$, n_i is the number of H-orbits of vertices at distance i from 1 in Γ, then

$$\Pi(\Gamma) = \begin{cases} 2\sum_{i=1}^{d} in_i, & \text{if } |H| \text{ is even or } |x| = 2, \\ \sum_{i=1}^{d} in_i, & \text{if } |H| \text{ is odd and } |x| > 2. \end{cases}$$

Moreover, $|H|$ is odd and $|x| > 2$ if and only if $n_1 = 2$.

3 Edge-Forwarding Indices of Frobenius Graphs of Non-p-groups

Let $G = K{:}H$ be a Frobenius group, where K is the Frobenius kernel and H is the Frobenius complement. Suppose $r(G) \leq 50$, we will give a description of the Frobenius graph of G and its edge-forwarding index. We say a group is a p-group for some prime number p if the order of this group is p^m for some integer number $m \geq 1$. In this section we assume K is not a p-group. We use h to denote the order of H.

Lemma 3.1 Let $G = K{:}H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. If $H \cong Z_2$, then

$$\Pi(\Gamma) = \begin{cases} \frac{n^2-1}{4}, & \text{if } |K| \text{ is odd}, \\ \frac{n^2}{4}, & \text{if } |K| \text{ is even}. \end{cases}$$

Proof: By Lemma 2.4, we know that $\Gamma = C_n$, $n = |K|$. ¿From Lemma 2.6, the result follows. From now on, we suppose $|H| > 2$ in the rest of this paper. □

Lemma 3.2 Let $G = K{:}H$ be a Frobenius group where K is the Frobenius kernel and H is the Frobenius complement. Suppose K is not a p-group and $r(G) \leq 50$, then K has two Sylow subgroups.

Proof: Suppose P is a Sylow subgroup of K, then by Scott [8, p.351], $P{:}H$ is a Frobenius group. So $|P| = th + 1$, for some integer $t \geq 1$. If K has at least three Sylow subgroups, then we have:

$$(h+1)(2h+1)(3h+1) \le |K| \le (50-1)h+1,$$

here we have $h \le 1$. This is impossible. So K has at most two Sylow subgroups. Considering K is not a p-group, K has two Sylow subgroups P_1 and P_2, so $K = P_1 \times P_2$. □

In the following, we can suppose $|P_1| \le |P_2|$ without any ambiguity. We divide the problem into four cases according to the numbers $|P_1|$ and $|P_2|$.

Case 1. Let $|P_1|=h+1$ and $|P_2|=2h+1$, then $|K|=(h+1)(2h+1) \le 49h+1$, we get $h \le 23$. All the possibilities of K and H are as follows:

h	H	K	h	H	K
18	Z_{18}	$Z_{19} \times Z_{37}$	6	Z_6	$Z_7 \times Z_{13}$
15	Z_{15}	$Z_2^4 \times Z_{31}$	4	Z_4	$Z_5 \times Z_3^2$
12	Z_{12}	$Z_{13} \times Z_5^2$	3	Z_3	$Z_2^2 \times Z_7$
8	Z_8	$Z_3^2 \times Z_{17}$			

Table 3.1

Suppose the Sylow subgroups of K is $P_1, P_2, \cdots, P_s$, then $K = P_1 \times P_2 \times \cdots \times P_s$. By Lemma 2.3-2.5, let $x = x_1x_2\cdots x_s$, $x_i \in P_i$, $1 \le i \le s$, be the element supplying for the conditions in Lemma 2.4, then $< x_i^H >= P_i$. By this rule, we can get the algorithm to calculate the type of a given Frobenius graph. We give an example to explain the algorithm.

Example. Let $G = K{:}H$ be a Frobenius group with Frobenius kernel $K \cong Z_{13} \times Z_5^2$ and Frobenius complement $H \cong Z_{12} \cong < \sigma >$. Suppose $x = x_1x_2$, $x_1 \in Z_{13}$ and $x_2 \in Z_5^2$, satisfying $< x^H >= K$, then $x_1^H \in Z_{13}$ and $x_2^H \in Z_5^2$. Suppose $x_1^\sigma = x_1^m$ for some $2 \le m \le 12$. So, $x_1^{m^{12}} = x_1$ and $|x_1^H| = 12$. We can get the proper numbers m. For x_2, suppose $x_2^\sigma = x_3$, then x_3 can not be expressed as x_2^i for any integer i because Z_5^2 has two disjoint generating elements. So, $Z_5^2 \cong < x_2 > \times < x_3 >$ and $K \cong < x_1 > \times < x_2 > \times < x_3 >$. Let $x_3^\sigma = x_2^{\sigma^2} = x_2^i x_3^j$, for some $0 \le i, j \le 4$. By $x_2^{\sigma^{12}} = x_2$ and $|x_2^H| = 12$, we get the proper numbers i, j and the corresponding

$$S = \{x^H\} = \cup_{i=0}^{11} x_1^{\sigma^i} x_2^{\sigma^i}.$$

Note that for different integers m, i, j, we may get different Cayley subset and non-isomorphic Frobenius graphs.

Algorithm:

```
>K:=Group<x_1, x_2, x_3 | x_1^13 = x_2^5 = x_3^5 = 1, x_1x_2 = x_2x_1, x_1x_3 = x_3x_1, x_2x_3 = x_3x_2> ;
> n := {[0, 0, 0]};
> for s in S do
> n join:={ [ExponentSum(s, x_1)mod 13, ExponentSum(s, x_2)mod 5, ExponentSum(s,
x_3)mod 5]};
> R := S;
> while #n lt 13 × 25 do
> n2 := {};
> R2 := {};
> for s in S do
> for r in R do
> y := rs;
> if [ExponentSum(y, x_1)mod 13, ExponentSum(y, x_2)mod 5, ExponentSum(y, x_3)mod
5]notin n
> then n2 join:={ [ExponentSum(y, x_1)mod 13, ExponentSum(y, x_2)mod 5,
ExponentSum(y, x_3)mod 5] };
> R2 join:={ y };
> end if;
> end for;
> end for;
> 'div'(#n2, 12);
> R:=R2;
> end while;
```

By the algorithm introduced above, we get Table 3.2 corresponding to Table 3.1. We use $|A|$ to denote the cardinality of the automorphism group of Γ and $|A_1|$ to denote the cardinality of the stabilizer of the identity of K.

H	K	type	$\Pi(\Gamma)$	$\lvert A\rvert$	$\lvert A_1\rvert$
Z_{18}	$Z_{19}\times Z_{37}$	$(1,8,26,4)$	222	12654	18
		$(1,8,24,6)$	226	12654	18
		$(1,8,27,3)$	220	12654	18
		$(1,8,21,9)$	232	12654	18
		$(1,8,22,8)$	230	12654	18
		$(1,7,21,10)$	236	12654	18
Z_{15}	$Z_2^4\times Z_{31}$	$(2,22,9)$	73	14880	30
		$(2,18,13)$	77	14880	30
Z_{12}	$Z_{13}\times Z_5^2$	$(1,6,13,7)$	160	3900	12
		$(1,5,15,6)$	160	3900	12
		$(1,5,12,9)$	166	3900	12
Z_8	$Z_3^2\times Z_{17}$	$(1,4,11,3)$	108	1224	8
		$(1,4,9,5)$	112	1224	8
Z_6	$Z_7\times Z_{13}$	$(1,2,3,4,5)$	110	546	6
		$(1,2,3,4,4,1)$	112	546	6
Z_4	$Z_5\times Z_3^2$	$(1,2,3,4,1)$	116	180	4
Z_3	$Z_2^2\times Z_7$	$(2,4,3)$	19	168	6

Table 3.2

Case 2. $|P_1| = h+1$, $|P_2| = 3h+1$. In this case, $|K| = (h+1)(3h+1) \leq 49h+1$, so $h \leq 15$. By the algorithm similar to that introduced in case 1, all the possibilities of K, H and the edge-forwarding indices of the corresponding Frobenius graphs in this case are listed in Table 3.3.

H	K	type	$\Pi(\Gamma)$	$\lvert A\rvert$	$\lvert A_1\rvert$
Z_{12}	$Z_{13}\times Z_{37}$	$(1,5,15,17,2)$	268	5772	12
		$(1,5,13,15,6)$	280	5772	12
		$(1,5,15,15,4)$	272	5772	12
		$(1,5,15,19)$	264	5772	12
Z_{10}	$Z_{11}\times Z_{31}$	$(1,5,13,14,1)$	222	3410	10
		$(1,5,15,13)$	216	3410	10
		$(1,5,13,15)$	220	3410	10
Z_8	$Z_3^2\times Z_5^2$	$(1,4,11,11,1)$	182	1800	8
H_8	$Z_3^2\times Z_5^2$	$(1,4,11,11,1)$	182	1800	8
Z_6	$Z_7\times Z_{19}$	$(1,2,3,4,5,5,2)$	244	798	6
		$(1,2,3,4,5,6,1)$	242	798	6
Z_4	$Z_5\times Z_{13}$	$(1,2,3,4,5,1)$	122	260	4
		$(1,2,3,4,3,2,1)$	128	260	4

Table 3.3

Case 3. $|P_1| = h+1$ and $|P_2| = 4h+1$. From $|K| = (h+1)(4h+1) \leq 49h+1$, we get $h \leq 11$. All the possible K, H and the types, the edge-forwarding indices of the corresponding Frobenius graphs are listed in Table 3.4.

H	K	type	$\Pi(\Gamma)$	$\|A\|$	$\|A_1\|$
Z_{10}	$Z_{11} \times Z_{41}$	$(1,5,15,20,4)$	312	4510	10
		$(1,5,15,21,3)$	310	4510	10
		$(1,5,15,23,1)$	306	4510	10
Z_7	$Z_2^3 \times Z_{29}$	$(2,14,17)$	81	3248	14
		$(2,14,15,2)$	83	3248	14
Z_6	$Z_7 \times Z_5^2$	$(1,2,3,4,5,6,7,1)$	296	1050	6
Z_4	$Z_5 \times Z_{17}$	$(1,2,3,4,5,3,2,1)$	190	340	4
		$(1,2,3,4,5,6)$	150	340	4
Z_3	$Z_2^2 \times Z_{13}$	$(2,4,6,5)$	48	312	6

Table 3.4

If $|P_1| = h+1$ and $|P_2| = 13h+1$, then $h \leq 2$. So we only consider $|P_2| = \delta h+1$, $\delta \leq 12$. When $5 \leq \delta \leq 12$, the corresponding Frobenius graphs and the edge-forwarding indices are listed in Table 3.5.

H	K	type	$\Pi(\Gamma)$	$\|A\|$	$\|A_1\|$
Z_8	Z_3^2	$(1,4,11,18,12)$	348	2952	8
		$(1,4,11,18,11,1)$	350	2952	8
Z_6	$Z_7 \times Z_{31}$	$(1,2,3,4,5,6,7,6,2)$	412	1302	6
		$(1,2,3,4,5,6,7,8)$	408	1302	6
Z_7	$Z_2^3 \times Z_{43}$	$(2,14,29,4)$	133	4816	14
		$(2,14,33)$	129	4816	14
Z_6	$Z_7 \times Z_{37}$	$(1,2,3,4,5,6,7,8,6,1)$	536	1554	6
		$(1,2,3,4,5,6,7,8,5,2)$	538	1554	6
Z_3	$Z_2^2 \times Z_{19}$	$(2,4,6,8,5)$	85	456	6
Z_4	$Z_5 \times Z_{29}$	$(1,2,3,4,5,6,5,4,3,2,1)$	432	580	4
		$(1,2,3,4,5,6,7,8)$	408	580	4
Z_3	$Z_2^2 \times Z_5^2$	$(2,4,6,8,9,4)$	129	1200	12
Z_4	$Z_5 \times Z_{37}$	$(1,2,3,4,5,6,7,8,4,3,2,1)$	608	740	4
		$(1,2,3,4,5,6,7,8,9,1)$	590	740	4
Z_3	$Z_2^2 \times Z_{31}$	$(2,4,6,8,10,9,2)$	178	744	6
Z_3	$Z_2^2 \times Z_{37}$	$(2,4,6,8,10,12,7)$	231	888	6

Table 3.5

Case 4. Let $|P_1| = t_1h+1$, $|P_2| = t_2h+1$ and $1 < t_1 < t_2$. Using $|K| = (t_1h+1)(t_2h+1) \leq 49h+1$, we can get the proper h. We list all the possible K and H and the corresponding types and edge-forwarding indices in Table 3.6.

H	K	type	$\Pi(\Gamma)$	$\|A\|$	$\|A_1\|$
Z_6	$Z_{13} \times Z_{19}$	$(1, 2, 3, 4, 5, 6, 7, 8, 5)$	498	1482	6
		$(1, 2, 3, 4, 5, 6, 7, 8, 4, 1)$	500	1482	6
Z_5	$Z_{11} \times Z_2^4$	$(2, 10, 20, 3)$	94	1760	10
Z_4	$Z_3^2 \times Z_{13}$	$(1, 2, 3, 4, 5, 6, 7, 1)$	296	468	4
Z_4	$Z_3^2 \times Z_{17}$	$(1, 2, 3, 4, 5, 6, 7, 4, 3, 2, 1)$	460	612	4
Z_3	$Z_7 \times Z_{13}$	$(2, 4, 6, 8, 10)$	110	546	6
		$(2, 4, 6, 8, 8, 2)$	112	546	6
Z_3	$Z_7 \times Z_4^2$	$(2, 4, 6, 8, 10, 7)$	152	672	6
Z_3	$Z_7 \times Z_{19}$	$(2, 4, 6, 8, 10, 12, 2)$	196	798	6
		$(2, 4, 6, 8, 10, 10, 4)$	198	798	6

Table 3.6

4 Edge-Forwarding Index of Frobenius Graphs of p-groups

Let $G = K{:}H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. In this section suppose K is a p-group for some prime number p and the Frattini subgroup of K is $\Phi(K) \neq 1$. Consider the quotient group $K/\Phi(K)$, we know by Scott([8], p.351) that $\Phi(K) : H$ and $(K/\Phi(K)) : H$ are all Frobenius groups. So we may suppose $|\Phi(K)| = t_1h+1$ and $|K/\Phi(K)| = t_2h + 1$ for some integers t_1 and t_2.

Case 1. $|\Phi(K)| = |K/\Phi(K)| = h+1$. In this case, $|K| = |\Phi(K)|\,|K/\Phi(K)| = (h+1)^2 \leq 49h + 1$, we get $h \leq 47$. All the possible H and K are listed in Table 4.1 and Table 4.2 respectively.

In Table 4.1, there are some vacaces because we only know the order of the Frobenius complement H but not certain about its structure. We will analyze Table 4.1 after Table 4.2. Look at Table 4.2, the order of K in the table are all p^2 for some prime number p. There are only two types of the groups with order p^2; one is the cyclic group and the other is the elementary abelian group. Under the condition $\Phi(K) \neq 1$, K are cyclic groups. Next, we will use an example to explain the algorithm to calculate the types of the Frobenius graphs in Table 4.2.

h	H	$\Phi(K)$	K	h	H	$\Phi(K)$	K
31	Z_{31}	Z_2^5	$K/Z_2^5 \cong Z_2^5$	8		Z_3^2	Z_9^2
26	Z_{26}	Z_3^3	Z_9^3	7	Z_7	Z_2^3	$K/Z_2^3 \cong Z_2^3$
24		Z_5^2	Z_{25}^2	3	Z_3	Z_2^2	$K/Z_2^2 \cong Z_2^2$
15		Z_2^4	$K/Z_2^4 \cong Z_2^4$				

Table 4.1

H	K	H	K	H	K	H	K
Z_{46}	Z_{47^2}	Z_{18}	Z_{19^2}	Z_{30}	Z_{31^2}	Z_{36}	Z_{37^2}
Z_{42}	Z_{43^2}	Z_{16}	Z_{17^2}	Z_{6}	Z_{7^2}	Z_{10}	Z_{11^2}
Z_{40}	Z_{41^2}	Z_{12}	Z_{13^2}	Z_{28}	Z_{29^2}	Z_{4}	Z_{5^2}
Z_{22}	Z_{23^2}						

Table 4.2

Example. Let $G = K{:}H$ be a Frobenius group with $K \cong Z_{25}$ and $H \cong Z_4$. By Lemma 2.4, assume $x \in K$ satisfy $< x^H >= K$ and $S = x^H$. Let $x^\sigma = x^i$, for some $s \leq i \leq 24$. Then $x^{i^4} = x$ and $|x^H| = 4$. Thus, we get all the proper numbers i and the corresponding $S = x^H$.

Algorithm:

```
> K := Group⟨x|x^25⟩;
> S := {x, x^i, x^(i^2 (mod 25)), x^(i^3 (mod 25))};
> n := {[0]};
> for s in S do
> n join:={ [ ExponentSum(s, x)mod 25 ]};
> end for;
> R := S;
> while #n lt 25 do
> n2 := {};
> R2 := {};
> for s in S do
> for y in R do
> y1 := ys;
> if [ExponentSum(y1, x)mod 25]notin n
> then n2 join:={[ExponentSum(y1, x)]};
> R2 join:={ y1};
> end if;
> end for;
> end for;
> ‘div’( #n2, 4 );
> n join:=n2;
> R := R2;
> end while;
```

We get the corresponding types and the edge-forwarding indices of Frobenius graphs derived from Table 4.2. We list them in Table 4.3.

H	K	type	$\Pi(\Gamma)$	$\|A\|$	$\|A_1\|$
Z_{46}	Z_{47^2}	$(1,23,24)$	238	101614	46
Z_{42}	Z_{43^2}	$(1,17,26)$	226	77658	42
Z_{40}	Z_{41^2}	$(1,20,21)$	208	67240	40
Z_{36}	Z_{37^2}	$(1,15,22)$	194	49284	36
Z_{30}	Z_{31^2}	$(1,12,19)$	164	28830	30
Z_{28}	Z_{29^2}	$(1,9,20)$	158	23548	28
Z_{22}	Z_{23^2}	$(1,11,12)$	118	11638	22
Z_{18}	Z_{19^2}	$(1,8,11)$	100	6498	18
Z_{16}	Z_{17^2}	$(1,6,10,1)$	94	4624	16
Z_{12}	Z_{13^2}	$(1,5,8)$	70	2028	12
Z_{10}	Z_{11^2}	$(1,4,6,1)$	62	1210	10
Z_6	Z_{7^2}	$(1,2,3,2)$	44	294	6
Z_4	Z_{25}	$(1,2,3)$	28	100	4

Table 4.3

To study the Frobenius groups possibly appear in Table 4.1, we first give some lemma.

Lemma 4.1 ([Zhang, 10, Theorem 5.1.10]) Suppose G is a 2-group, then the Frattini subgroup $\Phi(G)$ of G is $\Phi(G) = G^2$. In other words, $\Phi(G) = \{x^2 | x \in G\}$.

Look at Table 4.1. Because $\Phi(K){:}H$ is a Frobenius group, so H acts semi-regularly on $\Phi(K)/\{1\}$. When $|\Phi(K)| = |H| + 1$, we can know that the elements in $\Phi(K)/\{1\}$ have equal order. Consiquently, $\Phi(K)$ of K in Table 4.1 are all elementary abelian groups. By Gorenstein ([4], p.38 and p.339), when $|H|$ is even, K is abelian and H possesses a unique involution which necessarily is contained in $Z(H)$. Thus, we can determine the structure of some Frobenius groups when $|H|$ is even.

Lemma 4.2 Let $G = K{:}H$ be a Frobenius group. The Frobenius complement $H \cong Z_3$; the Frattini subgroup $\Phi(K) \cong Z_2^2$ and $K/\Phi(K) \cong Z_2^2$. Then K must be Z_4^2.

Proof: By Zhang ([11], Theorem7.1.3), Gorenstein ([4], p.339), and considering $|H| = 3$, we can get the conclusion. □

As to $h = 7$, $h = 15$, and $h = 31$, we do not examine all possible K and we only consider when K are abelian groups. But we conjecture that there is only one type Frobenius graph for each h, that is the type derived when K is abelian. We have the following Table.

H	$\Phi(K)$	$K/\Phi(K)$	K	type	$\Pi(\Gamma)$
Z_{31}	Z_2^5	Z_2^5	Z_4^5	(2,31)	64
Z_{15}	Z_2^4	Z_2^4	Z_4^4	(2,15)	32
Z_7	Z_2^3	Z_2^3	Z_4^3	(2,7)	16
Z_3	Z_2^2	Z_2^2	Z_4^2	(2,3)	8

Table 4.4

Case 2. If $|\Phi(K)| = |K/\Phi(K)| = 4h+1$, we get $h \leq 2$. So we need only consider $|\Phi(K)| = |K/\Phi(K)| = th+1$ for $t \leq 3$. When $2 \leq t \leq 3$, all the possible Frobenius groups and the corresponding types are listed below.

H	K	type	$\Pi(\Gamma)$	$\|A\|$	$\|A_1\|$
Z_{11}	Z_{23^2}	$(2,22,24)$	118	11638	22
Z_9	Z_{19^2}	$(2,16,22)$	100	6498	18
Z_8	Z_{17^2}	$(1,4,11,14,6)$	256	2312	8
Z_6	Z_{13^2}	$(1,2,3,4,5,6,7)$	280	1014	6
Z_5	Z_{11^2}	$(2,8,12,2)$	62	1210	10
Z_3	Z_{7^2}	$(2,4,6,4)$	44	294	6
Z_4	Z_{13^2}	$(1,2,3,4,5,6,7,8,3,2,1)$	524	676	4

Table 4.5

Case 3. When $|\Phi(K)| \neq |K/\Phi(K)|$, there are only two possibilities. One is $|\Phi(K)| = 16$, $K/\Phi(K) \cong Z_2^2$ and $H \cong Z_3$; the other is $K \cong Z_5^3$ and $H \cong Z_4$. For the first one we only consider when K is abelian. We get $K \cong Z_8^2$ and $H \cong Z_3$. The type of the relating Frobenius graph is $(2,4,6,7,2)$ and $\Pi(\Gamma) = 66, |A| = 768, |A_1| = 12$. For the second Frobenius group, the corresponding Frobenius graph has type $(1,2,3,4,5,6,4,3,2,1)$ and $\Pi(\Gamma) = 342, |A| = 500, |A_1| = 4$.

5 Affine Frobenius Graph, With Rank $r = 6$

Let $G = K{:}H$ be a Frobenius group whose Frobenius kernel K is elementary abelian. Let the rank of the G-Frobenius graph Γ is r and the order of H is h, then $|K| = (r-1)h+1$. So given an integer $r \geq 3$, there exist infinite pairs $(h, |K|)$ satisfying the above formula. In this section, we will analyze the Frobenius graphs of rank $r = 6$. For $r \leq 5$, we refer the reader to [3].

Lemma 5.1 Let $G = K{:}H$ be a Frobenius group. Suppose the rank of the G-Frobenius graph $\Gamma = Cay(K,S)$ for some Cayley subset S of K is 6, then K is a p-group for some prime number p.

Proof: If K is not a p-group, then $|K| = 5h+1 \geq (h+1)(2h+1)$. We get $h \leq 1$. This is impossible. So K is a p-group. □

If K is not an elementary abelian group, then we can suppose $|\Phi(K)| = t_1h+1$ and $|K/\Phi(K)| = t_2h+1$ for some integers $t_1, t_2 \geq 1$. From $|K| = (t_1h+1)(t_2h+1) = 5h+1$, we get $h = 3$. That is , $H \cong Z_3$, $\Phi(K) \cong Z_2^2$, and $K/\Phi(K) \cong Z_2^2$. According to the conclusions in Section 4, we know that in this case we get a Frobenius graph with type $(2, 3)$.

Lemma 5.2 Suppose Γ to be a G-Frobenius graph of Frobenius group $G = K{:}H$ with type $(2, n_2, ..., n_d)$. If the rank of G is $r = 6$, then K and H are isomorphic to the groups described above.

Proof: We need only prove that K is not an elementary abelian group. If the rank of Γ is $r = 6$, then we have $|K| = 5|H| + 1$. By Lemma 2.4, $|H|$ is odd and K is not an elementary abelian 2-group. But from $|K| = 5|H| + 1$ we know that $|K|$ is even, so K is not an elementary abelian group. □

Lemma 5.3 Given an integer $r = 6$, $G = K{:}H$ is a Frobenius group with rank 6 and Γ is the corresponding Frobenius graph. Then all the possible types of Γ are:

$$(1, 4), (2, 3), (1, 1, 3), (1, 2, 2), (1, 3, 1),$$
$$(1, 1, 1, 2), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 1, 1).$$

Proof: When the diameter of Γ is 2, the possible types of Γ is $(1, 4)$ and $(2, 3)$. Choose $K \cong Z_{101}$ and $H \cong Z_{20}$, we get a Frobenius group $G = K : H$. By the algorithm introduced in Section 3, we can get the type of the G-Frobenius graph is $(1, 4)$. The type $(2, 3)$ has appeared above.

When the diameter of Γ is 3, by Lemma 5.2 the possible types of Γ is $(1, 1, 3)$, $(1, 2, 2)$ and $(1, 3, 1)$. Choose $K \cong Z_{31}$ and $H \cong Z_6$, we get a Frobenius group $G = K : H$ whose G-Frobenius graph Γ has type $(1, 2, 2)$. Choose $K \cong Z_{41}$ and $H \cong Z_8$, we get a Frobenius graph with type $(1, 3, 1)$.

When the diameter of Γ is 4, by Lemma 5.2 the possible types of Γ is $(1, 1, 1, 2)$, $(1, 1, 2, 1)$, and $(1, 2, 1, 1)$. When the diameter is 5, Γ has only one possible type $(1, 1, 1, 1, 1)$. Choose $K \cong H_{11}$ and $H \cong Z_2$, we get a Frobenius graph with type $(1, 1, 1, 1, 1)$. □

We don't know whether there exist Frobenius graphs with types $(1, 1, 3)$, $(1, 1, 1, 2)$, $(1, 1, 2, 1)$, and $(1, 2, 1, 1)$ or not. We conjecture that these types don't exist in the following.

Conjecture: suppose G is a Frobenius group and there is a G-Frobenius graph with type $(n_1, n_2, ..., n_d)$. If there are two adjoining 1 in $(n_1, n_2, ..., n_d)$, say n_i and n_{i+1} for $1 \le i \le d-1$, then $n_j = 1$ for all $1 \le j \le d$.

In [3], the authors conjectured that no such Frobenius graphs with type $(1, 1, 2)$ exist. We prove that under some conditions this is true.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a cayley graph of group G with respect to the Cayley subset S. We call S irredundant(or minimal) if no proper subset of S can generates G.

Theorem 5.4 Let $G = K{:}H$ be a Frobenius group with Frobenius kernel K and Frobenius complement H. Γ is a G-Frobenius graph with type $(1, 1, t)$, where $\Gamma = \mathrm{Cay}(K, S)$ for some Cayley subset S of K. If S is irredundant, then $t = 1$.

Proof: Let $|H| = |S| = m$. We use I to denote the identity of K. By the assumption that S is irredundant, there are no edges between s_i and s_j for any $s_i, s_j \in S$, therefore the number of edges out from each vertex in $N_1(I)$ to $N_2(I)$ is $m - 1$. So, there are $m(m - 1)$ edges altogether out from $N_1(I)$ to $N_2(I)$.

By the condition that there are m vertex in $N_2(I)$ and suppose that the diameter of Γ is 3, then the number of edges out from each vertex in $N_2(I)$ to $N_3(I)$ is 1. As a result, there are at most m vertices in $N_3(I)$. On the other hand, there are at least $|S| = m$ vertices in $N_3(I)$. So, $t = m/|H| = 1$. □

6 Three Classes of Valency-4 Graphs

In this section, we investigated three classes of G-Frobenius graphs arising from the Frobenius group G, $G = K{:}H$ where $K = Z_t$ (or $Z_{t_1} \times Z_{t_2}$) and $H = Z_4$. These are valency-4 Frobenius graphs with type $(1, 2, ..., n)$, $(1, 2, ..., n - 1, n, n - 1, ..., 2, 1)$, and $(1, 2, ..., n, n, ...2, 1)$ respectively. We will calculate edge-forwarding index of these Frobenius graphs for $t = 2n^2 + 2n + 1, 4n^2 + 1$, and $t_1 = t_2 = 2n + 1$ respectively.

First, we have the class of Frobenius graphs with type $(1, 2, ..., n)$, $n \geq 2$ and $t = 2n^2 + 2n + 1$.

Lemma 6.1 Suppose $G = K{:}H$ is a Frobenius group with $K \cong Z_{2n^2+2n+1}$ and $H \cong Z_4$. Look K as an additive abelian group, then $S = \{\pm 1, \pm(2n+1)\}$ is a Cayley subset for some G-Frobenius graph supplying for Lemma 2.4.

Proof: Suppose H is generated by σ, that is $H \cong <\sigma>$. Consider the action of H on K, assume $1^{\sigma^4} = i^4 \equiv 1 (\mathrm{mod}\ 2n^2 + 2n + 1)$. It is easy to see that $i = 2n + 1$ is a proper choice. Thus $1^H = \{\pm 1, \pm(2n + 1)\}$, that is $S = \{\pm 1, \pm(2n + 1)\}$. □

In a graph Γ, for a vertex u we use $N_i(u)$ to denote the set of vertices in Γ with distance i from u and $d(u, v)$ the distance between u and v.

Lemma 6.2 Let $G = K{:}H$ be a Frobenius group with $K \cong Z_{2n^2+2n+1}$ and $H \cong Z_4$. If we choose the Cayley subset $S = \{\pm 1, \pm(2n + 1)\}$, then any vertex $k \in K$ and $k \in N_i(0)$ of the Frobenius graph $\Gamma = Cay(K, S)$ can be written in the form $k = x + (2n + 1)y$ for some integers x, y satisfying $|x| + |y| = d(0, k) = i$. On the contrary, if a vertex k has the form $k = x + (2n + 1)y$, then $d(0, k) \leq |x| + |y|$.

Proof: Because $K \cong Z_{2n^2+2n+1}$, we can write the elements of K as
$K = \{0, \pm 1, \pm 2, ..., \pm(n^2 + n)\}$. Given any $k \in K$, k can be expressed by the elements in S. To find a shortest way from 0 to k is to express k by the least number of elements in S. Noticing that if we choose $1\{-1\}$ we don't need to choose $-1\{1\}$ in order to acquire the

most brief expression. similarly, if we choose $(2n+1)\{-(2n+1)\}$, we don't need to choose $-(2n+1)\{(2n+1)\}$. So, if $k \in N_i(0)$, then k can be expressed as $k = x+(2n+1)y$ for some integers x, y with $|x|+|y| = i$.

On the contrary, if a vertex k has an expression $k = x + (2n+1)y$, then by analysis above, $d(k, 0) \leq |x| + |y|$. □

Theorem 6.1 G, K, H and S are as in Lemma 6.2, then the type of the Frobenius graph $\Gamma(K, S)$ is $(1, 2, ..., n)$ and the edge-forwarding index of Γ is $2\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{3}$.

Proof: Clearly, $N_1(0) = S$. We use an integer pair (x, y) to stand for a vertex $k = x + (2n+1)y$ in K. Of course, different pairs may stand for the same vertex. By Lemma 6.2, any vertex k in $N_i(0)$ has an accompanying pair (x, y) satisfying $k = x+(2n+1)y$ with $|x|+|y| = i$. Consider the pairs (x, y) with $|x|+|y| = i$, then $N_i(0) \subseteq T_i = \{x+(2n+1)y \,|\, |x|+|y| = i\}$. We use $(x, y) \longleftrightarrow x + (2n+1)y$ to stand for the pair (x, y) and its relating vertex $x + (2n+1)y$.

Consider $T_2(0)$, all the possible pairs of (x, y) and the corresponding vertices are:

$$\begin{array}{rcl}
(0, \pm 2) & \longleftrightarrow & \pm(4n+2) \\
\pm(1, 1) & \longleftrightarrow & \pm(4n+2) \\
\pm(1, -1) & \longleftrightarrow & \pm 2n \\
(\pm 2, 0) & \longleftrightarrow & \pm 2.
\end{array}$$

So $N_2(0) = T_2$ and $n_2 = 2$.

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If n is an even number, all the pairs (x, y) and the corresponding vertices are:

$$\begin{array}{rcl}
(0, \pm n) & \longleftrightarrow & \pm(n+1) \\
\pm(1, n-1) & \longleftrightarrow & \pm(3n+1) \\
\pm(1, 1-n) & \longleftrightarrow & \pm(3n+3) \\
 & \vdots & \\
\pm(\frac{n}{2} - 1, -(\frac{n}{2} + 1)) & \longleftrightarrow & \pm(n^2 - 1) \\
\pm(\frac{n}{2}, \frac{n}{2}) & \longleftrightarrow & \pm(n^2 + n)^* \\
\pm(\frac{n}{2}, -\frac{n}{2}) & \longleftrightarrow & \pm(n^2) \\
 & \vdots & \\
\pm(n-1, -1) & \longleftrightarrow & (n+2) \\
(\pm n, 0) & \longleftrightarrow & n
\end{array}$$

We can see that:

(1). $\pm(n^2+n)$ appear in $N_n(0)$, see the pair denoted by a *;

(2). Given any $k = x + (2n+1)y$, $|x|+|y| \le n$ and $(x,y) \ne \pm(\frac{n}{2}, \frac{n}{2})$, $|k\ (\bmod\ (2n^2+2n+1))| < n^2+n$. In other words, they are different from each other.

By now, we get $(1+2+3+...+n)/4+1 = 2n^2+2n+1$ elements of K, so we get all the elements. The type of Γ is $(1,2,3,...,n)$ and $\Pi(\Gamma) = \frac{n(n+1)(2n+1)}{3}$ by Lemma 2.6. When n is odd, we can prove it similarly. □

Refer to Figure 1 for the example. Some edges are omitted in the graph.

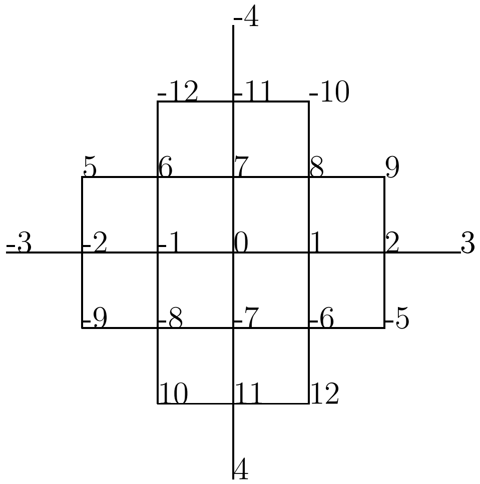

Figure 1. $n = 3, H \cong Z_4$ and $K \cong Z_{25}$

Now we turn to the class of Frobenius graphs with type $(1,2,...,n-1,n,n-1,...,2,1)$ and $t = 4n^2+1$.

Lemma 6.3 Let $G = K{:}H$ be a Frobenius group, where $K \cong Z_{4n^2+1}$ and $H \cong Z_4$. Look K as an additive abelian group, then $S = \{\pm1, \pm2n\}$ is a Cayley subset for some G-Frobenius graph satisfying for the condition of Lemma 2.4.

Proof: Suppose $H \cong <\sigma>$. Similar to Lemma 6.1, assume $1^\sigma = i$ for some integer $1 < i < 4n^2+1$, then $1^{\sigma^4} = i^4 \equiv 1(\bmod\ 4n^2+1)$. It is easy to check that $i = 2n$ fits for. Thus $S = \{\pm1, \pm2n\}$. □

Similar to Lemma 6.2, we have the following lemma.

Lemma 6.4 Let $G = K{:}H$ be a Frobenius group with $K \cong Z_{4n^2+1}$ and $H \cong Z_4$. If we choose the Cayley subset $S = \{\pm1, \pm2n\}$, then any vertex $k \in K$ and $k \in N_i(0)$ of the Frobenius graph $\Gamma = Cay(K,S)$ can be written in the form $k = x+2ny$ for some integers x, y satisfying $|x|+|y| = d(0,k) = i$. On the contrary, if a vertex k has the form $k = x+(2n+1)y$ then $d(0,k) \le |x|+|y|$.

Theorem 6.2 Let G, H, K, S are as in Lemma 6.3 and Lemma 6.4, then the type of the Frobenius graph $\Gamma = Cay(K, S)$ is $(1, 2, ..., n-1, n, n-1, ..., 2, 1)$. The edge-forwarding index of this graph is $2n^3$.

Proof: Clearly $N_1(0) = S$. By Lemma 6.4, $N_i(0) \subseteq T_i = \{x + 2ny \| x| + |y| = i\}$. So in order to find the vertices in $N_i(0)$ we only need consider the pairs (x, y) such that $|x|+|y| = i$.

All the pairs (x, y) in T_2 and the corresponding vertices are:

$$\begin{array}{rcl} (0, \pm 2) & \longleftrightarrow & \pm 4n \\ \pm(1,1) & \longleftrightarrow & \pm(2n+1) \\ \pm(1,-1) & \longleftrightarrow & \pm(2n-1) \\ (\pm 2, 0) & \longleftrightarrow & \pm 2 \\ & \cdot & \\ & \cdot & \\ & \cdot & \end{array}$$

All the possible pairs in T_n and the relating elements are:

$$\begin{array}{rcl} (0,\pm n) & \longleftrightarrow & \pm 2n^2 \\ \pm(1, n-1) & \longleftrightarrow & \pm(2n^2 - 2n + 1) \\ \pm(1, 1-n) & \longleftrightarrow & \pm(2n^2 - 2n - 1) \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \pm(n-1, 1) & \longleftrightarrow & \pm(3n-1) \\ \pm(n-1, -1) & \longleftrightarrow & \pm(n+1) \\ (\pm n,0) & \longleftrightarrow & \pm n \end{array}$$

By now, we can see that:

(1). $\pm 2n^2 \in N_n(0)$;

(2). When $|x| + |y| \leq n$ and $(x, y) \neq (0, \pm n)$,

$$|k = x + 2ny \pmod{4n^2 + 1}| < 2n^2.$$

So, the elements we get by now are different from each other. Consider the pairs and the related vertices in T_{n+1}, they are:

$$\begin{array}{rcl}
*(0,\pm(n+1)) & \longleftrightarrow & \pm(2n^2-2n+1) \\
*\pm(1,n) & \longleftrightarrow & \pm 2n^2 \\
\pm(1,-n) & \longleftrightarrow & \pm(2n^2-1) \\
\pm(2,n-1) & \longleftrightarrow & \pm(2n^2-2n+2) \\
 & \cdot & \\
 & \cdot & \\
 & \cdot & \\
\pm(n-1,-2) & \longleftrightarrow & \pm(3n+1) \\
\pm(n,1) & \longleftrightarrow & \pm 3n \\
*\pm(n,-1) & \longleftrightarrow & \pm n \\
*\pm(n+1,0) & \longleftrightarrow & \pm(n+1)
\end{array}$$

We can see that among the vertices listed above, the elements labelled by * have appeared in $N_n(0)$. So there are $4(n-1)$ elements altogether in $N_{n+1}(0)$.

When $i \geq 1$, all the pairs (x,y) and the corresponding elements in T_{n+i} are:

$$\begin{array}{rcl}
(0,\pm(n+i)) & \longleftrightarrow & \pm(2n^2-2ni+1) \in N_{n-i+1}(0) \\
\pm(1,n+i-1) & \longleftrightarrow & \pm(2n^2-2ni+2n) \in N_{n-i+1}(0) \\
\pm(-1,n+i-1) & \longleftrightarrow & \pm(2n^2-2ni+2n+2) \in N_{n-i+3}(0) \\
\pm(2,n+i-2) & \longleftrightarrow & \pm(2n^2-2ni+4n-1) \in N_{n-i+3}(0) \\
 & \cdot & \\
 & \cdot & \\
 & \cdot & \\
\pm(1-i,n+1) & \longleftrightarrow & \pm(2n^2-2n+i) \in N_{n+i-1}(0) \\
\pm(i,n) & \longleftrightarrow & \pm(2n^2-i+1) \in N_{n+i-1}(0) \\
\pm(-i,n) & \longleftrightarrow & \pm(2n^2-i) \in N_{n+i}(0) \\
 & \cdot & \\
 & \cdot & \\
 & \cdot & \\
\pm(n,i) & \longleftrightarrow & \pm(2ni+n) \in N_{n+i}(0) \\
\pm(n,-i) & \longleftrightarrow & \pm(2ni-n) \in N_{n+i-1}(0) \\
\pm(n+1,i-1) & \longleftrightarrow & \pm(2ni-n+1) \in N_{n+i-1}(0) \\
 & \cdot & \\
 & \cdot & \\
 & \cdot & \\
\pm(n+i-2,-2) & \longleftrightarrow & \pm(3n-i+2) \in N_{n-i+3}(0) \\
\pm(n+i-1,1) & \longleftrightarrow & \pm(3n+i-1) \in N_{n-i+3}(0) \\
\pm(n+i-1,-1) & \longleftrightarrow & \pm(n-i+1) \in N_{n-i+1}(0) \\
\pm(n+i,0) & \longleftrightarrow & \pm(n+i) \in N_{n-i+1}(0)
\end{array}$$

¿From the above calculation, the number of elements with distance $n+i$ from 0 to $4(n-i)$. So the type of Γ is $(1,2,...,n-1,n,n-1,...,2,1)$ and $\Pi(\Gamma) = 2(1+2^2+...+n^2+(n-1)n+(n-2)(n+1)+...+(2n-1)) = 2n^3$. □

For example, we refer the reader to Figure 2.

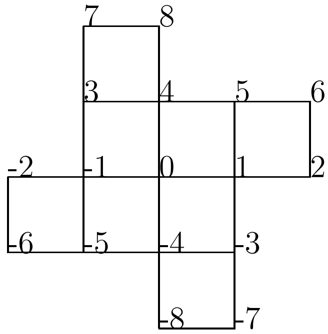

Figure 2. $n = 2, K \cong Z_{17}$ and $H \cong Z_4$

Finally, we study the class of Frobenius graphs of type $(1, 2, ..., n, n, ..., 2, 1)$ and $t_1 = t_2 = 2n + 1$.

Lemma 6.5 Suppose $G = K{:}H$ is a Frobenius group, where $K \cong Z_{2n+1} \times Z_{2n+1}$ and $H \cong Z_4$. If we see K as an additive abelian group, then $S = \{(\pm 1, 0), (0, \pm 1)\}$ is a Cayley subset of K satisfying Lemma 2.4.

Proof: Let $H \cong Z_4 \cong \langle \sigma \rangle$, Define an action of H on K as:

$$\sigma : (1, 0) \longmapsto (0, 1)$$
$$(0, 1) \longmapsto (-1, 0)$$

For any $(k_1, k_2) \in K$, define

$$(k_1, k_2)^\sigma = k_1(1, 0)^\sigma + k_2(0, 1)^\sigma = (-k_2, k_1).$$

It is easy to check that the definition of σ is proper. Choose $x = (1, 0) \in K$, then $\langle x^H \rangle = K$ and $|x^H| = 4 = |H|$. So, $S = x^H = \{(\pm 1, 0), (0, \pm 1)\}$ is a Cayley subset satisfying Lemma 2.4. □

Lemma 6.6 Let $G = K{:}H$ be a Frobenius group with $K \cong Z_{2n+1} \times Z_{2n+1}$ and $H \cong Z_4$. If we choose the Cayley subset $S = \{(\pm 1, 0), (0, \pm 1)\}$, then any vertex $k \in K$ and $k \in N_i(0, 0)$ of the Frobenius graph $\Gamma = Cay(K, S)$ can be written in the form $k = x(1, 0) + y(0, 1)$ for some integers x, y satisfying $|x| + |y| = d((0, 0), k) = i$. On the contrary, if a vertex k has the form $k = (x, 0) + (0, y)$ then $d((0, 0), k) \le |x| + |y|$.

Theorem 6.3 Let $G = K{:}H$ is a Frobenius group with Frobenius kernel $K \cong Z_{2n+1} \times Z_{2n+1}$ and Frobenius complement $H \cong Z_4$. If we choose the Cayley subset $S = \{(\pm 1, 0), (0, \pm 1)\}$, then we get a Frobenius graph $\Gamma = Cay(K, S)$ with type $(1, 2, ..., n, n, ..., 2, 1)$ and the edge-forwarding index of $\Pi(\Gamma) = n(n + 1)(2n + 1)$.

Proof: We use 0 to denote the identity $(0, 0)$ of K and

$$T_i = \{(x, y) || x| + |y| = i\}.$$

Obviously, $N_1(0) = S$.

For any $1 \leq m \leq n$, the elements of Γ in T_m are:

$$\pm(i, m-1), \pm(i, i-m),\ 0 \leq i \leq n.$$

We can see that:

(1). $\pm(n, 0)$ and $\pm(0, n)$ appear in T_n;

(2). All the elements we get by now are different from each other.

All the elements in $T_{n+i}(0)$, $1 \leq i \leq n$, are:

$$\pm(j, n+i-j), \pm(j, j-n-i),\ i \leq j \leq n$$

So, there are $4(n-i+1)$ elements in $N_{n+i}(0)$. From the above discussion, we can see that the type of Γ is $(1, 2, ..., n, n, ..., 2, 1)$ and the edge-forwarding index is $\Pi(\Gamma) = n(n+1)(2n+1)$. □

For the example, we refer the reader to Figure 3.

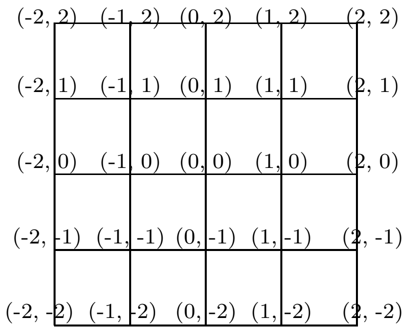

Figure 3. $n = 2, K \cong Z_5^2$ and $H \cong Z_4$.

7 Final Remarks

For a finite, simple and undirected graph, Γ, we use $V(\Gamma), E(\Gamma), A(\Gamma)$ *and* $Aut(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. An arc is an ordered pair of adjacent vertices. Γ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* or *symmetric*, if $Aut(\Gamma)$ acts transitively on $V(\Gamma), E(\Gamma)$, and $A(\Gamma)$ respectively. Furthermore, if $Aut(\Gamma)$ acts regularly on $A(\Gamma)$, then we say Γ is *arc-regular* or *one-regular.*

¿From the examples of Frobenius graphs in the preceding sections, we can see that we get a lot of 1-regular graphs. Suppose the type of Frobenius graph Γ is $(n_1, n_2, \ldots, n_d)$ then $n_1 = 1$ or $n_1 = 2$. If $n_1 = 1$, then Γ is arc-transitive because the Frobenius complement acts transitively on the Cayley subset. If $n_1 = 2$, then Γ is arc-transitive if the Frobenius kernel is abelian, by the fact that an edge-transitive Cayley graph is arc-transitive if the group is abelian. Γ is one-regular in case of the cardinality of the point stabilizer is equal to that of the Cayley subset of Γ.

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