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A Classification of Antichains of Finite Tournaments

by

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[1]Visitor

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ABSTRACT

Tournament embedding is an order relation on the class of finite tournaments. An antichain is a set of finite tournaments that are pairwise incomparable in this ordering. We say an antichain $\mathcal{A}$ can be extended to an antichain $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$. Those finite antichains that can not be extended to antichains of arbitrarily large finite cardinality are exactly those that contain a member of each of four families of tournaments. We give an upper bound on the cardinality of extensions of such antichains. This bound depends on the maximum order of the tournaments in the antichain.

Keywords: antichain, decidability, embedding, local order, subtournament, tournament

1 Introduction

A tournament T is a directed graph in which each pair of vertices is linked by a unique arc. We denote the vertex set of T by $V(T)$ and denote the cardinality of $V(T)$ by $|V(T)|$, which we also call the *order* of T. Except for the discussion in §5, in this paper we will consider only finite tournaments, that is, tournaments with finite vertex sets. For tournaments T_1 and T_2, if T_1 is isomorphic to a subtournament of T_2, we say that T_1 *embeds* in T_2, in symbols $T_1 \leq T_2$. Tournament embedding is an order relation on the class of tournaments. An *antichain* is a set of tournaments $\{T_i\}_{i \in I}$, for some index set I, such that $T_i \not\leq T_j$ for $i \neq j$. An antichain may be finite or infinite. We say an antichain $\mathcal{A}$ can be extended to an antichain $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$.

Given a finite antichain $\mathcal{A}$ of tournaments, there are three possibilities for extending $\mathcal{A}$. If the supremum of the cardinalities of antichains extending $\mathcal{A}$, which we denote by $n_{\mathcal{A}}$, is finite we say that $\mathcal{A}$ is *bounded*. If $\mathcal{A}$ cannot be extended to an infinite antichain but $n_{\mathcal{A}}$ is infinite, we say that $\mathcal{A}$ is *unbounded*. If $\mathcal{A}$ can be extended to an infinite antichain we say that $\mathcal{A}$ is *loose*. We now give examples of each of these three possibilities. If $\mathcal{A}$ consists of the transitive tournament on three vertices then $\mathcal{A}$ can be extended only to the antichain consisting of the transitive tournament on three vertices and the directed cycle on three vertices. In this case, $n_{\mathcal{A}} = 2$. More generally, if $\mathcal{A}$ is any antichain of tournaments that contains a transitive tournament, then $\mathcal{A}$ is bounded. This follows from a result of Erdös and Moser [5] or by the following simple argument: using induction on n, if a tournament T has at least 2^{n-1} vertices, then a transitive tournament on n vertices embeds in T. In the induction step, one considers any vertex a in a tournament T with at least 2^n vertices and partitions the vertices in $V(T) - \{a\}$ into the set of those that dominate a and the set of those that are dominated by a. One of these sets contains at least half the vertices and the induced tournament on this set embeds a transitive tournament on n vertices, L_n. The induced subtournament on $V(L_n) \cup \{a\}$ is a transitive tournament on $n+1$ vertices.

An antichain $\mathcal{A}$ can be changed from bounded to unbounded by making a small change to one of the tournaments. To construct an example, we define some specific tournaments.

IC_3 has vertex set $\{0, 1, 2, 3\}$ with arcs $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 2)$, $(2, 3)$, and $(3, 1)$.

C_3I has vertex set $\{0, 1, 2, 3\}$ with arcs $(1, 0)$, $(2, 0)$, $(3, 0)$, $(1, 2)$, $(2, 3)$, and $(3, 1)$.

The tournaments IC_3 and C_3I are shown in Figure 1.

Figure 1: IC_3 and C_3I

S_5 has vertex set $\{0, 1, 2, 3, 4\}$ and arc set $\{(i, j) : j - i \mod 5 \in \{1, 2\}\}$.

S_4 is the subtournament obtained from S_5 by deleting any vertex.

The tournament S_5 is shown in Figure 2.

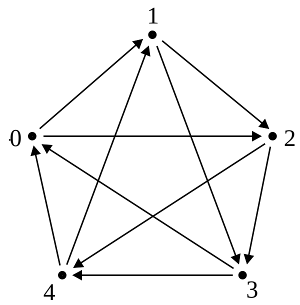

Figure 2: S_5

The tournaments that do not embed IC_3 or C_3I are called *local orders* by Cameron [1]. They are also described in [8], [9] and [17] with the names *vortex-free*, *locally transitive*, and *tournaments with property* $\mathcal{L}$, respectively. S_4 and S_5 are local orders. S_4 is the unique local order, up to isomorphism, on 4 vertices and embeds in all nontransitive local orders on 4 or more vertices. The antichain $\{IC_3, C_3I, S_4\}$ is bounded; it can only be extended by adding a transitive tournament to the set. On the other hand, the antichain $\{IC_3, C_3I, S_5\}$ is unbounded; it can be extended to the antichain

$$\{IC_3, C_3I, S_5\} \cup \{C_3(L_i, L_{n-i}, I)\}_{1 \leq i < n}$$

of cardinality $n + 2$, $n \geq 4$. The tournaments $C_3(L_i, L_{n-i}, I)$ are defined in §2. S_5 embeds in all nontransitive local orders of order at least 5 except those obtained by a composition operation: replace the vertices of a three cycle by three transitive tournaments. An antichain is obtained using the composition operation with appropriate choices for the orders of the transitive tournaments. These are the only possible antichains extending $\{IC_3, C_3I, S_5\}$ and each is finite. This shows that $\{IC_3, C_3I, S_5\}$ is not loose and is unbounded. It is shown in [10] that the antichain $\{S_5\}$ is loose. Additional examples are given in §4.

The problem that concerns us here is the following:

Main Problem: Is there an algorithm that determines whether a given finite antichain of tournaments is bounded?

Our main result settles this problem in the affirmative. We will define four infinite families of tournaments and show that a finite antichain $\mathcal{A}$ is bounded if and only if $\mathcal{A}$ contains a member of each one of these families. These families are not antichains and contain no infinite antichain but do contain arbitrarily large antichains. We will show that if $\mathcal{A}$ fails to contain a member of one of these families, then antichains that are a union of a subset of this "omitted" family together with $\mathcal{A}$ form extensions of arbitrarily large finite cardinality. Conversely, we show that if $\mathcal{A}$ does contain a member of each of these families then any tournament in an antichain extending $\mathcal{A}$ has so much structure that a bound can be computed on the number of such incomparable tournaments. This bound depends only on the maximum order of the tournaments in $\mathcal{A}$.

It is also of interest to distinguish between unbounded and loose antichains. This problem appears to be much more difficult and the known results are limited. Every subset of an infinite antichain is loose. The earliest published example of an infinite antichain of tournaments occurs in [6], though the existence of such antichains is noted in [7]. In general, given a finite antichain of tournaments, it is not known if there is an algorithm to determine whether or not it is loose, except in the case of antichains containing one or two tournaments. An explicit characterization of loose antichains that contain a single tournament is given in [10, 12, 14, 15]. We state the result here:

Theorem 1 $\{T\}$ *is not loose if and only if* T *is transitive, or* T *is isomorphic to a subtournament of* N_5, $IC_3(I, I, L_3)$, *or* $C_3(I, I, L_3)I$.

The tournament N_5 can be obtained from the transitive tournament on $\{1, \dots, 5\}$ by reversing the arcs between successive vertices. N_5 is shown in Figure 3.

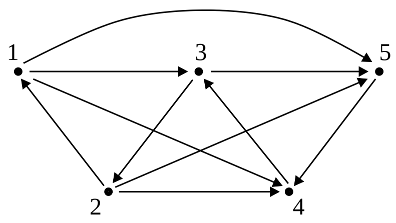

Figure 3: N_5

The tournament $IC_3(I, I, L_3)$ can be obtained from the tournament on IC_3 by replacing one of the vertices in the three cycle by a transitive tournament of order 3. $IC_3(I, I, L_3)$ is shown in Figure 4. $C_3(I, I, L_3)I$ is the *dual* (also known as the *converse*) of $IC_3(I, I, L_3)$, that is, the tournament obtained from $IC_3(I, I, L_3)$ by reversing the orientation of all the arcs.

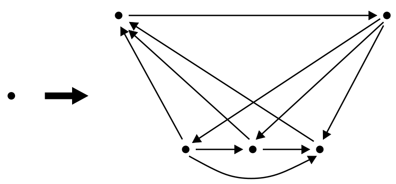

Figure 4: $IC_3(I, I, L_3)$

It is shown in [3, 4] that there is an algorithm that determines whether an antichain containing two tournaments is loose, though the proof is nonconstructive. There is strong evidence that for each positive integer k there is an algorithm that determines whether or not an antichain of cardinality k is loose, see, e.g., [3].

2 Families of Tournaments

If T is a tournament with $x, y \in V(T)$ and (x, y) an arc of T, we say x dominates y and use the notation $x \to y$. If W and W' are subtournaments of a tournament T and every vertex of W dominates every vertex of W' we use the notation $W \Longrightarrow W'$. We also say $V(W) \Longrightarrow V(W')$ and if $V(W) = \{x\}$, we say $x \Longrightarrow V(W')$.

Definition 2 *L_n is the transitive tournament on the vertex set $\{1, \dots, n\}$ whose arc relation is given by $i \to j$ if and only if $i < j$.*

We use I as well as L_1 to denote the tournament with a single vertex.

The three cycle, C_3, is the tournament on three vertices that forms a directed cycle. We construct more complex tournaments by replacing one or more vertices of C_3 by transitive tournaments. The arcs between the vertices of distinct transitive tournaments are induced by the arc relation of the three cycle. This gives us the following two definitions.

Definition 3 *$C_3(L_i, I, I)$ is the tournament with vertex set $\{1, \dots, i, a, b\}$ whose arc relation is given by $a \to b \to n \to a$, for $1 \leq n \leq i$ and $m \to n$ when $1 \leq m < n \leq i$.*

Definition 4 *$C_3(L_i, L_j, I)$ is the tournament with vertex set $\{1, \dots, i, i+1, \dots, i+j, a\}$ whose arc relation is given by $a \to m \to n \to a$, for $1 \leq m \leq i$, $i+1 \leq n \leq i+j$; $m \to n$ when $1 \leq m < n \leq i$ or $i+1 \leq m < n \leq i+j$.*

L_2 is a transitive tournament on two vertices. We construct more complex tournaments by replacing the vertices of L_2 by tournaments T_1 and T_2. The arcs between the vertices of T_1 and the vertices of T_2 are induced by the arc relation of L_2.

Definition 5 *By T_1T_2 we mean the tournament with vertex set $V(T_1) \mathbin{\dot\sqcup} V(T_2)$ that has subtournaments T_1, T_2 and $T_1 \Longrightarrow T_2$.*

The constructions we are defining here are a special case of a more general composition operation for tournaments. The special case suffices for our needs.

We now define four families of tournaments, which we will use to characterize the bounded antichains. Let $\mathcal{L} = \{L_i : i \geq 1\}$.

Definition 6 *Let $\mathcal{U}_i$ for $i = 1, 2, 3, 4$ be defined as follows:*

- $\mathcal{U}_1 = \{L_iC_3L_j : i, j \geq 0\} \cup \mathcal{L}$.
- $\mathcal{U}_2 = \{C_3(L_i, I, I)L_j : i \geq 1, j \geq 0\} \cup \mathcal{L}$.
- $\mathcal{U}_3 = \{L_jC_3(L_i, I, I) : i \geq 1, j \geq 0\} \cup \mathcal{L}$.
- $\mathcal{U}_4 = \{C_3(L_i, L_j, I) : i, j \geq 1\} \cup \mathcal{L}$.

Note that the four families are not disjoint. In fact, their intersection is infinite since it contains $\mathcal{L}$. Also $(\bigcap_{i=1}^{4} \mathcal{U}_i) \setminus \mathcal{L}$ is nonempty. It is possible for an antichain of cardinality 1 to contain a member of all four families, for example, $\mathcal{A} = \{C_3\}$.

We observe that each of the families $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, and $\mathcal{U}_4$ is downward closed under embedding. That is, if a tournament T embeds in a member of $\mathcal{U}_i$ for some $i = 1, 2, 3, 4$ then T is a member of $\mathcal{U}_i$.

3 Bounded Antichains

We have already observed that a sufficient condition for an antichain to be bounded is for the antichain to contain a transitive tournament. This is not a necessary condition since, for example, $\mathcal{A} = \{C_3\}$ is bounded and $n_{\mathcal{A}} = 2$. Other examples of bounded antichains that do not contain a transitive tournament are given in §4. Our main result shows that a necessary and sufficient condition for an antichain to be bounded is for the antichain to contain a member of each $\mathcal{U}_i$ for $i = 1, 2, 3, 4$.

Definition 7 *Let T be a tournament and W a subtournament of T. We say that W is* convex *if every vertex in $V(T) \setminus V(W)$ either dominates every vertex of W or is dominated by every vertex of W.*

Definition 8 *Define $r(n)$ to be the minimum integer such that every tournament of order at least $r(n)$ embeds L_n.*

The discussion in §1 shows that $r(n) \leq 2^{n-1}$.

For $\mathcal{A}$ a finite antichain, let $m_{\mathcal{A}} = \max\{|V(T)| \mid T \in A\}$. Recall that for a bounded antichain $\mathcal{A}$, $n_{\mathcal{A}}$ is the supremum of the cardinalities of antichains extending $\mathcal{A}$. We show that for a bounded antichain $\mathcal{A}$, $n_{\mathcal{A}}$ is a function of $m_{\mathcal{A}}$.

We first establish that if an antichain $\mathcal{A}$ contains elements of each of the $\mathcal{U}_i$'s and there is a sufficiently large tournament T in an antichain extending $\mathcal{A}$ then T has a large convex transitive subtournament.

For a tournament T and a set of vertices $S \subseteq V(T)$ we denote the subtournament induced by T on S by $T[S]$.

Lemma 9 *Let $\mathcal{A}$ be a finite antichain of finite tournaments and $\mathcal{A} \cap \mathcal{U}_i \neq \emptyset$ for all $i = 1, 2, 3, 4$. Let $k \geq 7$ be an integer. If there is a tournament T with $|V(T)| \geq r(km_{\mathcal{A}})$ such that $\{T\} \cup \mathcal{A}$ is an antichain, then there is a convex transitive subtournament $W \leq T$ and $|V(W)| \geq (k-6)m_{\mathcal{A}}$.*

Proof: Let $a = m_{\mathcal{A}}$. Suppose there is a tournament T whose vertex set has cardinality at least $r(ka)$ such that no member of $\mathcal{A}$ embeds in T. Let $p = \max\{n \mid L_n \leq T\}$. Then $p \geq ka \geq 7a$ by the choice of T and the definition of $r(ka)$. Let $\{v_1, \ldots, v_p\}$ be a subset of $V(T)$ such that the induced subtournament, $L = T[\{v_1, \ldots, v_p\}]$ is transitive and $v_i \to v_j$ for all $1 \leq i < j \leq p$. Let $W = T[\{v_{3a+1}, v_{3a+2}, \ldots, v_{p-3a}\}]$. We note that $|V(W)| = p - 6a \geq (k-6)a > 0$.

We claim that W is convex. It is certainly true that for all vertices $v \in V(L) \setminus V(W)$, either v dominates every vertex of W or is dominated by every vertex of W. Suppose, toward a contradiction, there is a vertex $x \in V(T) \setminus V(L)$ and vertices $v_i, v_j \in W$ such that $v_j \to x \to v_i$. We consider four cases based on the following:

- Does x dominate at least a of the $3a$ vertices in $\{v_1, \ldots, v_{3a}\}$?
- Does x dominate at least $2a$ of the $3a$ vertices in $\{v_{p-3a+1}, \ldots, v_p\}$?

We will show that in each case either $T[\{x\} \cup L]$ is transitive, contradicting the maximality of p, or $T[\{x\} \cup L]$ embeds one of the four tournaments:

$$\begin{aligned} S_1 &= L_a C_3 L_a \\ S_2 &= C_3(L_a, I, I) L_a \\ S_3 &= L_a C_3(L_a, I, I) \\ S_4 &= C_3(L_a, L_a, I). \end{aligned}$$

Each of these four tournaments embeds every tournament of the corresponding $\mathcal{U}_i$ whose vertex set has cardinality at most a. Let $R_i \in A \cap \mathcal{U}_i$. Then $R_i \leq S_i \leq T$, for some $1 \leq i \leq 4$, contradicting our choice of T as embedding no member of $\mathcal{A}$.

We now analyze our four cases. Suppose x dominates at least a of the vertices in $\{v_1, \ldots, v_{3a}\}$. If x dominates at least $2a$ of the vertices in $\{v_{p-3a+1}, \ldots, v_p\}$ then

$$C_3(L_a, I, I)L_a \leq T[\{v_1, \ldots, v_{3a}\} \cup \{v_j, x\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

Otherwise, at least a of the vertices in $\{v_{p-3a+1}, \ldots, v_p\}$ dominates x, and

$$C_3(L_a, L_a, I) \leq T[\{v_1, \ldots, v_{3a}\} \cup \{x\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

Suppose at least $2a$ of the vertices in $\{v_1, \ldots, v_{3a}\}$ dominate x. If at least a of the vertices in $\{v_{p-3a+1}, \ldots, v_p\}$ dominate x then

$$L_a C_3(L_a, I, I) \leq T[\{v_1, \ldots, v_{3a}\} \cup \{x, v_i\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

Otherwise, x dominates at least $2a$ of the vertices in $\{v_{p-3a+1}, \ldots, v_p\}$, and we must do a more careful analysis. If $T[\{v_i, v_j, x\}]$ is a three cycle, then

$$L_a C_3 L_a \leq T[\{v_1, \ldots, v_{3a}\} \cup \{v_i, v_j, x\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

We may assume $T[\{v_i, v_j, x\}]$ is transitive. Suppose there is a vertex v_k, $1 \leq k \leq 3a$ such that $x \to v_k$. There are at least $2a$ vertices in $v_1, \ldots, v_{3a}$ that dominate x. If at least a of these vertices have index less than k then

$$L_a C_3 L_a \leq T[\{v_1, \ldots, v_{3a}\} \cup \{x, v_k, v_j\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

Otherwise, at least a of the vertices in $\{v_1, \ldots, v_{3a}\}$ that dominate x have index greater than k, and

$$C_3(L_a, I, I)L_a \leq T[\{v_{k+1}, \ldots, v_{3a}\} \cup \{x, v_k\} \cup \{v_{p-3a+1}, \ldots, v_p\}].$$

There is a dual argument if there is a vertex v_k with $p - 3a + 1 \leq k \leq p$, such that $v_k \to x$. We conclude that $T[\{x\} \cup L]$ is transitive, contradicting the maximality of p. ■

We now establish the structure of the tournament T more explicitly.

Lemma 10 *Let $\mathcal{A}$ be a finite antichain of finite tournaments and $\mathcal{A} \cap \mathcal{U}_i \neq \emptyset$ for all $i = 1, 2, 3, 4$. If there is a tournament T with $|V(T)| \geq r(7m_{\mathcal{A}})$ such that $\{T\} \cup \mathcal{A}$ is an antichain, then T has subtournaments R, S, and W such that the following hold:*

- $V(T) = V(R) \,\dot\sqcup\, V(W) \,\dot\sqcup\, V(S)$,
- $R \Longrightarrow W \Longrightarrow S$,
- *W is a transitive subtournament,*
- $|V(R) \cup V(S)| < r(10m_{\mathcal{A}})$.

Proof: Let $a = m_{\mathcal{A}}$. Let $R_i \in A \cap U_i$. Suppose there is a tournament T whose vertex set has cardinality at least $r(7a)$ and no member of $\mathcal{A}$ embeds in T. By the previous lemma, T has a convex transitive subtournament W and $|V(W)| \geq a$. Without loss of generality, we may assume that W is a maximum convex transitive subtournament of T.

We now claim that the cardinality of $V(T) \setminus V(W) < r(10a)$. Otherwise, by the previous lemma, $T[V(T) \setminus V(W)]$ has a convex transitive subtournament W' and $|W'| \geq 4a$. By the Pigeonhole Principle, there is a subtournament $W'' \leq W'$ of cardinality at least $2a$ and $W'' \Longrightarrow W$ or $W \Longrightarrow W''$. We consider the case $W'' \Longrightarrow W$. The argument for the other case is similar.

We partition the vertices $V(T) \setminus [V(W') \cup V(W)]$ into four disjoint subsets as follows:

$$\begin{array}{rcl} X &=& \{x \in V(T) \mid x \Longrightarrow [V(W') \cup V(W)]\} \\ Y &=& \{y \in V(T) \mid V(W') \Longrightarrow y \Longrightarrow V(W)\} \\ Z &=& \{z \in V(T) \mid [V(W') \cup V(W)] \Longrightarrow z\} \\ N &=& \{n \in V(T) \mid V(W) \Longrightarrow n \Longrightarrow V(W')\}. \end{array}$$

We claim that $N = \emptyset$. Otherwise, $C_3(L_a, L_a, I) \leq T[N \cup V(W) \cup V(W'')]$ and $R_4 \leq C_3(L_a, L_a, I) \leq T$, a contradiction.

We claim that if there is a vertex $x \in X$ then $x \to y$ for every $y \in Y$. Otherwise, $C_3(L_a, I, I)L_a \leq T[\{x, y\} \cup V(W'') \cup V(W)]$ and $R_2 \leq C_3(L_a, I, I)L_a \leq T$, a contradiction.

Similarly, if there is a vertex $z \in Z$ then $y \to z$ for every $y \in Y$. Otherwise, $L_aC_3(L_a, I, I) \leq T[\{y, z\} \cup V(W'') \cup V(W)]$ and $R_3 \leq L_aC_3(L_a, I, I) \leq T$, a contradiction.

We have shown that $X \Longrightarrow V(W') \cup V(W) \cup Y \Longrightarrow Z$, i.e., the induced subtournament $T' = T[V(W') \cup V(W) \cup Y]$ is convex. By the maximality of W, T' is not transitive. There are vertices u, v, and w of T' forming a three cycle. As W is convex, at most one vertex of the three cycle is in $V(W)$. As $T[V(W') \cup Y]$ is transitive, at most two vertices of the three cycle are in $V(W') \cup Y$. We may assume $u \to v \to w \to u$, with $w \in V(W)$, $v \in V(W') \cup Y$, and $u \in V(W')$. The vertex u is dominated by every vertex of W and the vertex v dominates every vertex of W. Since there are at least $2a$ vertices of W' that dominate every vertex of W, namely, the vertices of W'', by the Pigeonhole Principle at least one of the following occur:

- there are a vertices of W' that dominate u, v, and every vertex of W,
- there are a vertices of W' that are dominated by u and dominate every vertex of W.

In the first case, $R_3 \leq L_a C_3(L_a, I, I) \leq T' \leq T$, a contradiction. In the second case, $R_4 \leq C_3(L_a, L_a, I) \leq T' \leq T$, a contradiction.

Every tournament T of order at least $r(7a)$ that embeds no member of $\mathcal{A}$ has a convex transitive subtournament with at least a vertices and $|V(T) \setminus V(W)| < r(10a)$. Such a tournament T has the structure $R \Longrightarrow W \Longrightarrow S$, where R is the induced subtournament on those vertices of T that dominate every vertex of W and S is the induced subtournament on those vertices of T that are dominated by every vertex of W. ■

Theorem 11 *Let $\mathcal{A}$ be a finite antichain of finite tournaments. Then $\mathcal{A}$ is unbounded if and only if $\mathcal{A}$ does not contain a member of $\mathcal{U}_i$ for some $i = 1, 2, 3, 4$.*

Proof: Let $a = m_{\mathcal{A}}$. Suppose $\mathcal{A}$ does not contain a member of $\mathcal{U}_1$. For all integers $t \geq a$, $\mathcal{A} \cup \{L_s C_3 L_{t-s} \mid 0 \leq s \leq t\}$ is an antichain of cardinality $|\mathcal{A}| + t + 1$ that extends $\mathcal{A}$. A similar argument holds for $\mathcal{U}_2$, $\mathcal{U}_3$, and $\mathcal{U}_4$.

Conversely, we suppose $\mathcal{A}$ contains a member of $\mathcal{U}_i$ for all $i = 1, 2, 3, 4$. We have established with the previous lemma that every tournament T that does not embed any member of $\mathcal{A}$ can be obtained by specifying a tournament D of order at most $r(10a)$, identifying a subtournament R within D, and fixing an integer $n \geq 0$. T is taken to be the tournament with vertex set $V(L_n) \cup V(D)$, with subtournaments L_n and D, and structure $R \Longrightarrow L_n \Longrightarrow D \backslash R$. Since any two tournaments constructed from the same D and R are comparable in the embedding ordering, the number of such tournaments in any antichain extending $\mathcal{A}$ is bounded by the number of choices for D times the number of choices for the subtournament R. The maximum number of incomparable tournaments of order at most $r(10a)$ is the maximum number of tournaments in an antichain in the poset of tournaments whose vertex sets are the subsets of $r(10a)$. Dilworth's Theorem says that the maximum number of elements in an antichain in a poset is the minimum number of chains needed to cover that poset. To cover the poset of tournaments we need one chain for each tournament of order $r(10a)$. By Dilworth's Theorem, one of the antichains of maximum cardinality consists of the tournaments of order $r(10a)$. The number of such tournaments is bounded by $2^{\binom{r(10a)}{2}}$. The number of choices for R is bounded by the number of subsets of $V(D)$, which is bounded by $2^{r(10a)}$.

■

Corollary 12 *Let $\mathcal{A}$ be a finite antichain of finite tournaments with $\mathcal{A} \cap \mathcal{U}_i \neq \emptyset$ for all $i = 1, 2, 3, 4$. Let $a = m_{\mathcal{A}}$. Then $n_{\mathcal{A}} \leq 2^{r(10a)} 2^{\binom{r(10a)}{2}}$.*

Proof: This is immediate from the proof of the previous theorem. ■

4 Additional Examples and Remarks

We have already seen that the antichain $\{L_3, C_3\}$ is bounded. In fact, it cannot be extended. We have also seen that $\{IC_3, C_3I, S_4\}$ is bounded and can be extended to an antichain of cardinality at most 4. See also the discussion in [11].

Theorem 11 shows that the following antichains are bounded:

$$\{L_2C_3, C_3(L_4, I, I)\} \text{ and } \{IC_3I, C_3(L_4, I, I)\}.$$

It is shown in [13] that any antichain of the form $\{C_3(L_a, L_b, L_c), L_pC_3L_q\}$, for $a, b, c \geq 1$, $a+b+c \geq 4$, and any $p, q \geq 0$, is not loose. By adding the additional constraints $a, b \geq 2$ and $p+q \geq 1$ we obtain an infinite family of unbounded and not loose antichains. For example, the following antichains are unbounded and not loose:

$$\{L_2C_3, C_3(L_2, L_2, I)\} \text{ and } \{IC_3I, C_3(L_2, L_2, I)\}.$$

Finally, it is shown in [10] that the antichains

$$\{L_2C_3, C_3L_2, IC_3I\} \text{ and } \{C_3(L_2, L_2, I), C_3(I, I, C_3)\}$$

are each loose.

5 Infinite Tournaments

In this paper we have completely characterized antichains of finite tournaments that can be extended to antichains of finite tournaments of arbitrary finite cardinality. We now consider countable tournaments, that is, tournaments whose vertex sets are countably infinite. For an antichain $\mathcal{A}$ of countable tournaments, let $n_{\mathcal{A}}$ be the supremum of the cardinalities of antichains extending $\mathcal{A}$. We say an antichain $\mathcal{A}$ is *finitely bounded* if $n_{\mathcal{A}}$ is finite. We say an antichain $\mathcal{A}$ is *countably bounded* if $\mathcal{A}$ can not be extended to an infinite antichain but $n_{\mathcal{A}}$ is infinite. We say that $\mathcal{A}$ is *uncountably bounded* if $\mathcal{A}$ can be extended to a countably infinite antichain but not to an uncountably infinite antichain. Otherwise, we say that $\mathcal{A}$ is *uncountably unbounded.* There are antichains of countable tournaments that are finitely bounded. For example, by an application of Ramsey's Theorem, every countable tournament embeds a countable transitive tournament. Let P be the countable transitive tournament with vertex set the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, and arc relation defined by the natural ordering, that is, $i \to j$ if and only if $i < j$. Let N have vertex set $\{0, -1, -2, \ldots\}$, with the same ordering. The antichain $\{N, P\}$ can not be extended. It is not known if there are antichains of countable tournaments that are countably bounded.

On the other hand, we can construct a countable antichain of countable tournaments. We say a tournament is *strongly connected* if it is not of the form T_1T_2 with $V(T_1) \neq \emptyset$ and $V(T_2) \neq \emptyset$. Let $\{B_i\}_{i<\omega}$ be an infinite antichain of finite strongly connected tournaments. Some examples of such antichains are given in [10, 13]. Define the countable tournament T_i as B_iP. Then $\{T_i\}_{i<\omega}$ is a countably infinite antichain of countable tournaments. Any subset of $\{T_i\}_{i<\omega}$ can be extended to a countably infinite antichain. It is not known if there are antichains of countable tournaments that are uncountably bounded.

We can construct an uncountable antichain of countable tournaments. For S a subset of $\mathbb{N}$, define T_S to be the tournament obtained by replacing vertex i of P by the tournament B_i for all $i \in S$. For example, if $S = \{3, 4, 5\}$, then T_S would be the tournament

$$\{0, 1, 2\} \Longrightarrow B_3 \Longrightarrow B_4 \Longrightarrow B_5 \Longrightarrow \{6, 7, \dots\} \text{ and } \{0, 1, 2\} \Longrightarrow B_4, \text{etc.}$$

A family $\mathcal{S}$ of subsets of $\mathbb{N}$ is called *Sperner* if no member of $\mathcal{S}$ properly contains any other. Let $\mathcal{S}$ be an uncountable Sperner family of subsets of $\mathbb{N}$. Then $\{T_S\}_{S\in\mathcal{S}}$ is an uncountable antichain of countable tournaments. Any subset of $\{T_S\}_{S\in\mathcal{S}}$ can be extended to a uncountable infinite antichain.

A problem that remains to be explored is the following:

Open Problem: Can one determine whether a given antichain $\mathcal{A}$ of countable tournaments is finitely bounded, countably bounded, uncountably bounded, or uncountably unbounded?

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