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On Nash-solvability in pure stationary strategies of finite games with perfect information which may have cycles [1]

by

E. Boros[2] V. Gurvich[1]

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[2]RUTCOR, Rutgers University, 640 Bartolomew road, Piscataway, New Jersey 08854-8003, USA ({boros,gurvich}@rutcor.rutgers.edu).

ABSTRACT

Let g be a positional game with perfect information modelled by a directed graph $G = (V, E)$ which is finite but may have directed cycles. A local cost function $f(i, e)$ is given for every player $i \in I$ and for every move $e \in E$. The players are allowed only pure stationary strategies, that is the move in any position is deterministic, and may depend only on this present position, not on the preceding positions or moves. Hence, the resulting play p is a directed path which begins in the initial position v_0 and either (i) ends in a final position or (ii) results in a simple directed cycle. Given a play p, for every player $i \in I$ the effective cost $f(i, p)$ is defined as the sum of all local costs along the path $f(i, p) = \sum_{e \in p} f(i, e)$ in case of (i), or as $f(i, p) = +\infty$ in case of (ii). Let us call a local cost function f and the corresponding game terminal if $f(i, e) = 0$ for every player $i \in I$ and for every move e which is not leading to a final position. The players wish to minimize their effective costs, thus in particular they should try to avoid cycles. A game is called Nash-solvable if it has at least one Nash equilibrium in pure stationary strategies and properly Nash-solvable if the corresponding play results in a final position, not in a cycle. It is easy to demonstrate that already zero-sum games with two players may not be solvable. Yet, Nash-solvability turns into an exciting open problem under the following simple additional condition: ($\clubsuit$) all local costs are non-negative, i.e. $f(i, e) \geq 0$ for every player $i \in I$ for every move $e \in E$, or under the seemingly weaker but in fact equivalent condition: ($\spadesuit$) $\sum_{e \in c} f(i, e) \geq 0$, for every player $i \in I$ and for every simple directed cycle c in G.
In all examples, which we were able to analyze, games satisfying condition ($\clubsuit$) turned out to be properly Nash-solvable, yet the Nash-solvability of such games is an open problem. In this paper we prove proper Nash-solvability for the following special cases: (a) play-once games, i.e. games in which every player controls exactly one position, (b) terminal games with only 2 players, and (c) terminal games with only 2 terminal moves. We also show that in each of these cases a Nash equilibrium can be constructed in polynomial time in the size of the graph G.

Keywords: cycle, cyclic game, directed graph, directed cycle, directed path, effective cost, local cost, Nash equilibrium, perfect information, positional game, potentials, pure strategies, saddle point, stationary strategies, strong equilibrium, shortest path.

1 Introduction

1.1 Basic notions

Let $G = (V, E)$ be a directed graph (digraph) with a fixed vertex $v_0 \in V$. For a vertex $v \in V$ we denote by $\Gamma^-(v) \subseteq E$ the set of all directed edges (u, v) which end at v, and by $\Gamma^+(v) \subseteq E$ the set of all edges (v, w) leaving vertex v. Let $V_T \subseteq V$ denote the vertices with no outgoing edges, i.e. $V_T = \{v \in V \mid \Gamma^+(v) = \emptyset\}$. The vertices of V will be called *positions*, $v_0 \in V$ being a fixed *initial position*, while vertices of V_T are called *final* or *terminal positions*. Edges in $\Gamma^+(v)$ are called the possible *moves* in position $v \in V$. A move e is called *terminal* if it leads to a terminal position, i.e. if $e \in \Gamma^-(v)$ for some terminal position $v \in V_T$. Note that such a move cannot belong to a directed cycle.

Furthermore, let $I = \{0, 1, \ldots, n\}$ denote the set of *players* and let $\phi : (V \setminus V_T) \rightarrow I \cup \{\emptyset\}$ be a mapping which assigns a player (or $\emptyset$) to every non-final position. Then, for $i \in I$ the set $V_i = \phi^{-1}(i)$ denotes the set of positions *controlled* by player i, and $V_\emptyset = \phi^{-1}(\emptyset)$ denotes the set of *random* positions. In this paper we shall focus on games with no random moves, with the exception of example 1.1.

We define a *strategy* s_i of player i as a mapping which assigns a possible move $e = s_i(v) \in \Gamma^+(v)$ to every position $v \in V_i$.

Remark 1 *In this paper we restrict the players by their pure and stationary strategies, that is every move of a player may depend only on the present position but not on the preceding positions or moves.*

Let $S_i = \bigotimes_{v \in V_i} \Gamma^+(v)$ denote the set of all strategies of player $i \in I$ and let $S = \bigotimes_{i \in I} S_i$ be the direct product of all these sets. The n-tuples of strategies $s = \{s_i \mid i \in I\} \in S$ are called *situations*. To every situation $s \in S$ we associate a sequence of vertices $p(s) = (v_0, v_1, ...)$, where v_0 is the initial position, and $s(v_j) = (v_j, v_{j+1}) \in E$ is the move in position v_j of player $i = \phi(v_j)$ for $j = 0, 1, ...$ Such a sequence $p(s)$ will be called a *play*. Since the strategies are stationary, for every nonterminal position v there exists exactly one edge $e = s(v)$ leaving vertex v (and none for a terminal positions), thus the play $p(s)$ is either a directed path in G ending at a terminal position, or it ends in a directed cycle $c(s)$ (through some nonterminal positions).

Let A denote the set of *outcomes*, consisting of all simple directed paths starting from the initial positions and ending in a terminal position, together with a special symbol C. Furthermore, let us define a *game form* $\pi : S \longrightarrow A$ by $\pi(s) = p(s)$ if $p(s)$ is a directed path, and let $\pi(s) = C$ if $p(s)$ ends in a cycle. In the first case we will call the situation s *proper* or *finite*, while in the second case it is called *cyclic* or *infinite*.

Finally, let $f : I \times E \rightarrow \mathbb{R}$ denote a *local cost function*, where the value $f(i, e)$ indicates how much player $i \in I$ pays for the move $e \in E$. Such a function will be called *terminal* if $f(i, e) = 0$ for every move $e \in E$ which is not terminal and for every player $i \in I$. Let us then associate to f the *payoff* function $u : I \times A \longrightarrow \mathbb{R} \cup \{-\infty\}$ by setting $u(i, C) = -\infty$, and $u(i, a) = -\sum_{e \in E(a)} f(i, e)$, for all outcomes (paths) $a \in A$ and for all players $i \in I$, where $E(a)$ denotes the set of edges in the simple directed path $a \in A$.

Let us also introduce the *effective cost function* $F : I \times S \to \mathbb{R} \cup \{+\infty\}$ by defining $F(i,s) = -u(i,\pi(s))$.

All players are assumed to behave rationally, i.e. they try to maximize their respective payoff, or equivalently, minimize their respective effective cost. In particular, they try to avoid directed cycles, or in other words, they wish to reach a terminal position in V_T.

Remark 2 *The above goals, to avoid cyclic plays and to minimize the cost of the resulting finite play, may sometimes be contradictory. Suppose for example that every terminal move, $e = (v, v')$ for $v' \in V_T$, is expensive for the player $i = \phi^{-1}(v)$ who can choose this move. Yet, if nobody makes such a move then the play will end in a directed cycle. For example, we can interpret such a terminal move as an unpleasant but necessary work which may be postponed by someone, if it could be done by somebody else. However, if no player makes such an unpleasant move, then a directed cycle will appear, penalizing all of them.*

We will call the tuple $g = (G, v_0, I, \phi, f)$ a *finite positional game with perfect information which may have directed cycles* or just a *game* for short. The mapping π defines the corresponding *game form*, and the pair (π, F) is the *normal form* of g. A game g is called *acyclic* if the directed graph G has no directed cycles, *play-once* if the mapping ϕ is a one-to-one correspondence between players I and non-terminal positions $V \setminus V_T$ (or in other words, if every player can control the moves in only one position), and *terminal* if the local cost function f is terminal.

Following standard terminology, a situation $s = \{s_i \mid i \in I\}$ is called a *Nash equilibrium* if no single player $i \in I$ can profit by changing his strategy $s_i \in s$ to some other strategy $s'_i \in S_i$, in other words, if

$$F(i, s \setminus \{s_i\} \cup \{s'_i\}) \geq F(i,s) \ \ \forall i \in I, \text{ and } \forall s'_i \in S_i.$$

A Nash equilibrium s is called *cyclic* if $\pi(s) = C$, and is called *proper* or *finite* otherwise. In principle, a cyclic Nash equilibrium may exist but it hardly can be considered "attractive" for the players, because the effective cost in this case is $+\infty$ for every player.

Remark 3 *We may assume, without any loss of generality that there is exactly one terminal position v_T in the considered game. Indeed, if $V_T = \emptyset$ then all the situations are cyclic and all the players will pay $+\infty$ anyway, while if $|V_T| > 1$ we can identify all the position in V_T. Such a transformation will not change the normal form of the game.*

Remark 4 *We can also assume that there exists a simple directed path from v_0 to v_T, otherwise the game contains no finite situation again. Moreover, we can assume that every vertex $v \in V$ and every edge $e \in E$ belongs to such a directed path, because otherwise we could delete this vertex or edge without any change in the normal form.*

1.2 Directed cycles and Nash-solvability

Let us start with a simple claim.

Theorem 1 *Every acyclic game (even with random moves) has a Nash equilibrium in pure stationary strategies.*

This result is perhaps well known, since e.g. it could be derived from the results of [17] with some minor modification of his arguments: A Nash equilibrium in this case can be obtained by backward induction. All positions of an acyclic game g are partially ordered by the structure of G. For every position $v \in V$ let us denote by $g(v)$ the sub-game of g for which v is the initial position. Let $i_0 = \phi^{-1}(v)$ be a player who moves in v and $(v, v_1), \dots , (v, v_k)$ be all his possible moves in v. Further, let s^j be a Nash equilibrium in sub-game $g(v_j)$ and $F^j = (F^j_i, i \in I)$ be the corresponding effective cost vector, $j = 1, \dots , k$. Take $\min_{1\leq j\leq k} \left(F^j_{i_0} + f(i_0, (v, v_j))\right)$ and let player i_0 make in v a move which realizes this minimum. Clearly, the obtained situation is a Nash equilibrium in sub-game $g(v)$. We proceed backward along the partial order of the positions starting with the terminal position $v = v_T$ until $v = v_0$. (This construction works even with random moves, just instead of $\min_{1\leq j\leq k} \left(F^j_{i_0} + f(i_0, (v, v_j))\right)$ we assign to a random position $v \in V_0$ the expectation $Exp\left[F^j_i + f(i, (v, v_j)) \mid j = 1, \dots , k\right]$ for all players $i \in I$.)

Yet, if the directed graph G of the game contains directed cycles then the positions are not partially ordered naturally, and backward induction fails. Moreover, Nash-solvability itself can fail, too.

Example 1.1 *Consider the game g in Figure 1. It consists of 5 positions v_0, v_1, v_2, v_3, v_T and 9 moves $e_{01} = (v_0, v_1)$, $e_{02} = (v_0, v_2)$, $e_{03} = (v_0, v_3)$, $e_{12} = (v_1, v_2)$, $e_{1T} = (v_1, v_T)$, $e_{23} = (v_2, v_3)$, $e_{2T} = (v_2, v_T)$, $e_{31} = (v_3, v_1)$, $e_{3T} = (v_3, v_T)$. There are 3 players $I = \{1, 2, 3\}$ who control positions v_1, v_2, v_3 respectively, while position v_0 is random, with probability distribution P_1, P_2, P_3 for the moves $e_{01} = (v_0, v_1)$, $e_{02} = (v_0, v_2)$, $e_{03} = (v_0, v_3)$, respectively.*

The local cost function is terminal: the values $f(i, (v_j, v_T))$ for the terminal moves are given in the following table

	$f(i,e)$		
	$i = 1$	$i = 2$	$i = 3$
$e_{1T} = (v_1, v_T)$	*2*	*3*	*1*
$e_{2T} = (v_2, v_T)$	*1*	*2*	*3*
$e_{3T} = (v_3, v_T)$	*3*	*1*	*2*

and $f(i, e) = 0$ for all other moves.

This is a Condorcet-type preference profile: for each player $i \in I = \{1, 2, 3\}$ making the terminal move yields the average result, and he is better off if the next player $i+1(mod\ 3)$ makes the terminal move, and worse if the previous one $i+2(mod\ 3)$ makes such a move.

Each player has only 2 possible strategies: either to make a terminal move or to move along the cycle. Hence there are 8 situations in the normal form. One can easily verify

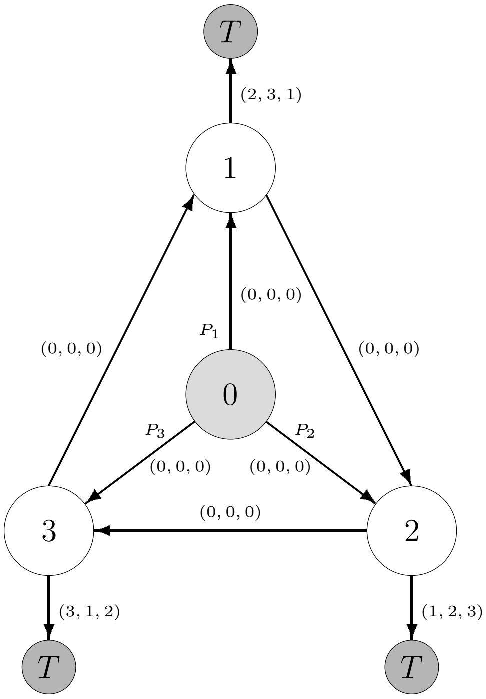

Figure 1: A play-once example with 3 players and a random initial position. The cost vectors $(f(1,e), f(2,e), f(3,e))$ are indicated along the edges.

that none of them is a Nash equilibrium. In the following table we display all 8 situations, and the corresponding costs for the players, and the player which can choose a different strategy to ensure a smaller costs for himself.

Situation	*Expected cost for player*			*Possible change*
	$i = 1$	$i = 2$	$i = 3$	$\xrightarrow{i} s^k$
$s^1 = \{e_{12}, e_{23}, e_{3T}\}$	$3P_1 + 3P_2 + 3P_3$	$P_1 + P_2 + P_3$	$2P_1 + 2P_2 + 2P_3$	$\xrightarrow{1} s^2$
$s^2 = \{e_{1T}, e_{23}, e_{3T}\}$	$2P_1 + 3P_2 + 3P_3$	$3P_1 + P_2 + P_3$	$P_1 + 2P_2 + 2P_3$	$\xrightarrow{3} s^3$
$s^3 = \{e_{1T}, e_{23}, e_{31}\}$	$2P_1 + 2P_2 + 2P_3$	$3P_1 + 3P_2 + 3P_3$	$P_1 + P_2 + P_3$	$\xrightarrow{2} s^4$
$s^4 = \{e_{1T}, e_{2T}, e_{31}\}$	$2P_1 + P_2 + 2P_3$	$3P_1 + 2P_2 + 3P_3$	$P_1 + 3P_2 + P_3$	$\xrightarrow{1} s^5$
$s^5 = \{e_{12}, e_{2T}, e_{31}\}$	$P_1 + P_2 + P_3$	$2P_1 + 2P_2 + 2P_3$	$3P_1 + 3P_2 + 3P_3$	$\xrightarrow{3} s^6$
$s^6 = \{e_{12}, e_{2T}, e_{3T}\}$	$P_1 + P_2 + 3P_3$	$2P_1 + 2P_2 + P_3$	$3P_1 + 3P_2 + 2P_3$	$\xrightarrow{2} s^1$
$s^7 = \{e_{12}, e_{23}, e_{31}\}$	$+\infty$	$+\infty$	$+\infty$	$\xrightarrow{1} s^3$
$s^8 = \{e_{1T}, e_{2T}, e_{3T}\}$	$2P_1 + P_2 + 3P_3$	$3P_1 + 2P_2 + P_3$	$P_1 + 3P_2 + 2P_3$	$\xrightarrow{1} s^6$

In fact, every player can improve situations s^7 and s^8. Thus every situation can be improved by at least one of the players, implying that there exists no Nash equilibrium *(as long as all 3 probabilities P_1, P_2, P_3 are strictly positive). Note that the above game is play-once, and the local cost function is terminal and non-negative.*

¿From now on we will restrict ourselves to games with no random moves. Still, Nash equilibria in pure stationary strategies may fail to exist.

Example 1.2 *Consider the game g in Figure 2. It consists of 4 positions v_0, $v_{1'}$, $v_{1''}$ and v_T, and there are 6 possible moves $e_{01'} = (v_0, v_{1'})$, $e_{01''} = (v_0, v_{1''})$, $e_{1'1''} = (v_{1'}, v_{1''})$, $e_{1''1'} = (v_{1''}, v_{1'})$, $e_{1'T} = (v_{1'}, v_T)$, and $e_{1''T} = (v_{1''}, v_T)$. There are two players $I = \{0, 1\}$ who control positions v_0 and $v_{1'}$, $v_{1''}$, respectively.*

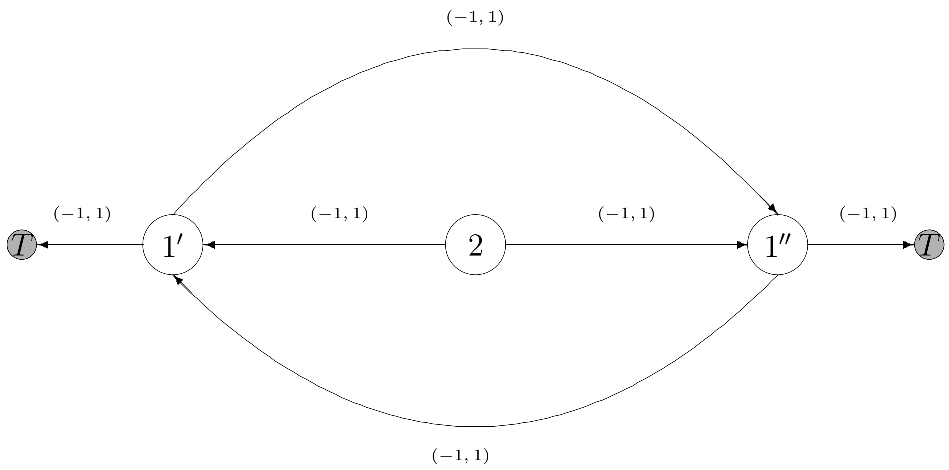

Figure 2: A simple game with 2 players and a negative cycle. Initial position is v_0 controlled by player 0, while player 1 controls positions $v_{1'}$ and $v_{1''}$. Cost vectors $(f(0, e), f(1, e))$ are indicated along the edges.

The local cost function is defined as follows: player 0 pays \$1 for every move, while player 1 gets this \$1. By definition, if the directed cycle between $(v_{1'}, v_{1''})$ appears both players pay $+\infty$, yet we can substitute $+\infty$ by 0, in order to make the game zero-sum, and still in both cases Nash equilibria do not exist.

Indeed, player 1 *has four possible strategies, and player* 0 *has two, thus we have* 8 *possible situations, and as the table below shows, in each of those one of the players can change its strategy to achieve a smaller cost for himself.*

Situation	Cost for players		Possible change
	$i=0$	$i=1$	$\xrightarrow{i} s^k$
$s^1 = \{e_{1'T}, e_{1''T}, e_{01'}\}$	2	-2	$\xrightarrow{1} s^2$
$s^2 = \{e_{1'1''}, e_{1''T}, e_{01'}\}$	3	-3	$\xrightarrow{0} s^3$
$s^3 = \{e_{1'1''}, e_{1''T}, e_{01''}\}$	2	-2	$\xrightarrow{1} s^4$
$s^4 = \{e_{1'T}, e_{1''1'}, e_{01''}\}$	3	-3	$\xrightarrow{0} s^5$
$s^5 = \{e_{1'T}, e_{1''1'}, e_{01'}\}$	2	-2	$\xrightarrow{1} s^2$
$s^6 = \{e_{1'T}, e_{1''T}, e_{01''}\}$	2	-2	$\xrightarrow{1} s^4$
$s^7 = \{e_{1'1''}, e_{1''1'}, e_{01'}\}$	∞	∞	$\xrightarrow{1} s^2$
$s^8 = \{e_{1'1''}, e_{1''1'}, e_{01''}\}$	∞	∞	$\xrightarrow{1} s^4$

(or we can substitute ∞ in the last two rows by 0 in order to make the game zero-sum). In either case, Nash equilibria do not exist.

The above game contains only one directed cycle which total cost is $1+1=2$ for player 0 and $(-1)+(-1)=-2$ for player 1. So we can say that there exists a directed cycle of negative total cost for a player. Surprisingly, in every example which we were able to analyze and in which there was no such a "negative" directed cycle, a proper Nash equilibrium did exist.

1.3 Main problem and partial results

Let us call a game *positive* if its local cost function satisfies the condition

$$\sum_{e\in E(c)} f(i,e) \geq 0 \;\text{ for every player } i\in I \;\text{ and for every directed cycle } c \text{ in } G. \qquad (\spadesuit)$$

In particular, the last condition holds if all the local costs $f(i,e)$ are non-negative, i.e. if

$$f(i,e)\geq 0 \;\text{ for all } i\in I \;\text{ and for all } e\in E. \qquad (\clubsuit)$$

In fact it is well-known that these two conditions are equivalent. In order to see this in a constructive way, let us introduce a *potential function* $U_i : V \to \mathbb{R}$ for every player $i \in I$, and let us define a new local cost for every edge $e=(v,v')\in E$ by $f_U(i,e) = f(i,e)+U_i(v)-U_i(v')$. Then, the normal forms of the two games $g=(G,v_0,I,\phi,f)$ and $g_U=(G,v_0,I,\phi,f_U)$ differ just slightly. Indeed, for any finite situation $s\in S$ the effective costs of the situation s in games g and g_U are respectively

$$F(i,s) = \sum_{e\in E(\pi(s))} f(i,e) \qquad (1a)$$

and

$$F_U(i,s) = \sum_{e=(v,v')\in E(\pi(s))} [f(i,e) + U(v) - U(v')] = F(i,s) + U_i(v_0) - U_i(v_T). \qquad (1b)$$

In other words, they just differ by the constant term $U_i(v_0) - U_i(v_T)$ for each player $i \in I$. Obviously, such a transformation does not change the set of Nash equilibria in S, and consequently, games g and g_U are equivalent.

It readily follows e.g. by the results of [15] that for any local cost function f satisfying ($\spadesuit$) there exists a potential transformation U, such that f_U satisfies ($\clubsuit$). Obviously, on its turn ($\clubsuit$) implies immediately ($\spadesuit$), and therefore, these two assumptions are indeed equivalent.

Let us also note that all terminal local cost functions are positive. Indeed, a terminal cost function automatically satisfies ($\spadesuit$), since no directed cycle can contain a terminal edge, and $f(i,e) = 0$ for every other edges $e \in E$ and for every player $i \in I$.

Main Problem. Is it true that every positive game (without random moves) has a proper Nash equilibrium?

Example 1.1 shows that the answer is negative if random moves are allowed, even for play-once terminal (and hence, positive) games.

We consider games with no random moves in the sequel, and prove, as the main result of this paper, that such games in three special cases are properly Nash-solvable, that is we prove the existence of a proper Nash equilibrium for those cases.

Theorem 2 *Every game which is (i) play-once and positive, or (ii) terminal with only two players, or (iii) terminal with only three distinct outcomes (or equivalently, with only two terminal moves) have a proper Nash equilibrium. Furthermore, such a proper Nash equilibrium can be found in polynomial time in all of the above cases.*

Remark 5 *In fact case (iii) includes every terminal game in which the cost function takes at most two different values, since we can modify the original game by introducing two new "pre-terminal" positions v'_T, and v''_T and two new terminal moves $e' = (v'_T, v_T)$, and $e'' = (v''_T, v_T)$ so that we can identify every terminal move of the original game with e' or e''.*

Let us also remark that in case (iii) the obtained proper Nash equilibrium is in fact a strong equilibrium, i.e. it is an equilibrium with respect to all coalitions $K \subseteq I$, not only with respect individual players $i \in I$ (see Section 4).

An attempt to solve the above main problem in the affirmative, in general, was recently made by [1], where unfortunately the "proof" (an induction by the number of the players) contains obvious and irreparable flaws. In fact, the authors do not make use of condition (♣), which is essential, as we shall see in Section 2.1. Moreover, exactly the same arguments would "prove" that every game in normal form has a Nash equilibrium in pure strategies. Nevertheless, their idea, to allow directed cycles and assume that these are the worst outcomes for all players, is certainly very interesting.

Let us point out here that the natural idea of *local improvements* does not necessarily converge to a Nash equilibrium, even in play-once games with nonnegative local costs: One can regard games as an optimization problem, where each of the players are trying to optimize their own objectives. In this setting a Nash equilibrium can be regarded as a local optimum, i.e. a situation where none of the players (alone) can improve on their own objective. A widely used procedure in the optimization literature is the so called *local improvement*'s algorithm, which generates a sequence of situations $s^0 \xrightarrow{i_0} s^1 \xrightarrow{i_1} s^2 \xrightarrow{i_2} \cdots \xrightarrow{i_m} s^m$, such that situation s^{j+1} in this sequence is obtained from s^j by changing the strategy of player i_j whose own objective improves by this change. The algorithm stops, when there are no such players who can improve, i.e. the last situation s^m is a Nash equilibrium.

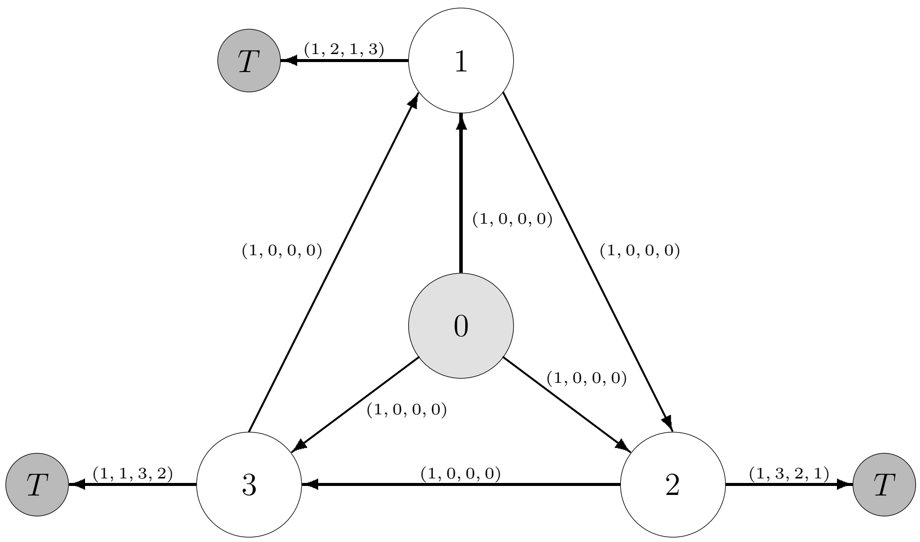

Figure 3: An example with 4 players; the local cost vector $(f(0,e), f(1,e), f(2,e), f(3,e))$ is indicated along each edge e.

The main problem with this idea is that the above procedure may cycle, without ever reaching a Nash equilibrium. The example in Figure 3 show such a simple 4-player play-once game with nonnegative local costs. In this game, player 0, which controls v_0, has a unit cost on each of the edges of the cycle $c = \{1, 2, 3\}$, and zero otherwise, i.e. his objective is to find a shortest path to the terminal position from v_0. The other three players have nonzero costs only on terminal moves, in a counter-clockwise Condorcet fashion, i.e. each of them prefers the most the terminal move from the previous position along the cycle $c = \{1, 2, 3\}$, and prefers the least the terminal move from the next position. It is easy to see that, even though this game has a Nash equilibrium (e.g., $s = \{(01), (1T), (23), (31)\}$), yet the local improvement method may cycle through nine situations, none of which is a Nash equilibrium. The table below lists these nine situations, with costs for each of the players and with arrows showing the player whose switch of strategy resulted in the transition to the next situation, in cyclical order of these nine situations.

situation	cost for players				local improvement
	$i=0$	$i=1$	$i=2$	$i=3$	$\xrightarrow{i} s^k$
$s^1 = \{(01), (12), (23), (3T)\}$	2	1	3	2	$\xrightarrow{2} s^2$
$s^2 = \{(01), (12), (2T), (3T)\}$	1	3	2	1	$\xrightarrow{0} s^3$
$s^3 = \{(03), (12), (2T), (3T)\}$	0	1	3	2	$\xrightarrow{3} s^4$
$s^4 = \{(03), (12), (2T), (31)\}$	2	3	2	1	$\xrightarrow{1} s^5$
$s^5 = \{(03), (1T), (2T), (31)\}$	1	2	1	3	$\xrightarrow{0} s^6$
$s^6 = \{(02), (1T), (2T), (31)\}$	0	3	2	1	$\xrightarrow{2} s^7$
$s^7 = \{(02), (1T), (23), (31)\}$	2	2	1	3	$\xrightarrow{3} s^8$
$s^8 = \{(02), (1T), (23), (3T)\}$	1	1	3	2	$\xrightarrow{0} s^9$
$s^9 = \{(01), (1T), (23), (3T)\}$	0	2	1	3	$\xrightarrow{1} s^1$

For finite *acyclic* games with perfect information Nash-solvability in pure strategies was proved long ago. For zero-sum games of two players we can refer to [23] and for games of n-players to [17].

Remark 6 *Both authors modelled the games not by acyclic digraphs but by trees. Clearly, for every acyclic digraph G there exists a tree G', such that the families of all plays in G and G' are isomorphic. Yet, substituting G by G' we lose the notion of a stationary strategy. However, the same arguments (backward induction) easily generalize to prove the existence of a Nash equilibrium in pure stationary strategies for acyclic games with perfect information modelled by finite acyclic digraphs (moreover, these games may even include random moves). Yet, the existence of directed cycles make the problem of Nash-solvability much more difficult (see Examples 1.1 and 1.2).*

In Figure 4 we show two examples for some simple graphs with only 3 players, which are indeed Nash solvable, as a quite lengthy derivation shows, but which do not belong to any of the classes for which we could prove Nash solvability, in general.

1.4 Nash-solvability of cyclic games and games with cycles

It may be interesting to compare the considered games, which may have cycles, with the so-called *cyclic games* defined as follows. Let us assume that $V_T = \emptyset$. Then every situation is cyclic, that is for every situation $s \in S$ the path $p(s)$ ends in a directed cycle

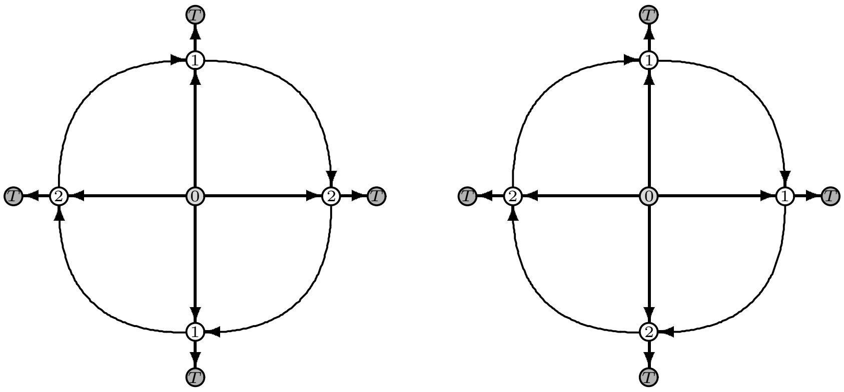

Figure 4: Simple directed graphs with 3 players, in which a Nash equilibrium exists for all positive local cost functions.

$c(s)$. The effective cost is defined as the average of the local costs of this directed cycle, that is $F(i, s) = |c(s)|^{-1} \sum_{e \in E(c(s))} f(i, e)$, or in other words, this is the cost per one move.

Zero-sum cyclic games of two players are Nash-solvable, that is every such game has a saddle point in pure stationary strategies. First, it was proved by [18, 19] for complete bipartite digraphs, then by [3] for arbitrary bipartite digraphs (in both cases two players move alternatingly), and then by [13] for arbitrary digraphs. Some further generalizations are suggested by [16] and by [21]. The proof by [13] is constructive. An algorithm is suggested which computes the value and a saddle-point of a given cyclic game, and this algorithm is proved to be finite, whenever all the local costs are rational. In the same paper, the concept of ergodic classes are studied and the existence theorem is announced for more general classes of games: (a) with discounted costs, (b) with random positions. Interestingly, the computational complexity of solving zero-sum cyclic games is still not known, though it is known that the problem belongs to both NP and co-NP, simultaneously (an observation made by [16]). The algorithm by [13] is pseudo-polynomial, shown recently by [21].

All the papers on cyclic games mentioned above make use of the same approach of potential transformations of local costs (see section 1.3), which do not change the normal form of a cyclic game, and hence do not change the set of Nash equilibria, either.

However, for non-zero-sum cyclic games Nash-solvability fails already in case of 2 players. An example is given by [13] (see also [8] for more details). In this example two players move alternatingly in the complete bipartite digraph $K_{3,3}$. In a way, this example is minimal, because for $K_{2,k}$ bipartite digraphs Nash-sovability already holds (see [10]). Some other classes of cyclic game forms, Nash-solvable in a stronger sense, were characterized by [4, 5, 6]. These characterizations are based on a general criterion of Nash-solvability for normal game forms of two players by [7, 9].

1.5 Some related difficult algorithmic problems

First of all let us note that in case of one player, $|I| = 1$, we get the well-known algorithmic problem of finding a *shortest path*: given $g = (G, v_0, f)$, find a simple directed path p from v_0 to v_T of the minimum length (cost). This problem is NP-hard in general. Indeed, let $f(e) = -1$ for all $e \in E$ then the decision problem (P)

(P) Is there a directed path in G from v_0 to v_T of cost at most $1 - |V|$?

is obviously equivalent with the well-known NP-hard problem

(P') Is there a Hamiltonian directed path in G from v_0 to v_T?

Yet, the shortest path problem is polynomial whenever G contains no directed cycle of negative cost (see [15]).

Let us remark that in this case the only player can choose any situation, and hence the only Nash equilibria are the shortest paths in G. Such optimality of Nash equilibria does not hold, in general, for games with multiple players.

Let us next consider games of two players, $I = \{0, 1\}$, such that $v_0 \in V_0$, $|V_1| = 1$ (that is $V_1 = \{v_1\}$) and for any vertex $v \in V_0$ there is an edge from v_1 to v and also two more edges from v to v_1 and to v_T. (Here $V_i = \phi^{-1}(i)$, $i = 0, 1$.) The local costs are defined as $f(i, e) = 1$ for edges e from V_i to v_T, and $f(i, e) = 0$ in all other cases. In other words, a player has to pay \$1 (let us say, to pay for the game) if he is one who makes the last (terminal) move, otherwise every move is free. This game of two players is not exactly zero-sum but the sum is constant (namely, \$1) for any finite play. However, each player pays $+\infty$ if the play is cyclic. The solution of this game depends on the following decision problem.

(Q) Does the sub-digraph $G[V_0]$, induced by V_0, contain a Hamiltonian directed path p which starts at v_0?

In case the answer to (Q) is YES, player 0 can win. Indeed, he just have to follow the Hamiltonian path p, and then finally move to v_1. If after this player 1 returns to V_0 then a directed cycle will appear. Thus player 1 should rather move to v_T and pay \$1 for the game.

In case the answer to (Q) is NO, player 1 can win. Indeed, after several possible moves within V_0, player 0 has the following alternatives: either he makes a cycle (which he will prefer not to do), or he moves to v_T and pays \$1 for the game, or he moves to v_1. In the latter case player 1 still has an opportunity to move to a position in V_0, which did not appear in the play yet, thus can pass the turn to player 0 again, with no cycle induced. Thus, player 0 sooner or later has to move to v_T and pay \$1 for the game, because otherwise he will be forced to make a cycle.

It is well known that decision problem (Q) is NP-hard. Nevertheless, our proof of Nash-solvability, given below, is constructive, and it will provide a polynomial algorithm which finds a special Nash equilibrium. In particular, such an equilibrium exists in the above example, too.

2 Nash-solvability of positive play-once games

In this section we focus on the special case of play-once positive games, for which the existence of a Nash equilibrium can always be guaranteed.

Theorem 3 *Every positive play-once game has a proper Nash equilibrium in pure stationary strategies.*

We shall show this statement by breaking the graph into further special substructures. The first simplifying assumption, which we can make without any loss of generality is the following:

Every directed cycle c and directed path p from v_0 to v_T (a play) have a vertex in common. $(\heartsuit)$

Indeed, in the case of the play-once games we can assume this without any loss of generality. Let us suppose that there exist vertex-disjoint pair, p and c. Then, let us delete the directed cycle c from G (i.e. remove all the vertices of c together with all the incident edges), and let $s' \in S'$ be a Nash equilibrium in the obtained sub-game g'. To extend s' to a Nash equilibrium $s \in S$ in the original game g, it is sufficient for all the players of c to apply the strategies which prescribe them to move along the directed cycle c. Thus, every Nash equilibrium in the sub-game g' corresponds to a Nash equilibrium in the original game g, and hence we can restrict ourselves to g'.

In what follows we discuss three special subcases, which will help us in proving Theorem 3.

2.1 Nash-solvable examples

2.1.1 St. George games

Let us consider a digraph $G = (V, E)$ consisting of $n + 2$ vertices $V = \{v_0, v_1, \ldots, v_n, v_T\}$ and $3n$ edges $E = \{(v_0, v_1), \ldots, (v_0, v_n), (v_1, v_2), (v_2, v_3), \ldots (v_{n-1}, v_n), (v_n, v_1), (v_1, v_T), \ldots, (v_n, v_T)\}$. In other words, G contains a directed cycle c of length n, the initial position v_0 and the final position v_T. For any vertex v of the directed cycle there is a move from v_0 to v, and a move from v to v_T. Let us assume also that, without exactly specifying the local costs, for each of the players v_i along the cycle the terminal move (v_i, v_T) is much worse than staying in the cycle.

The following is then a possible interpretation of such games. Let us assume that a town is conquered by a dragon who demands, as a ransom, to marry one of the n most beautiful local girls. The mayor chooses one of them and then she has to make a decision: either she accepts the proposal and then the game is over (the dragon is happy ever after, etc.), or she rejects passing the responsibility of deciding to the next girl in line, in a given pre-specified circular order. However, if all n girls reject (i.e. the play results in a cycle), then the dragon will get angry and eat everybody, including the mayor. There are only $n+1$ players in this game including the girls and the mayor, since St. George is not on the scene yet, and the dragon is a dummy, who may have some preferences but no strategies.

These very simple looking special games offer now a good opportunity to explain why backward induction does not work in games with cycles. Consider a sub-game g_i in which position v_i along the cycle is the initial position. In this sub-game the optimal move for player i in v_i will be (v_i, v_{i+1}). However, the same holds for every players $i = 1, \ldots, n$ in the cycle. Thus by backward induction we would obtain the cycle in the original game g, regardless of the first move from v_0.

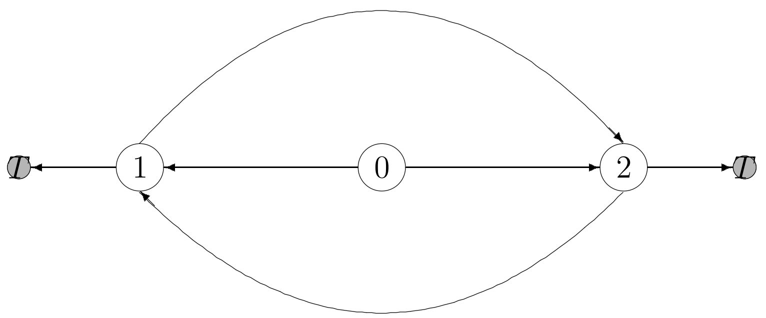

Figure 5: St. George game with $n = 2$.

As the simplest cases, let us consider first St. George games with $n = 2$ and $n = 3$ as shown respectively on Figures 5 and 6. (To simplify notation we shall use $0, 1, \ldots, n, T$ instead of $v_0, v_1, \ldots, v_n, v_T$, respectively.)

A simple analysis shows that for $n = 2$ the corresponding St. George game is Nash-solvable for any cost function (even if assumptions (♠) and (♣) do not hold). Each of the three players (which includes 0, the mayor) have two distinct strategies, and thus we have in total 8 possible situations in this game, out of which two are cyclic. To prove our claim, we shall assume that for some local cost function none of the remaining 6 situations, namely $s_1 = \{(01), (1T), (21)\}$, $s_2 = \{(02), (12), (2T)\}$, $s_3 = \{(02), (1T), (21)\}$, $s_4 = \{(01), (12), (2T)\}$, $s_5 = \{(01), (1T), (2T)\}$, and $s_6 = \{(02), (1T), (2T)\}$, is a Nash equilibrium, and derive from this a contradiction with the existence of such local costs. Let us list in a table below for each of these situations which other situations can be obtained from them, by changing the strategy of one of the players (indicating above the transition arrow the player whose strategy change results in this transition):

$s_1 = \{(01), (1T), (21)\} \xrightarrow{0} s_3$	$s_2 = \{(02), (12), (2T)\} \xrightarrow{0} s_4$
$s_3 = \{(02), (1T), (21)\} \xrightarrow{0} s_1$	$s_4 = \{(01), (12), (2T)\} \xrightarrow{0} s_2$
$s_3 = \{(02), (1T), (21)\} \xrightarrow{2} s_6$	$s_4 = \{(01), (12), (2T)\} \xrightarrow{1} s_5$
$s_5 = \{(01), (1T), (2T)\} \xrightarrow{0} s_6$	$s_6 = \{(02), (1T), (2T)\} \xrightarrow{0} s_5$
$s_5 = \{(01), (1T), (2T)\} \xrightarrow{1} s_4$	$s_6 = \{(02), (1T), (2T)\} \xrightarrow{2} s_3$

Looking at the left column we can observe from the first line that if s_1 is not a Nash equilibrium then s_3 must be better than s_1 for player 0. Thus, using the next two lines and that s_3 is not a Nash equilibrium, we obtain that s_6 must be better than s_3 for player 2. Therefore, by the last two lines and our assumption that s_6 is not a Nash equilibrium, either, we can conclude that s_5 must be better then s_6 for player 0. Using the right column, and a similar chain of implications we can conclude that s_6 must be better than s_5 also for player 0. These two final implications clearly cannot hold simultaneously, therefore one of these six situations must indeed be a Nash equilibrium for any local cost.

Let us consider next a St. George game with $n = 3$. In this case assumptions ($\spadesuit$), ($\clubsuit$) are already essential to guarantee the existence of a Nash equilibrium. To see this, let us consider the game shown in Figure 6, where we indicated along each edge the local costs. Clearly, in this game $f(0,(1,2)) + f(0,(2,3)) + f(0,(3,1)) = -3$ violating condition ($\spadesuit$). We claim that there is no Nash equilibrium in this game.

Each of the players along the cycle have two possible strategies, while player 0 has three, thus we have, in total, 24 different situations. Three of these are cyclic, and each of the players in the cycle can improve those by switching to a terminal move, hence these are obviously not Nash equilibria. For the remaining 21 situations we list in the following table the costs for each of the players, and the player who can improve the situation for himself, together with the resulted new situation:

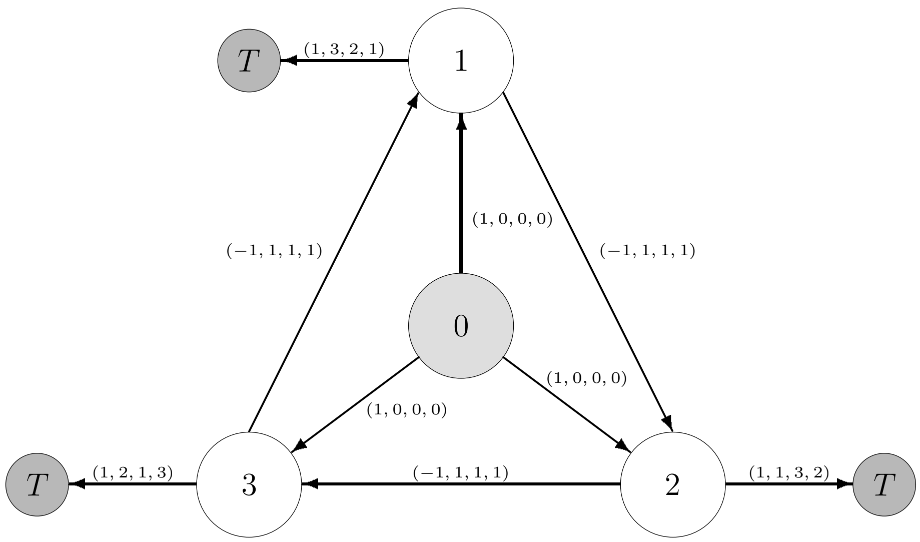

Figure 6: St. George game with $n = 3$. The local cost vector $(f(0,e), f(1,e), f(2,e), f(3,e))$ is indicated along each edge e. In this example there is a negative cycle, $c = (1,2,3)$ for player 0, and it has no Nash equilibrium.

Situation	Cost for players				Possible change
	$i=0$	$i=1$	$i=2$	$i=3$	$\xrightarrow{i} s^k$
$s^1 = \{(01),(12),(23),(3T)\}$	-2	3	1	2	$\xrightarrow{1} s^5$
$s^2 = \{(01),(12),(2T),(31)\}$	-1	1	2	3	$\xrightarrow{0} s^{16}$
$s^3 = \{(01),(12),(2T),(3T)\}$	-1	1	2	3	$\xrightarrow{2} s^8$
$s^4 = \{(01),(1T),(23),(31)\}$	0	2	3	1	$\xrightarrow{0} s^{11}$
$s^5 = \{(01),(1T),(23),(3T)\}$	0	2	3	1	$\xrightarrow{0} s^{12}$
$s^6 = \{(01),(1T),(2T),(31)\}$	0	2	3	1	$\xrightarrow{0} s^{20}$
$s^7 = \{(01),(1T),(2T),(3T)\}$	0	2	3	1	$\xrightarrow{1} s^3$
$s^8 = \{(02),(12),(23),(3T)\}$	-1	3	1	2	$\xrightarrow{0} s^1$
$s^9 = \{(02),(12),(2T),(31)\}$	0	1	2	3	$\xrightarrow{0} s^{16}$
$s^{10} = \{(02),(12),(2T),(3T)\}$	0	1	2	3	$\xrightarrow{0} s^3$
$s^{11} = \{(02),(1T),(23),(31)\}$	-2	2	3	1	$\xrightarrow{2} s^{13}$
$s^{12} = \{(02),(1T),(23),(3T)\}$	-1	3	1	2	$\xrightarrow{3} s^{11}$
$s^{13} = \{(02),(1T),(2T),(31)\}$	0	1	2	3	$\xrightarrow{0} s^{20}$
$s^{14} = \{(02),(1T),(2T),(3T)\}$	0	1	2	3	$\xrightarrow{2} s^{12}$
$s^{15} = \{(03),(12),(23),(3T)\}$	0	3	1	2	$\xrightarrow{0} s^1$
$s^{16} = \{(03),(12),(2T),(31)\}$	-2	1	2	3	$\xrightarrow{3} s^{17}$
$s^{17} = \{(03),(12),(2T),(3T)\}$	0	3	1	2	$\xrightarrow{0} s^3$
$s^{18} = \{(03),(1T),(23),(31)\}$	-1	1	2	3	$\xrightarrow{0} s^{11}$
$s^{19} = \{(03),(1T),(23),(3T)\}$	0	3	1	2	$\xrightarrow{0} s^{12}$

This table shows that none of the 21 possible non-cyclic situations is a Nash equilibrium, implying that the game is not Nash-solvable.

In fact it can be shown that the above game is not Nash solvable for any local cost function f satisfying the conditions

$$f(0,(0,1)) = f(0,(0,2)) = f(0,(0,3)) = f(0,(1,T)) = f(0,(2,T)) = f(0,(3,T)) = 0$$
$$f(0,(1,2)) = f(0,(2,3)) = f(0,(3,1)) = -1$$
$$f(1,(1,2)) + f(1,(2,T)) \ < \ f(1,(1,T)) \ < \ f(1,(1,2)) + f(1,(2,3)) + f(1,(3,T))$$
$$f(2,(2,3)) + f(2,(3,T)) \ < \ f(2,(2,T)) \ < \ f(2,(2,3)) + f(2,(3,1)) + f(2,(1,T))$$
$$f(3,(3,1)) + f(3,(1,T)) \ < \ f(3,(3,T)) \ < \ f(3,(3,1)) + f(3,(1,2)) + f(3,(2,T))$$

We shall see that the lack of Nash equilibria in these games is due to the existence of a negative cycle (namely $(12),(23),(31)$ for player 0).

We shall prove that a Nash equilibrium always exists in St. George games, whenever conditions (♠) holds. For this, we shall consider a somewhat more general class of games.

2.1.2 Enter-exit games

Let us consider graphs G consisting of $n+2$ vertices v_0, v_T and $V' = \{v_1, \ldots, v_n\}$, "entry" edges $\{(v_0, v) | v \in A \subseteq V'\}$, "exit" edges $\{(v, v_T) | v \in B \subseteq V'\}$, as well as a set E' of some other edges, which are not incident with neither v_0 nor v_T. We assume that there is an edge $(v', v) \in E'$ for every $v' \in V'$, or in other words, that there are no forced moves to v_T and every player can prolong the game. An enter-exit game turns into a St.Geogre game if $A = B = V'$ and the edges of E' form a Hamiltonian directed cycle in V'. We also assume that every non-terminal position is controlled by one of the players (there are no random positions), and that every player controls exactly one position (play-once). Let us call such games *enter-exit*.

Proposition 1 *Any enter-exit game is Nash solvable, provided $A \subseteq B$ and (♠) holds.*

Proof. Clearly $A, B \neq \emptyset$, because otherwise there is no path from v_0 to v_T. Since $A \subseteq B$, we can assign to every vertex $v \in A$ a path $p_v = ((v_0, v), (v, v_t))$ and a situation s^v defined as follows. Player 0 moves from v_0 to $v \in A$, player v moves from position v to v_T, while all other players move arbitrarily, but not to v_T. (This is possible by definition of the enter-exit games.) We prove that one of these $|A|$ situations is a Nash equilibrium for every local cost function satisfying (♠).

Clearly, only two players, 0 and v can change s^v, the strategies of all others do not influence the value of the game. It is also clear that player v can not improve s^v because a directed cycle will appear if v will chose any other move in place of (v, v_T). Thus only player 0 could effectively improve (for himself) situation s^v, for all $v \in A$.

Let us assume, indirectly that none of the strategies s^v, $v \in A$ is a Nash equilibrium, i.e. that for every $v \in A$ there exists a $u = u(v) \in A$ and a path $q_{u,v}$ from u to v such that

$$f(0,(v_0,u)) + \sum_{e \in E(q_{u,v})} f(0,e) + f(0,(v,v_T)) \ < \ f(0,(v_0,v)) + f(0,(v,v_T)). \qquad (\bigstar)$$

Then, there must exist a subset $\{v_1, ..., v_k\} \subseteq A$ such that $u(v_{i+1}) = v_i$ for $i = 1, ..., k-1$, and $u(v_1) = v_k$, i.. such that the concatenated paths $q_{v_i,v_{i+1}}$, $i = 1, ..., k-1$ and q_{v_k,v_1} form a directed cycle c in G. Summing up the inequalities ($\bigstar$) for $(u,v) = (u(v_i), v_i)$, $i = 1, ..., k$ then yields

$$\sum_{e \in E(c)} f(0,e) \ < \ 0$$

since the terms $f(0,(v_0,v_i))$ and $f(0,(v_i,v_T))$ for $i = 1, ..., k$ appear exactly once on both sides of those inequalities. Therefore, by the above construction, the total cost of c for player 0 is strictly negative. Though cycle c may not be simple, that is it may pass several times the same vertex or even the same edge, still c must contain a simple directed cycle of negative cost for player 0, contradicting ($\spadesuit$). $\square$

Remark 7 *Let us note that the costs of the terminal moves (v, v_T) do not matter at all in the above arguments. This property is important, because it will enable us to divide a play-once game into two parts: "inside" of a given cycle and "outside" of it, (see section 2.2.3).*

Let us also note that the Nash equilibrium guaranteed by the above proof will include the moves (v_0, v) and (v, v_T) for some position v even if these have a very large cost. Suppose that in every path p_v one of its two edges (v_0, v) or (v, v_T) is very expensive, while all other edges are fairly inexpensive for all players. Still one of these $|A|$ paths will realize an equilibrium, provided ($\spadesuit$) holds. At the same time, there may exist two edges (v_0, v), (v', v_T) and a directed path $q_{v,v'}$ from v to v' which costs much lest for all players.

Remark 8 *Assuming ($\clubsuit$) instead of ($\spadesuit$), we can simplify the above proof. Consider the situation s in which player 0 chooses an edge (v_0, v) of minimum cost for himself, player v chooses the edge (v, v_T), while the other players choose non-terminal moves. It is easy to see then that s is a Nash equilibrium. Indeed, if player v chooses another edge then a directed cycle appears, which cannot be better for player v. If player 0 chooses another edge then his cost can not decrease, because any other move (v_0, v') costs to him not less than move (v_0, v) and any directed path from v' to v has a non-negative cost, due to ($\clubsuit$). Finally, the strategies of all other players do not influence the play.*

2.1.3 Sausage games

Let $m \geq 1$, and let us consider $m+2$ vertices v_0, b_j for $j = 1, ..., m$, and v_T. Let G_j $j = 1, ..., m$ be acyclic digraphs with b_j as the only sink and b_{j-1} as the only source of G_j, $j = 1, ..., m$, $m \equiv 0$. The graphs G_j, $j = 1, ..., m$ are vertex disjoint otherwise. In other words, G_j and G_{j-1} have only one common vertex b_j, for $j = 1, \ldots, m$, and all other pairs $(G_j, G_{j+t}, t = 2, \ldots, m-1)$ are vertex-disjoint. (Here and further all indices and their sums are taken modulo m and they take values $1, \ldots, m$). If $m = 1$, then G_1 is not acyclic, but it is a directed graph in which every directed cycle contains vertex b_1.

Furthermore, let G_0 be another acyclic directed graph with v_0 as its only source, and which has its terminal vertices A in the set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_m)$, and which is vertex disjoint otherwise from G_1, ..., G_m.

Let us finally define G as the union of these graphs G_0, ..., G_m, and the additional edges (b_j, v_T) for $j = 1, ..., m$. We shall call such a directed graph G a *sausage graph*, and any game $g = (G, v_0, f)$ defined on it a *sausage game*.

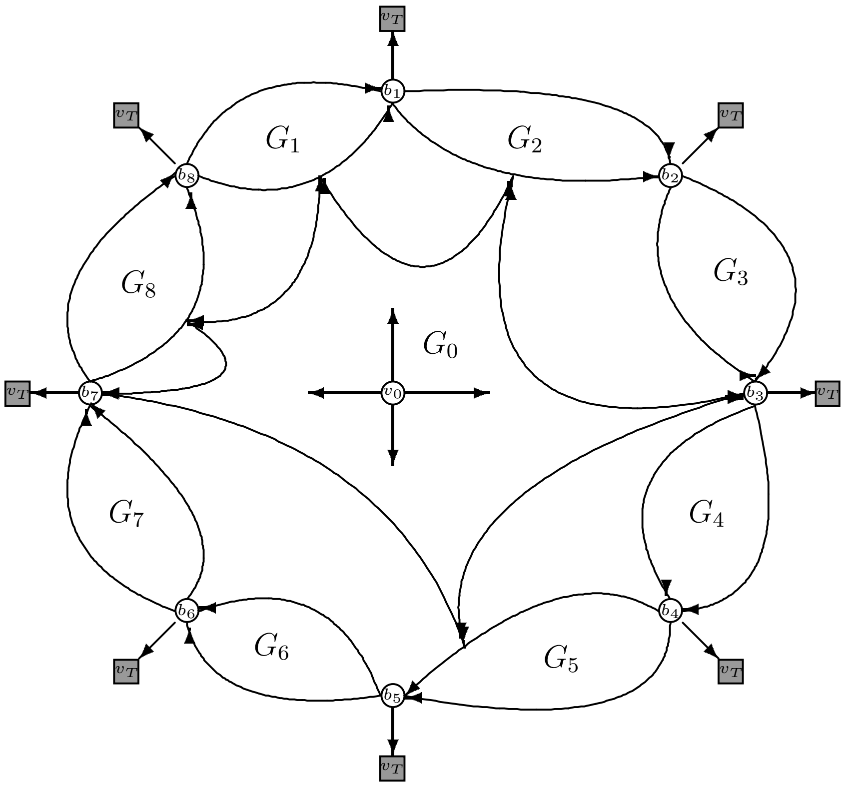

Figure 7: Sausage game with $m = 8$.

Clearly, St. George games are special cases of sausage games, in which every graph G_j, $j = 1, ..., m$ consists of only one edge, and G_0 is a star. Sausage games will play a major role in our proof of Theorem 3.

Proposition 2 *Any positive sausage game is Nash-solvable.*

Proof. Since Nash solvability of a game does not change if we apply a potential transformation, we may as well assume, without any loss of generality that conditions (♣) hold.

Let us consider first the case of $m = 1$. In this case vertex b_1 is a "bottleneck", that is b_1 belongs to every play. Hence the edges of the form (b_1, v) cannot belong to any play, unless $v = v_T$. Let us thus delete all the edges going out from b_1 except (b_1, v_T). Obviously, the normal form of the game does not change (see Remark 4) and the obtained game is acyclic, therefore the claim follows by Theorem 1.

Let us assume in the sequel that $m \geq 2$. Let us consider for every $j = 1, \ldots, m$ the acyclic (sub)game $g_j = (G_j, b_{j-1}, f)$ in which b_j $(m \equiv 0)$ is the only terminal position. Since this (sub)game is acyclic, we can solve it by backward induction and obtain a Nash equilibrium, denoted by s^j. Situation s^j is known to be *perfect* equilibrium of g_j, that is it defines a Nash equilibrium not only in g_j but also in every of its (sub)games $g_j(a)$, obtained by choosing a position $a \in V(G_j)$ as a new initial position (see e.g. [20]). Let us denote the corresponding equilibrium play from a to b_j by $p(s^j, a)$. In particular, $p(s^j, b_{j-1})$ is the equilibrium play in game g_j. The union of all these paths for $j = 1, \ldots, m$ form a directed cycle c^* in G.

Let us next consider the digraph G_0, and fix one of its terminal positions $a \in A$. Then, there exists a unique $j = j(a) \in \{1, \ldots, m\}$, such that $a \in V(G_j) \setminus \{b_{j-1}\}$. Let us extend the digraph G_0 by the directed paths $p(s^{j(a)}, a)$ from a to $b_{j(a)}$ for every $a \in A$ and denote the obtained digraph by G'_0, and the corresponding game by g'_0. In this game all the moves on $p(s^{j(a)}, a)$ between a and $b_{j(a)}$ are forced, and the cost function for this moves is the same as in $g_{j(a)}$, for all $a \in A$. The initial position of g'_0 is v_0 and the set of its final positions is $B \subseteq \{b_1, \ldots, b_m\}$. (Of course, we could identify all these final positions, according to remark 3.) Clearly, game g'_0 is acyclic. Let us chose an arbitrary Nash equilibrium s'_0 in it, and let $p(s'_0)$ denote the corresponding path leading from v_0 to a terminal position $b_{j_0} \in B$. Now let us define a situation s in the original sausage game g as a combination of s'_0 and $s^1, \ldots, s^m$, e.g., in every position of the directed cycle c^* (and in particular, in $b_1, \ldots, b_m$) s prescribes to move along c^*, except for only one position b_{j_0}, where by definition let s finish the game by the move (b_{j_0}, v_T).

It is not difficult to check that s is a Nash equilibrium in game g. Indeed, if in b_{j_0} the player changes his strategy then the play results in directed cycle c^*. If in another position $b_j \in B$ a player changes his strategy and moves to v_T, instead of following c^* then the play does not change, because it reaches b_{j_0} before b_j and from b_{j_0} it comes at once to v_T. The simple case analysis shows that if a player v changes his strategy then the play either remains the same, or still arrives at c^* then moves along c^* to b_{j_0} and then to v_T. The first part can not give a profit to v, because he has deviated from an equilibrium in a game g_j for some $j \in \{0, 1, \ldots, m\}$. The second part of the play also can not be profitable to v, because the local costs of all the edges, and in particular the edges of c^*, are not negative. □

Remark 9 *We can notice again that the obtained Nash equilibrium s has some very special characteristics. The play defined by s arrives to a directed cycle, namely to c^*, and then it leaves this directed cycle for v_T as soon as possible, namely at b_{j_0}. Yet, except for b_{j_0}, all other players along c^* are playing a sort of "punishing strategy". As a result, no player can get a profit by a deviation from s, because the play will still arrive to c^* and then to b_{j_0}. In particular, the costs of the edges incident to v_T, that is (b_j, v_T) for $j = 1, \dots, m$, are irrelevant. Indeed, s prescribes to move along c^* in every position of c^*, except for b_{j_0}. Thus all these edges, except (b_{j_0}, v_T), "are cut" by s anyway.*

These properties of the obtained Nash equilibrium s will enable us to reduce Nash-solvability of any play-once game to the case of sausage games (see section 2.2.3). Given a sausage game $g = (G, v_0, f)$, let us substitute in G the final position v_T by an arbitrary digraph G_T, we can also add arbitrary edges from $V(G_T)$ to $V(G)$ and define arbitrary local costs for these edges and for $E(G_T)$. Let us denote the obtained larger game, with the same initial position v_0, by g^+. Now let us consider in g^+ a sub-game g' induced by the set of vertices $V(G_T) \cup \{b_{j_0}\}$, where b_{j_0} is the initial position. Any Nash equilibrium s' in g' combined with the Nash equilibrium s in g will result in a Nash equilibrium in g^+. Indeed, the two stages of game g^+, before and after b_{j_0}, are fully independent, while any attempt to return to G on the second stage results in a directed cycle, because situation s will lead the play to b_{j_0} from every position of G, and b_{j_0} has already appeared as the end of the first stage and the beginning of the second one.

2.2 Proof of Theorem 3

2.2.1 Partitioning of digraphs by directed cycles

Given a game $g = (G, v_0, f)$ and a directed cycle c in G, we partition the set $V = V(G)$ of all the positions in four subsets: $on(c), in(c), out(c)$ and $cut(c)$. The definition of $on(c)$ is the simplest of the four, $on(c)$ is just the set of all vertices of the directed cycle, i.e. $on(c) = V(c)$. Let us consider in G all the directed paths from v_0 to c, (respectively, from c to v_T) which have exactly one common vertex with c; we define $in(c)$ (respectively, $out(c)$) as the union of the vertices of all such directed paths minus set $on(c)$. In other words, $v \in in(c)$ (respectively, $v \in out(c)$) if and only if there is a directed path from v_0 to v (respectively, from v to v_T) disjoint from c. Obviously, $in(c) \neq \emptyset$ and $out(c) \neq \emptyset$, for example $v_0 \in in(c)$, and $v_T \in out(c)$ for every directed cycle c. Let us note that set $cut(c) = V \setminus (on(c) \cup \in (c) \cup out(c)$ may be non-empty. By definition it consists of all vertices $v \in V \setminus on(c)$, such that every directed path from v_0 to v and every directed path from v to v_T intersects c.

According to ($\heartsuit$), every play p and every directed cycle c have a vertex in common. Hence $in(c) \cap out(c) = \emptyset$ and thus the four sets $on(c), in(c), out(c)$ and $cut(c)$ form a partition of V.

Lemma 1 *Given two vertex-disjoint directed cycles c' and c'', every play p (a $v_0 \longrightarrow v_T$ path) meets them in the same order: either the first intersection of p and c' precedes the* *first intersection of* p *and* c'' *and then the last intersection of* p *and* c' *precedes the last intersection of* p *and* c'' *too, or vice versa. In other words, if* $on(c') \cap on(c'') = \emptyset$ *then either* $on(c') \subseteq in(c'')$ *and then* $on(c'') \subseteq out(c')$*, or vice versa* $on(c'') \subseteq in(c')$ *and then* $on(c') \subseteq out(c'')$*.*

Proof. Suppose indirectly that there is a play (path) p' meeting c' first and then c'' second, while some other play p'', on the contrary, meets c'' first and then c' second. Let us denote by v the last vertex along the path p'' belonging to $on(c') \cup on(c'')$. If $v \in on(c'')$, then let us consider the path p consisting of the segment p'' from v_0 to the first intersection of p'' with c'', then the segment of c'' leading to v, and finally the segment of p'' after v. Clearly, p is also a play, and it has no intersection with c', a contradiction with ($\heartsuit$). If $v \in on(c')$, then let us consider the path p consisting the initial segment of p' until it intersects c', then follow c' until v, and the follow p''. This path p is again a play, which avoids now c', contradicting ($\heartsuit$). We obtained in both cases a contradiction, proving the lemma. An analogous contradiction can be derived to prove the statement about the order of the last intersections. □

More generally, given a system $\mathcal{C}$ of k pairwise vertex-disjoint directed cycles $\mathcal{C} = \{c_1, \dots, c_k\}$ and a path p, let us introduce two permutations $\sigma'(p, \mathcal{C})$ and $\sigma''(p, \mathcal{C})$ of $\{1, \dots, k\}$, these permutations show in which order appear in p the first and respectively the last intersections with each of these k cycles.

Lemma 2 *Equality* $\sigma'(p, \mathcal{C}) = \sigma''(p, \mathcal{C})$ *holds for every* p *and* $\mathcal{C}$*, moreover, this permutation does not depend on* p *for a fixed* $\mathcal{C}$*.*

Proof. For $k = 1$ this claim is trivial, for $k = 2$ this is exactly Lemma 1, and in general it follows from Lemma 1 by induction. □

It is easy to check that no similar claim to Lemma 1 can be stated for directed cycles c and c' which may have a common vertex. Plays can come to and leave such directed cycles in different orders. Yet, the next important claim holds.

Lemma 3 *The following two inclusions are equivalent:*
(i) $on(c') \subseteq on(c) \cup in(c) \cup cut(c)$ *and (ii)* $out(c) \subseteq out(c')$*.*

Proof. Suppose (i) holds and $v \in out(c)$, i.e. there is a path p from v to v_T disjoint from c. Then p is disjoint from c' as well, because otherwise p would intersect $in(c) \cup cut(c)$ and then $on(c)$. Thus $v \in out(c')$. Vice versa, assume that (i) does not hold, then $on(c') \cap out(c) \neq \emptyset$ and hence $out(c')$ could not contain $out(c)$. □

Given a directed cycle c, it will be convenient to introduce one more set $onout(c) \subseteq on(c)$. By definition a vertex $b \in on(c)$ belongs to $onout(c)$ if and only if there exists a directed path from b to v_T, which intersects $on(c)$ only in b. Clearly, $onout(c) \neq \emptyset$ for any c, because every play intersects $on(c)$, according to ($\heartsuit$). We can strengthen Lemma 3 as follows.

Lemma 4 *Suppose (i) of Lemma 3 holds. Then inclusion (ii) in Lemma 3 is an equality if and only if* $onout(c) \subseteq on(c')$

Proof. Suppose there is a vertex $b \in onout(c) \setminus on(c')$. Then there is a directed path p from b to v_T, which intersects $on(c)$ only in b. Clearly, p can not intersect c', because $on(c') \subseteq in(c) \cup on(c) \cup cut(c)$ and hence after intersecting c' the directed path p must come back to c again. Thus $b \in out(c') \setminus out(c)$ and hence $out(c') \not\subseteq out(c)$, Vice versa, let $out(c') \not\subseteq out(c)$, and let $v \in out(c') \setminus out(c)$. By definition, there is a path p from v to v_T, which meets c but do not meet c'. Let b be the last vertex of p, which belongs to c. Clearly, $b \in onout(c) \setminus on(c')$ □

The above two Lemmas enable us to find a directed cycle c whose set $out(c)$ is inclusion-maximal, i.e. strong inclusion $out(c) \subset out(c')$ holds for no directed cycle c'. Let us chose such a directed cycle c. If there exists a directed cycle c' such that $on(c') \subseteq in(c) \cup on(c) \cup cut(c)$ and $onout(c) \not\subseteq on(c')$ then we obtain a strong inclusion $out(c) \subset out(c')$. If there is no such directed cycle c' then the set $out(c)$ is already inclusion-maximal.

2.2.2 A characterization of the sausage games and sub-games

Let G be a sausage digraph. Typically, G contains many directed cycles, which may have different sets $on(c)$, $in(c)$ and $cut(c)$, yet obviously, $out(c) = \{v_T\}$ for every directed cycle c in G. However, if we add to G a directed path from $V(G_j)$ to $V(G_{j+t})$, where $t \neq 0($ mod $m)$, then a directed cycle c' appears, such that set $out(c')$ contains v_T and b_j. The only exception is a directed path from b_{j-1} to b_j, in which case $t = 1$, $b_{j-1} \in V(G_j)$, and $b_j \in G_{j+t}$. Let us also note that $onout(c) = \{b_1, \dots, b_m\}$ for every directed cycle c of a sausage game, and all the directed cycles pass these m vertices in the same cyclic order. The inverse is also true. Moreover, the above property is a characteristic one for the sausage digraphs.

Lemma 5 *Let $g = (G, v_0, f)$ be a game whose final position v_T, is incident to exactly m vertices $B = \{b_1, \dots, b_m\}$, The following claims are equivalent:*

(i) $onout(c) = B$ for every directed cycle c in G;

(ii) all directed cycles of G contains B and pass vertices $b_1, \dots, b_m$ in the same cyclic order

(iii) G is a sausage digraph.

Proof. Clearly, (iii) implies (ii) which on its turn implies (i).

Suppose next that (i) holds, and consider two directed cycles c' and c'' which contain all m vertices each but pass them in different orders. Then it is easy to combine a new directed cycle c, such that $on(c) \subseteq on(c') \cup on(c'')$ and $B \not\subseteq on(c)$. Thus (i) implies (ii).

Suppose finally that (ii) holds, Without loss of generality we can assume that the cyclic order of B is $b_1, \dots, b_m$. Let $m > 1$, and fix an index $j \in \{1, \dots, m\}$ and consider the set P_j of all the directed paths from b_{j-1} to b_j. The union of all these directed paths form a digraph G_j. This digraph is acyclic, because a cycle in it could not contain B. (In case $m = 1$, we get directed cycles instead of directed paths, $j = j - 1 \mod m$ and

G_1 has directed cycles all of which contain b_1.) As we already mentioned, G contains no directed path from $V(G_j)$ to $V(G_{j+t})$, where $t \neq 0(\mod m)$, except for directed paths of P_j. Now consider the set P_0 of all the directed paths from v_0 to $\bigcup_{j=1}^{m} V(G_j)$. The union of all these directed paths form a digraph G_0. This digraph is acyclic too, because a cycle in it could not contain B. Thus G is a sausage digraph, i.e. (iii) holds. $\square$

Given an arbitrary game $g = (G, v_0, f)$, Let us chose a directed cycle c in G whose set $out(c)$ is inclusion-maximal. Let us add to G a new vertex v_T and the edges (b, v_T) for all $b \in onout(c)$.

Lemma 6 *The set of vertices* $on(c) \cup in(c) \cup \{v_T\}$ *induces a sausage digraph* G'.

Proof. According to Lemma 5 it is sufficient to show that every directed cycle in G' contains $onout(c)$. Suppose not, i.e. there exists a directed cycle c' in G' and a vertex $b \in onout(c)$, such that $b \notin on(c')$. Yet, $b \in out(c')$ and hence $out(c) = \{v_T\} \subset \{b, v_T\} \subseteq out(c')$, which contradicts the maximality of $out(c)$. $\square$

2.2.3 Reduction of an arbitrary play once game to sausage games

Let $g = (G, v_0, f)$ be a game. If the digraph G is acyclic then there are Nash equilibria in g and one of them can be found by the backward induction. If there are directed cycles in G we make use of the following reduction. Let us chose in G a directed cycle c, whose set $out(c)$ is inclusion minimal. Let $onout(c) = B = \{b_1, \ldots, b_m\}$. Further, let us add a new vertex v_T and m edges (b_j, v_T) for every $b_j \in B$, define arbitrary local costs for these edges and consider the sub-game $g' = (G', v_0, f')$ induced by the vertices $on(c) \cup in(c) \cup \{v_T\}$. This sub-game is a sausage game, according to Lemma 6. Thus, according to Proposition 2, g' has a special Nash equilibrium s', such that there is a cycle c^* in G', along which s' recommends to move in all positions from $on(c^*)$, except for one position $b_{j_0} \in onout(c^*)$, where the move to v_T is recommended by s'. (According to Lemma 5, $onout(c) = onout(c^*) = onout(c')$ for every directed cycle c' in G'.)

Let us now consider game $g'' = (G'', b_{j_0}, f'')$ where G'' is the digraph induced by vertices $(V(G) \setminus V(G')) \cup \{b_{j_0}\}$ and the local cost function f'' is induced by f. Suppose, we obtain a Nash equilibrium s'' in game g''. Then combination of s' and s'' generates a Nash equilibrium s in the original game $g = (G, v_0, f)$. Indeed, if a player v' deviates somewhere in a position $v' \in V(G')$ then the play still comes to b_{j_0} and player v' can not get a profit because s' is a Nash equilibrium in g'. If a player v'' deviates somewhere in a position $v'' \in V(G'')$ then there are two cases. If v'' made a move (v'', v), where $v \in on(c) \cup in(c) \cup cut(c)$, then the play returns to b_{j_0} and thus a cycle appears. Otherwise, the play comes to v_T, and in this case player v'' can not get a profit, because s'' is a Nash equilibrium in g''. Thus in our search of Nash equilibria we reduced the original game $g = (G, v_0, f)$ to the sub-game $g'' = (G'', b_{j_0}, f'')$ and the sausage sub-game $g' = (G', v_0, f')$. Since we already know that the last one is Nash-solvable, this will conclude the proof of Theorem 3.

Now let us analyze the conditions, which we have to assume, to prove the existence of a Nash equilibrium s' in g'. Clearly, (♠) is sufficient. Yet, as we know, directed cycles of G' have a special structure. They all pass all vertices of B in the same order $b_1, \ldots, b_m$. Such a structure enable us to check (♠) easily. Consider a vertex $v \in V(G') \setminus \{v_T\} = on(c) \cup in(c)$ and the corresponding player $i = i(v)$. According to (♠), for every directed cycle c' in G' the sum $\sum_{e \in E(c')} f(i, e)$ must be non-negative. Yet, it is easy to obtain a directed cycle for which the above sum takes the minimum value. For every $j = 1, \ldots, m$ find the shortest path between b_{j-1} and b_j in G', with respect to the cost function $f(i, \cdot)$ of player i. The union of all these directed paths is the shortest directed cycle. Let us also note that it is not necessary to check the above inequalities for $v \in B$, because these vertices belong to all the directed cycles of G'.

Our next step of the reduction, instead of $G' = G_1$, we will construct the next sausage sub-digraph G_2. Let us consider the sub-digraph induced by set $(V(G) \setminus V(G_1)) \cup \{b^1_{j_0}\}$, where a directed cycle c^2 is chosen such that $out(c^2)$ is inclusion-maximal. Thus, we get the next sausage sub-digraph G_2, and so on, until we get rid of all the directed cycles. Let $G_1, \ldots, G_k$ be the sequence of the obtained sausage sub-digraphs, and $\mathcal{C}_1, \ldots, \mathcal{C}_k$ be the sets of all their directed cycles. By the above construction, any two directed cycles from different sets ($c_{j'} \in \mathcal{C}_{j'},\ c_{j''} \in \mathcal{C}_{j''};\ \ j' \neq j''$) are vertex-disjoint. According to Lemma 2, for every path p the first intersections with $\mathcal{C}_1, \ldots, \mathcal{C}_k$ as well as the last intersections with them appear in the same order $1, \ldots, k$.

These observations and the above proof shows that in fact a Nash equilibrium can be found in polynomial time in this case.

Remark 10 *Finally, let us note that though every sausage game is solved by backward induction, yet, the obtained sequence of sausage games $g_1, \ldots, g_k$ is solved by "forward induction". We first solve g_1 and get equilibrium s^1 and path $p(s^1)$, which starts in v_0 and finishes by edge (b^1, v^1_T). Then we solve g_2 and get equilibrium s^2 and path $p(s^2)$, which starts in b^1 and finishes by the edge (b^2, v^2_T), etc. Thus we solve k sausage games and then the last game g^{k+1}, which is acyclic.*

2.3 The complexity of finding a Nash equilibrium in positive play-once games

In this section we show that the above proof in fact can be implemented so that a Nash equilibrium can always be found in polynomial time (in the input size of the graph G). According to the above remark, it is enough to show that both testing condition (♡), and finding the sausage decomposition (in case (♡) holds) can be done in polynomial time.

Let us observe first that testing condition (♡) is in fact equivalent with testing the existence of a 2-commodity flow in an associated graph, which therefore can be solved in polynomial time [14]. Indeed, given vertices v_0, v, and v_T in G, the existence of a directed cycle c and a $v_0 \longrightarrow v_T$ directed path p such that p and c are vertex disjoint and c passes through v is equivalent with the existence of arc disjoint paths $s_1 \longrightarrow t_1$ and

$s_2 \longrightarrow t_2$ in the network G^v, where G_v is defined as follows: Let us define the vertex set $V(G^v) = \{u', u'' \mid u \in V(G) \setminus \{v_0, v_T\}\} \cup \{v_0, v_T\}$, and let $(u'u'')$ for $u \in V(G) \setminus \{v_0, v_T\}$ and (u'', w') for $(u, w) \in E(G)$ be the arcs of G^v. Let us further define the capacities of all the arcs in G^v to be equal 1, and let $s_1 = v_0$, $t_1 = v_T$, $s_2 = v''$ and $t_2 = v'$.

Thus, solving a 2-commodity flow problem in G^v for each $v \in V(G) \setminus \{v_0, v_T\}$ we can decide in polynomial time whether ($\heartsuit$) holds, or not.

Let us note next that if ($\heartsuit$) holds in G, then finding a cycle c for which $out(c)$ is maximal is again a polynomially solvable task. Indeed, finding the strong components of a directed graph G can be done in $O(|E(G)|)$ time ([22]), and thus the acyclicity of G can be recognized, and if not acyclic, a cycle c can be found in linear time.

To determine $out(c)$ amounts to finding $v \longrightarrow v_T$ paths in the subgraph induced by $(V(G) \setminus on(c)) \cup \{v\}$ for every $v \in V(G)$, a task, which can clearly be accomplished in $O(|V(G)||E(G)|)$ time. Furthermore, the same computation will also determine the set $onout(c)$.

To test the maximality of $out(c)$ we need to test the acyclicity of the subgraph induced by $V \setminus (out(c) \cup \{v\})$ for all $v \in onout(c)$, which can also be completed in $O(|V(G)||E(G)|)$. If for some $v \in onout(c)$ we find this a cycle c' in the corresponding induced subgraph, then clearly $out(c') \supset out(c)$.

Thus, repeating all the above at most $|V(G)|$ times, we should find a cycle c in polynomial time, such that $out(c)$ is maximal.

Summarizing the above we can state that the first sausage subgraph in G can be identified in polynomial time. Thus, we can proceed as described in Remark 10. Since the contraction of the first sausage subgraph decreases the size of G by at least one, the repetition of all these at most $|V(G)|$ will provide us with the desired Nash equilibrium, as in Remark 10, in polynomial time, as claimed in Theorem 2.

3 Nash-solvability of terminal games of two players

In this section we focus on terminal games with two players, and prove (ii) of Theorem 2.

Theorem 4 *Every terminal game with two players has a proper Nash equilibrium in pure stationary strategies. Moreover, such a strategy can be found in polynomial time in the size of the graph G on which the game is defined.*

The proof of this theorem will be based on a general criterion of Nash-solvability obtained in [7, 9] for arbitrary game forms with two players. In section 3.2 we apply this criterion, and prove Nash-solvability for terminal game forms with two players. In section 3.3 we recall from [2] a constructive proof for the cited criterion of [7, 9], and show that this constructive proof provides a Nash equilibrium in polynomial time, completing the proof of Theorem 4.

3.1 Tight game forms and Nash-solvability

In order to be able to state the result of [7, 9], which will be instrumental in our proof, we shall need recall some definitions and notations.

Let $I = \{1, \dots, n\}$ denote the set of players, S_i be the finite set of pure strategies of player $i \in I$. The direct product $S = \prod_{i \in I} S_i$ be the set of situations (or strategies), and let $A = \{a_1, \dots, a_k\}$ denote the set of outcomes. A *game form* $\pi : S \to A$ is defined as a mapping which assigns an outcome $\pi(s) \in A$ to every situation $s = (s_i \mid i \in I) \in S$. This mapping may not be injective, that is an outcome $a \in A$ may be assigned to different situations. Further, let $u : I \times A \to \mathbb{R}$ be a *payoff* function, i.e. $u(i, a)$ is the profit of player $i \in I$ in case outcome $a \in A$ is realized. (Instead, we could equivalently consider the effective *cost* function $-u$.) A *game in normal form* is defined as a pair of mappings $g = (\pi, u)$ (or rather as a quadruple $g = (I, A, \pi, u)$). A game form π is called *Nash-solvable* if for every payoff u the obtained game $g = (\pi, u)$ has at least one Nash equilibrium in pure strategies.

The subsets of players $K \subseteq I$ are called coalitions, and the subsets of outcomes $B \subseteq A$ are called blocks. We say that coalition K is effective for block B, if K can guarantee that an outcome from B will be chosen, in other words, if members of K have a joint strategy $s_K = (s_i \mid i \in K)$ such that $\pi(s_K, s_{I \setminus K}) \in B$ for every joint strategy $s_{I \setminus K} = (s_j \mid j \in I \setminus K)$ of the remaining players. The *effectivity function* $\mathcal{E} : 2^I \times 2^A \longrightarrow \{0, 1\}$ is a Boolean function, for which $\mathcal{E}(K, B) = 1$ for $K \subseteq I$ and $B|subseteq A$ if and only if the coalition K is effective for the block B.

Clearly, if K_1 and K_2 are disjoint coalitions, and $\mathcal{E}(K_1, B_1) = \mathcal{E}(K_2, B_2) = 1$, then $B_1 \cap B_2 \neq \emptyset$, since e.g. $\pi(s_{K_1}, s_{K_2}, s_{I \setminus (K_1 \cup K_2)}) \in B_1 \cap B_2$ by the above definition, for an arbitrary pair of strategies s_{K_1} and s_{K_2} of coalitions K_1 and K_2, respectively, and for any strategy $s_{I \setminus (K_1 \cup K_2)}$ of the rest of the players. In particular, $\mathcal{E}(K, B) = 1$ implies $\mathcal{E}(I \setminus K, A \setminus B) = 0$. A game form π is called *tight* if the inverse implication holds, too, i.e. if

$$\mathcal{E}(K, B) = 1 \Longleftrightarrow \mathcal{E}(I \setminus K, A \setminus B) = 0. \tag{2}$$

Let us note that tightness of a game form is defined solely in terms of its effectivity function. Such effectivity functions in the literature are called *maximal* [20].

Let us note that in a terminal game the payoff of the players depend only on the terminal moves. Thus, to simplify notations, we can equivalently define the set of outcomes A for a terminal game form consisting of all the terminal moves $e = [v, v_T)$, for $v \in V(G)$, and one more special outcome C, which appears if the play results in a directed cycle. We shall consider payoff functions which rank C as the worst outcome for all the players.

A useful result of [7, 9] claims that for two player game forms Nash-solvability and tightness are equivalent.

Theorem 5 ([7, 9]) *A game form of two players is Nash-solvable if and only if it is tight.*

Remark 11 *Note that tightness implies the existence of a Nash equilibrium for every payoff (cost, preference) function on the set of all outcomes, so we could rank C arbitrarily, not necessarily as the worst outcome.*

Unfortunately, Theorem 5 does not generalize to the case of $n > 2$ players. Already for 3 players tightness is neither a necessary nor a sufficient condition for Nash-solvability of general game forms (see [9]). However, the example constructed there (for a tight but not Nash-solvable game form) cannot be realized as a normal form of a positional game with cycles. So for such game forms the question about (proper) Nash-solvability remains open.

For the sake of completeness, let us recall here this example in three players $I = \{1, 2, 3\}$ and with three outcomes $A = \{a_1, a_2, a_3\}$. The strategy of player i consists of a pair of numbers $s_i = (\alpha_i, \beta_i)$, where $\alpha_i \in \{0, 1, 2\}$ and $\beta_i \in \{0, 1\}$, $i \in I$, i.e. each player has 6 possible strategies. The mapping π is defined by $\pi((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)) = a_j$, where

$$j = 1 + \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 & \pmod 3 \quad \text{if} \quad \beta_1 = \beta_2 = \beta_3 \\ \alpha_1 + \alpha_2 & \pmod 3 \quad \text{if} \quad \beta_1 > \beta_2 \\ \alpha_2 + \alpha_3 & \pmod 3 \quad \text{if} \quad \beta_2 > \beta_3 \\ \alpha_3 + \alpha_1 & \pmod 3 \quad \text{if} \quad \beta_3 > \beta_1 \end{cases}$$

It is easy to verify that this game form is indeed tight (any one of the players is effective only for the entire set A of outcomes, but any two or more are effective for any single outcome) and not Nash-solvable (e.g. for a Condorcet preference $u(1, a_1) > u(1, a_2) > u(1, a_3)$, $u(2, a_2) > u(2, a_3) > u(2, a_1)$ and $u(3, a_3) > u(3, a_1) > u(3, a_2)$).

3.2 Tightness of terminal games with two players

Proposition 3 *Every terminal game form with two players is tight.*

Proof. Let $A = V_T \cup \{C\}$ be the set of outcomes of a terminal game form $\pi : S \to A$. We prove that π is tight, that is $\mathcal{E}(1, B) = 0 \Rightarrow \mathcal{E}(2, A \setminus B) = 1$ for every subset $B \subseteq A$.

We shall consider two cases (i) $C \notin B$ and (ii) $C \in B$, and prove that they are in fact equivalent. Indeed, to make them more symmetric introduce a partition $A = B' \cup B'' \cup \{C\}$. In case (i, $B = B'$) tightness fails iff $\mathcal{E}(1, B') = \mathcal{E}(2, B'' \cup \{C\}) = 0$, while in case (ii, $B = B' \cup \{C\}$) tightness fails iff $\mathcal{E}(1, B' \cup \{C\}) = \mathcal{E}(2, B'') = 0$. Obviously, these two claims are symmetric, one transforms to the other if we substitute 1 by 2 and B' by B''.

Thus, it is sufficient to consider only case (i). Suppose $\mathcal{E}(1, B') = 0$, that is no strategy of player 1 guarantees that the play will result in a terminal position from B'. Let us treat B' as the winning positions of player 1 and apply backward induction. Mark every position from B' by $+$. If player 1 moves in a position v and can reach a position marked by $+$ then mark v by $+$, too. If player 2 moves in a position v and every move leads to a position marked by + then mark v by +, too. Apply this procedure recursively. Clearly, not every position will be marked by +, for example initial position v_0 will not, because otherwise player 1 would be able to force B' and $\mathcal{E}(1, B')$ would be equal to 1, and not 0. Consider all positions which are not marked by +. By construction, in every position not marked by + player 2 has a move which does not lead to a +. Let him apply a strategy which chooses such moves in all these positions. By construction, player 1 can not reach B' if player 2 applies this strategy. In other words, player 2 can enforce $B'' \cup \{C\}$, which then completes our proof of tightness. □

3.3 Complexity of finding a Nash equilibrium in terminal games with two players

While Proposition 3 and Theorem 5 together implies the Nash solvability of every terminal game form with two players, it is not (yet) clear how can we find in some efficient way a Nash equilibrium in a given two player terminal game.

To overcome this shortcoming, and to prove that a Nash equilibrium can indeed be found always in polynomial time, we shall recall from [2] the main idea of a constructive proof for Theorem 5, and show that it in fact constructs a Nash equilibrium in polynomial time in our case.

The computational difficulty comes from the fact that the effectivity function of a terminal game form is not given explicitly, and even if we had a method to generate it completely, this may take exponential space and time in the input size of the game form.

It is a much more general approach to consider game forms for which the effectivity function is given by an *oracle*. More specifically, let us assume that for any given coalition K and block B the oracle returns the value of $\mathcal{E}(K, B)$. Furthermore, if $\mathcal{E}(K, B) = 1$, then it also provides us with a strategy $s_K = (s_i \mid i \in K)$ such that $\pi(s_K, s_{I \setminus K}) \in B$ for every strategy $s_{I \setminus K} = (s_j \mid j \in I \setminus K)$ of the other players. Let us call such an oracle a *P-oracle*.

Proposition 4 *Every terminal game form with two players has a P-oracle.*

Proof. Let $B \subseteq A$ and let us assume that we need to determine $\mathcal{E}(1, B)$.

Let us observe first that $C \notin B$ can be assumed, since otherwise we would compute $\mathcal{E}(1, B)$ by the equality $\mathcal{E}(1, B) = 1 - \mathcal{E}(2, A \setminus B)$, which is due to tightness. Let us now apply backward induction as follows: Let us start with $H = B$, (i.e. with a (sub)set of terminal vertices of G, since $C \notin B$ is assumed). Then, increment H as long as possible by joining a vertex $u \notin H$ to H, if (i) u is controlled by player 1 and there is an arc (u, v) for some vertex $v \in H$ (let us mark such an arc as e_u); or (ii) u is controlled by player 2 and for all outgoing arcs (u, v) from u, the end vertex v belongs to H.

It is immediate to see that, on the one hand, $\mathcal{E}(1, B) = 1$ holds if and only if $v_0 \in S$ at the end of this process. If this is the case, then let us define a strategy s_1 of player 1 by choosing the arcs e_u for all vertices in $H \cap V_1$, and by choosing an arbitrary outgoing arc from every other vertex in $V_1 \setminus H$ (recall that by V_1 we denoted the set of vertices controlled by player 1). Clearly, this strategy s_1 assures player 1 with an outcome in B, no matter what strategy player 2 chooses.

On the other hand, $v_0 \notin H$ at the end of the above process occurs if and only if $\mathcal{E}(1, B) = 0$, i.e. iff $\mathcal{E}(2, A \setminus B) = 1$ by the tightness of $\mathcal{E}$. In this case, for every vertex u from which there is an arc entering H we must have vertex u belonging to V_2. Furthermore, for every vertex $u \in V_2$ we must have an arc $e_u = (u, v)$ such that $v \notin H$. Let us define a strategy s_2 for player 2 now by choosing the arcs e_u for all $u \in V_2$. Clearly, strategy s_2 guarantees for player 2 an outcome from $A \setminus B$.

Let us finally add that the above computations, and the construction of the respective strategies, can be implemented to run in $O(|V| + |E|)$ steps, completing hence the proof. $\square$

Proposition 5 (c.f. [2, 11, 12]) *A Nash equilibrium can be constructed via $O(|A|)$ oracle calls in every two player game which has a tight game form and a P-oracle .*

Proof. The main idea of the proof is to construct two blocks B_1 and B_2 such that $\mathcal{E}(1, B_1) = \mathcal{E}(2, B_2) = 1$, $B_1 \cap B_2 = \{a\}$ and outcome a is the best outcome both for player 1 in B_2 and for player 2 in B_1. Let us denote by s_i a strategy for player i which guarantees an outcome in B_i $(i = 1, 2)$, and let $s = (s_1, s_2)$. Clearly, for this situation we have $\pi(s) \in B_1 \cap B_2 = \{a\}$, i.e. $\pi(s) = a$. We claim that s is a Nash equilibrium. Indeed, if player 1 would try to change his strategy, all he can hope for is another outcome from B_2, since s_2 is unchanged and it guarantees an outcome within B_2. Since in B_2 outcome a is player 1's best, he is not interested to change. Due to the symmetry in the above definitions, an analogous argument shows that player 2 is not interested in change, either, which then proves that s is a Nash equilibrium.

Let us show next that such blocks B_1 and B_2 can indeed be found. Let us initialize $B_1 = B_2 = \emptyset$, $B_0 = A$, and let us incrementally enlarge the sets B_1 and B_2, taking outcomes from B_0, while keeping $\mathcal{E}(1, B_1) = 0 = \mathcal{E}(2, B_2)$. Since the game form is tight, $\mathcal{E}(1, B_1) = 0$ implies that $\mathcal{E}(2, A \setminus B_1) = 1$. Here $B_2 \subseteq A \setminus B_1$, and $\mathcal{E}(2, B_2) = 0$, thus $B_0 = A \setminus (B_1 \cup B_2) \neq \emptyset$ follows for such pairs of blocks.

Let us sort the outcomes $B_0 = \{b_1, b_2, ..., b_p\}$ by the payoff of player 2, such that b_1 is his worst outcome, and b_p is the best for him (in B_0), and let k be the largest index for which $\mathcal{E}(1, B_1 \cup \{b_1, b_2, ..., b_k\}) = 0$. Let $B_1 = B_1 \cup \{b_1, b_2, ..., b_k\}$ for this k, and set $B_0 = \{b_{k+1}, ..., b_p\}$. Clearly $B_0 \neq \emptyset$ after this step, since $\mathcal{E}(1, B_1 \cup B_0) = 1$ is implied by the tightness of the game form and by $\mathcal{E}(2, B_2) = 0$.

Let us next resort the outcomes left in $B_0 = \{a_1, ..., a_q\}$ by the payoff of player 1, such that a_1 is his worst, and a_q is his best (in B_0), and let l be the largest index for which $\mathcal{E}(2, B_2 \cup \{a_1, ..., a_l\}) = 0$. Let $B_2 = B_2 \cup \{a_1, ..., a_l\}$ for this l, and set $B_0 = \{a_{l+1}, ..., a_q\}$. Clearly $B_0 \neq \emptyset$ after this step either, since $\mathcal{E}(2, B_2 \cup B_0) = 1$ by the tightness of the game form and by $\mathcal{E}(1, B_1) = 0$.

Let us repeat the above two steps, as long as there is a change in B_1 or B_2. Since the elements are attempted to be added to either B_1 or B_2, one-by-one, the above computations can be completed by at most $|A|$ calls to the oracle.

When the above procedure stops (after $O(|A|)$ oracle calls) we end up with a partition of A into blocks B_1, B_2 and B_0 such that $\mathcal{E}(1, B_1) = 0$, $\mathcal{E}(2, B_2) = 0$ (which then implies $B_0 \neq \emptyset$ by tightness). Furthermore, by the above selection procedure, player 1 prefers every outcome in B_0 to any element of B_2, and similarly, player 2 prefers every outcome in B_0 to those in B_1.

Let us now denote by $a \in B_0$ the outcome which is player 1's worst in B_0, and let $b \in B_0$ the outcome player 2 prefers the least in B_0. Clearly,

$$\mathcal{E}(2, B_2 \cup \{a\}) = 1 = \mathcal{E}(1, B_1 \cup \{b\}) \tag{3}$$

since otherwise the above two-step procedure could be continued. Let us further denote by $B_3 \subseteq B_0$ the set of all outcomes from B_0 which are preferred by player 2 less than a (B_3 maybe empty), and let us set finally $B_1 = B_1 \cup B_3 \cup \{a\}$ and $B_2 = B_2 \cup \{a\}$.

We claim that the blocks B_1 and B_2 (constructed in $O(|A|)$ oracle calls) indeed have the desired properties, i.e. $B_1 \cap B_2 = \{a\}$, $\mathcal{E}(1, B_1) = 1 = \mathcal{E}(2, B_2)$, and outcome a is the best for both player 1 in B_2 and player 2 in B_1.

Since $a \in B_2$, we have $\mathcal{E}(2, B_2) = 1$ by (3). It is also clear that $b \in B_3 \cup \{a\}$, since it contains all outcomes form B_0 which are not better for player 2 than a, and b is the worst form player 2 in B_0. Thus, $b \in B_1$, and hence $\mathcal{E}(1, B_1) = 1$ also follows by (3). Furthermore, we have $B_1 \cap B_2 = \{a\}$ by the above construction. Finally, a is the best for both player 1 in B_2 and player 2 in B_1, since any outcome of B_0 was preferred by player 1 to all outcomes in B_2, and since we included in B_1 from B_0 only worse outcomes for player 2 than a. □

It is immediate to see that the above proposition proves also Theorem 5.

3.4 Proper Nash solvability of terminal games with two players

Let us finally show that not only two player terminal game forms are Nash solvable, but also a proper Nash equilibrium can be found in such games.

Proposition 6 *Every terminal game with two players has a proper Nash equilibrium.*

Proof. Clearly, if at most one player can enforce $\{C\}$ then every Nash equilibrium is proper, because a cycle is the worst outcome for both players. Hence in this case Nash-solvability automatically implies proper Nash-solvability. Yet, the case when each player can enforce $\{C\}$ is more difficult. Suppose that player 1 can, i.e. $\mathcal{E}(1, \{C\} = 1$. Then obviously, $\mathcal{E}(2, V_T) = 0$, i.e. player 2 can reach no terminal position. This means that player 1 has a strategy which restricts player 2 by a set of positions $M = V \setminus V_T$, that is player 1 can always remain in M, while player 2 can not leave M. Let M be a minimal such set. Then player 2 has no move $e = [v', v'')$, such that $v' \in M$, $v'' \notin M$, because otherwise we could exclude v' from M in contradiction to the minimality of M. Hence only player 1 can leave M and on the other hand he can always stay in M. Denote by E_M the set of all edges from M to $V_T \setminus M$. Let us contract M, that is substitute M by a single vertex v_0^* but keep all edges from E_M and assume that player 1 moves in v_0^*. Denote the obtained game by g'. We will proceed by induction on the number of positions. Game g' contains less positions than g. Indeed, $|M| > 1$ because M contains a directed cycle. Hence there is a proper equilibrium s' in g'. The corresponding play $\pi(s')$ starts with a move e_0' of the player 1. Consider the corresponding move $e_0 \in E_M$, chose in M an arbitrary directed path p from v_0 to e_0 and define situation s in g as follows: (i) in the beginning both players follow p, (ii) then player 1 leaves M by e_0, (iii) in $V_T \setminus M$ both players follow s', (iv) in $M \setminus p$ player 1 does not leave M and player 2 plays arbitrarily.

We claim that this situation s is a proper equilibrium in g. Indeed, by construction, play $\pi(s)$ is finite. For both players it would be too late to try to improve after M, because s' is an equilibrium in g'. And in M none of them can improve either. Indeed, player 2 can avoid e_0 only by creating a directed cycle. Payer 1, perhaps, can change his strategy and leave M by some other move e', instead or e_0. Yet, if by such a change he could improve his result then he could also do the same in game g', by choosing e' as his first move. $\square$

4 Nash-solvability of terminal games with three outcomes

The following theorem is a consequence of the above results.

Theorem 6 *Every terminal game with at most two terminal moves has a proper strong equilibrium in pure stationary strategies.*

Proof. Such a game has at most 3 outcomes $A = \{b_1, b_2, C\}$, i.e. two terminal moves and a cycle. We can also assume that no player has a move leading directly to his best outcome, since such moves could be fixed, recursively, in advance.

Let us then choose an arbitrary path p from v_0 to b_1, and consider the strategy s defined by choosing the arcs in this path for the vertices in the path, and choosing an arbitrary arc (u, v) for every other vertex u such that $v \neq b_2$.

We claim that this is a proper Nash equilibrium, which is also a strong equilibrium.

By its definition, it is clear that $\pi(s) = b_1$. For players along the path p, who prefer b_1 to b_2 this strategy provides their best, and hence they are not interested in changing it. If the other players would try to change along the path p, the resulted situation s' would still not have a move into b_2, by our preprocessing assumption, and hence they cannot obtain anything better for themselves either (C is assumed to be the worst for all players). $\square$

Let us remark that the above claim would also follow form the previous section, since players who prefer b_i to b_j ($\{i, j\} = \{1, 2\}$) could be identified, and the above game could be reduced to a terminal game with only two players.

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