DIMACS Technical Report 2002-13
April 2002

Graph Ramseian bipartitions and weighted stability

by

I. E. Zverovich [1]
RUTCOR–Rutgers Center for Operations Research
Rutgers, The State University of New Jersey
640 Bartholomew Rd.
Piscataway, NJ 08854-8003, U.S.A.
igor@rutcor.rutgers.edu

I. I. Zverovich
ASA Institute of Technology,
Brooklyn, New York, NY 11201, U.S.A.

[1]Supported by DIMACS Winter Research 2001–2002.

ABSTRACT

Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary classes of graphs. The ordered pair $(\mathcal{P}, \mathcal{Q})$ is called *Ramseian* if both $\mathcal{P}$ and $\mathcal{Q}$ are polynomially recognizible, $\mathcal{P}$ is α-bounded, and $\mathcal{Q}$ is ω-bounded. Let $(\mathcal{P}, \mathcal{Q})$ be an ordered pair of graph classes. We denote by $\mathcal{P} * \mathcal{Q}$ the class of all graphs G such that there exists a partition $A \cup B = V(G)$ with

- $G(A) \in \mathcal{P}$ and
- $G(B) \in \mathcal{Q}$,

where $G(X)$ denotes the subgraph of G induced by a set $X \subseteq V(G)$.
A class of graphs $\mathcal{C}$ is called *α_w-polynomial* if there exists a polynomial-time algorithm for calculation the weighted stability number $\alpha_w(G)$ for all graphs $G \in \mathcal{C}$. A class of graphs $\mathcal{C}$ is called *α_w-complete* if the corresponding decision problem is NP-complete for graphs in $\mathcal{C}$.
Our main result is the following theorem.

Theorem *Let $(\mathcal{P}, \mathcal{Q})$ be a Ramseian pair.*
(i) *If $\mathcal{Q}$ is an α_w-polynomial class then the class $\mathcal{P} * \mathcal{Q}$ is also α_w-polynomial.*
(ii) *If $\mathcal{Q}$ is an α_w-complete class then the class $\mathcal{P} * \mathcal{Q}$ is also α_w-complete.*

A similar results for ω_w-polynomial classes and ω_w-complete classes are easily follow ($\omega_w(G)$ is the weighted clique number of a graph G). Finally, a recent result of Alekseev and Lozin (2002) is a particular case of our main theorem.

Keywords: Hereditary class, forbidden induced subgraphs, Ramseian partition, weighted stability number.

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is called a *stable set* if it induces an edgeless subgraph. The largest cardinality of a stable set in G is the *stability number* $\alpha(G)$ of G. A set $K \subseteq V$ is called a *clique* if it induces a complete subgraph. The largest cardinality of a clique in G is the *clique number* $\omega(G)$ of G.

Definition 1. *A class of graphs $\mathcal{C}$ is* α-bounded *if there exists a constant m such that $\alpha(G) \leq m$ for each graph in $G \in \mathcal{C}$. Similarly, $\mathcal{C}$ is* ω-bounded *if there exists a constant n such that $\omega(G) \leq n$ for each graph in $G \in \mathcal{C}$.*

A class of graphs $\mathcal{C}$ is *polynomially recognizible* if there exists a polynomial-time algorithm for recognizing graphs in $\mathcal{C}$.

Definition 2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be hereditary classes of graphs. The ordered pair $(\mathcal{P}, \mathcal{Q})$ is called* Ramseian *if*

(R1) *both $\mathcal{P}$ and $\mathcal{Q}$ are polynomially recognizible,*

(R2) *$\mathcal{P}$ is α-bounded, and $\mathcal{Q}$ is ω-bounded.*

Definition 3. *Let $(\mathcal{P}, \mathcal{Q})$ be an ordered pair of graph classes. We denote by $\mathcal{P} * \mathcal{Q}$ the class of all graphs G such that there exists a partition $A \cup B = V(G)$ with*

- *$G(A) \in \mathcal{P}$ and*
- *$G(B) \in \mathcal{Q}$,*

where $G(X)$ denotes the subgraph of G induced by a set $X \subseteq V(G)$.

If $(\mathcal{P}, \mathcal{Q})$ is a Ramseian pair then the partition $A \cup B$ above is called a Ramseian $(\mathcal{P}, \mathcal{Q})$-partition.

The most general result on Ramseian partitions was obtained by Zverovich [2] (as a consequence of a theorem on bipartite bihypergraphs). A hereditary class $\mathcal{C}$ of graphs is called *finitely generated* if the number of minimal forbidden induced subgraphs for $\mathcal{C}$ is finite.

Theorem 1 (Zverovich [2]). *Let $(\mathcal{P}, \mathcal{Q})$ be a Ramseian pair such that both $\mathcal{P}$ and $\mathcal{Q}$ are finitely generated. Then the class $\mathcal{P} * \mathcal{Q}$ is also finitely generated.*

Obviously, if a hereditary class is finitely generated then it is polynomially recognizible. The converse is not true. However, an analogue of Theorem 1 for Ramseian pairs can be easily proved.

Proposition 1. *If $(\mathcal{P}, \mathcal{Q})$ is a Ramseian pair then the class $\mathcal{P} * \mathcal{Q}$ is polynomially recognizible.*

Moreover, given a graph $G \in \mathcal{P} * \mathcal{Q}$, it is possible to find a Ramseian $(\mathcal{P}, \mathcal{Q})$-partition for G in polynomial time.

Proof. Let G be a graph. Suppose that we have disjoint sets $A_k, B_k \subseteq V(G)$ ($k \in \{0, 1, \ldots, |V(G)|\}$) such that $G(A_k) \in \mathcal{P}$ and $G(B_k) \in \mathcal{Q}$. Initially we may put $k = 0$ and $A_0 = B_0 = \emptyset$. Recall that, by (R1), membership in $\mathcal{P}$ and $\mathcal{Q}$ can be tested in polynomial time. Also, it follows from (R2), that there exist constants m and n such that

- $\alpha(H) < m$ for each $H \in \mathcal{P}$, and
- $\omega(H) < n$ for each $H \in \mathcal{Q}$.

If $A_k \cup B_k = V(G)$ then $A_k \cup B_k$ is a Ramseian $(\mathcal{P}, \mathcal{Q})$-partition for G, and $G \in \mathcal{P} * \mathcal{Q}$. Therefore we may assume that $A_k \cup B_k \neq V(G)$.

If there exists a Ramseian $(\mathcal{P}, \mathcal{Q})$-partition $A \cup B$ for G, then

- $G(A \cap B_k) \in \mathcal{P}$ (since $A \cap B_k \subseteq A$, $G(A) \in \mathcal{P}$, and $\mathcal{P}$ is a hereditary property), and
- $G(A \cap B_k) \in \mathcal{Q}$ (since $A \cap B_k \subseteq B_k$, $G(B_k) \in \mathcal{Q}$, and $\mathcal{Q}$ is a hereditary property).

Thus, $G(A \cap B_k) \in \mathcal{P} \cap \mathcal{Q}$. Similarly, $G(B \cap A_k) \in \mathcal{P} \cap \mathcal{Q}$. By Ramsey's theorem, $|A \cap B_k| \leq r(m, n)$ and $|A \cap B_k| \leq r(m, n)$, where $r(m, n)$ is the Ramsey number.

Now we perform the following total search.

Total Search [for (A_k, B_k)]

- We generate all pairs of sets $X \subseteq A_k$ and $Y \subseteq B_k$ with $|X| \leq r(m, n)$ and $|Y| \leq r(m, n)$.
- For each pair (X, Y) we put $A'_k = (A_k \backslash X) \cup Y$ and $B'_k = (B_k \backslash Y) \cup X$.
- If both $G(A'_k) \in \mathcal{P}$ and $G(B'_k) \in \mathcal{Q}$ hold then we choose an arbitrary vertex $u \notin A_k \cup B_k$, and
  - we check
  $$\text{(A)} \quad G(A'_k \cup \{u\}) \in \mathcal{P};$$
  - if (A) holds, we put $A_{k+1} = A'_k \cup \{u\}$ and $B_{k+1} = B'_k$, and STOP;
  - if (A) does not hold, we check
  $$\text{(B)} \quad G(B'_k \cup \{u\}) \in \mathcal{Q};$$

  - if (B) holds, we put $A_{k+1} = A'_k$ and $B_{k+1} = B'_k \cup \{u\}$, and STOP;
  - if (B) does not hold, we consider then we consider the next pair (X, Y) (if any) or STOP.

- If either $G(A'_k) \notin \mathcal{P}$ or $G(B'_k) \notin \mathcal{Q}$ then we consider next pair (X, Y) (if any) or STOP.

The search either produces (in polynomial time) disjoint sets A_{k+1} and B_{k+1} with $G(A_{k+1}) \in \mathcal{P}$, $G(B_{k+1}) \in \mathcal{Q}$, and

$$|A_{k+1} \cup B_{k+1}| = |A_k \cup B_k| + 1,$$

or concludes that there are no such sets. In the former case we have either

- $A_{k+1} \cup B_{k+1} = V(G)$ and hence $G \in \mathcal{P} * \mathcal{Q}$, or
- $A_{k+1} \cup B_{k+1} \neq V(G)$ and we continue the search (with A_{k+1} and B_{k+1} instead of A_k and B_k, respectively).

The latter case implies that $G \notin \mathcal{P} * \mathcal{Q}$.

Since the search runs in polynomial time, and the maximum possible number of searches is $|V(G)|$, our algorithm for recognizing $\mathcal{P} * \mathcal{Q}$ (and constructing the corresponding partition) is polynomial. □

Suppose that G is a graphs with non-negative weight function $w : V(G) \to [0, +\infty)$. As usual, the *weight of a subset* $X \subseteq V(G)$ is the sum of $w(x)$ taken over all $x \in X$. The *weighted stability number* $\alpha_w(G)$ of G is the maximum weight of a stable set in G.

Definition 4. *A class of graphs $\mathcal{C}$ is called* α_w-polynomial *if there exists a polynomial-time algorithm for calculation $\alpha_w(G)$ for all graphs $G \in \mathcal{C}$.*

A class of graphs $\mathcal{C}$ is called α_w-complete *if the corresponding decision problem is* NP*-complete for graphs in $\mathcal{C}$.*

The following theorem is our main result.

Theorem 2. *Let $(\mathcal{P}, \mathcal{Q})$ be a Ramseian pair.*

(i) *If $\mathcal{Q}$ is an α_w-polynomial class then the class $\mathcal{P} * \mathcal{Q}$ is also α_w-polynomial.*

(ii) *If $\mathcal{Q}$ is an α_w-complete class then the class $\mathcal{P} * \mathcal{Q}$ is also α_w-complete.*

Proof. (i) Let G be a graph in $\mathcal{P} * \mathcal{Q}$. By Proposition 1, we can construct a Ramseian partition $A \cup B = V(G)$ in polynomial time. Let us fix such a partition. Since $\mathcal{P}$ is an α-bounded class, we can generate in polynomial time the list L of all stable sets in $G(A)$, including the empty set.

For each stable set $S \in L$ we consider the graph $H = G(B) - N(S)$ [i.e., H is obtained from $G(B)$ by deleting all vertices that are adjacent to at least one vertex of S]. Clearly H is an induced subgraph of $G(B)$. Since $G(B) \in \mathcal{Q}$ and $\mathcal{Q}$ is a hereditary class, we have $H \in \mathcal{Q}$. The class $\mathcal{Q}$ is an α_w-polynomial, therefore we can find $\alpha_w(H)$ in polynomial time. Now we define $a(S) = w(S) + \alpha_w(H)$. We put $a(L) = \max\{a(S) : S \in L\}$.

It remains to check that $a(L) = \alpha_w(G)$ (and to use the fact that $a(L)$ was computed in polynomial time). Since $a(L)$ is the weight of a stable set in G, we have $a(L) \leq \alpha_w(G)$. Now let I be a maximum weight stable set in G. Clearly, the set $S = I \cap A$ belongs to L. The set $S' = I \backslash S$ is a stable set disjoint from A, i.e., $S' \subseteq B$. Moreover, no vertex of S' is adjacent to a vertex of S (since $I = S \cup S'$ is a stable set). Therefore S' is a stable set in $H = G(B) - N(S)$. It follows that

$$\alpha_w(G) = |I| = w(S) + w(S') \leq w(S) + \alpha_w(H) = a(L).$$

We have proved that $a(L) = \alpha_w(G)$.

(ii) The statement follows from the inclusion $\mathcal{Q} \subseteq \mathcal{P} * \mathcal{Q}$ and the condition that $\mathcal{Q}$ is an α_w-complete class. □

Let G be a graph and let w be a non-negative weight function. The *weighted clique number* $\omega_w(G)$ of G is the maximum weight of a clique in G.

Remark 1. a) *The condition that $\mathcal{P}$ and $\mathcal{Q}$ are polynomially recognizible classes is inessential in the statement* (ii) *of Theorem 2. If we omit this condition, the class $\mathcal{P} * \mathcal{Q}$ can be not polynomially recognizible.*

b) *A similar situation arises if we omit the condition that $\mathcal{P}$ is a polynomially recognizible class in the statement* (i) *of Theorem 2. In this case an easy modification of the proof required.*

Definition 5. *A class of graphs $\mathcal{C}$ is called* ω_w-polynomial *if there exists a polynomial-time algorithm for calculation $\omega_w(G)$ for all graphs $G \in \mathcal{C}$.*

A class of graphs $\mathcal{C}$ is called ω_w-complete *if the corresponding decision problem is* NP-*complete for graphs in $\mathcal{C}$.*

Now we can state a "complementary" form of Theorem 2.

Corollary 1. *Let $(\mathcal{P}, \mathcal{Q})$ be a Ramseian pair.*

(i) *If $\mathcal{P}$ is an ω_w-polynomial class then the class $\mathcal{P} * \mathcal{Q}$ is also ω_w-polynomial.*

(ii) *If $\mathcal{P}$ is an ω_w-complete class then the class $\mathcal{P} * \mathcal{Q}$ is also ω_w-complete.*

Proof. For a class of graphs $\mathcal{C}$, its complement is defined as $\overline{\mathcal{C}} = \{\overline{G} : G \in \mathcal{C}\}$. It is clear that the complement of an α-bounded class (respectively, an ω-bounded class) is an ω-bounded class (respectively, an α-bounded class). Also, a class $\mathcal{C}$ is polynomially recognizible if and only if $\overline{\mathcal{C}}$ is polynomially recognizible. Thus, a pair $(\mathcal{P}, \mathcal{Q})$ is Ramseian if and only if the pair $(\overline{\mathcal{Q}}, \overline{\mathcal{P}})$ is Ramseian.

Finally, a class is α_w-polynomial (respectively, α_w-complete) if and only if its complement is ω_w-polynomial (respectively, ω_w-complete). Now the result follows from Theorem 2. □

Definition 6. *A graph G is called* (p, q)-colorable *if $V(G)$ can be partitioned into $p + q$ subsets*

$$(X_1 \cup X_2 \cup \cdots \cup X_p) \cup (Y_1 \cup Y_2 \cup \cdots \cup Y_q)$$

(some of them may be empty) such that

- *each X_i induces a complete subgraph, and*
- *each Y_j is a stable set.*

We denote by $\mathcal{C}(p, q)$ be the class of all (p, q)-colorable graphs.

In other words,

$$\mathcal{C}(p, q) = \overline{\mathcal{C}}(p) * \mathcal{C}(q),$$

where $\mathcal{C}(r)$ is the class of all colorable graphs, and $\overline{\mathcal{C}}(r)$ is its complement.

Corollary 2 (Alekseev and Lozin [1]).

(i) *If $q \leq 2$ then $\mathcal{C}(p, q)$ is an α_w-polynomial class.*

(ii) *If $q \geq 3$ then $\mathcal{C}(p, q)$ is an α_w-complete class.*

Remark 2. *We can strengthen the statement* (i) *of Corollary 2. As it follows from the proof below, a more general class is α_w-polynomial, namely, $\mathcal{P}_p * \mathcal{C}(2)$, where $\mathcal{P}_p = \{G : \alpha(G) \leq p\}$.*

Proof. (i) The class $\mathcal{C}(2)$ consists of all bipartite graphs, it is polynomially recognizible, ω-bounded and α_w-polynomial. The $\overline{\mathcal{C}}(p)$ class is contained in the polynomially recognizible α-bounded class $\mathcal{P}_p = \{G : \alpha(G) \leq p\}$. By Theorem 2(i), the class $\mathcal{P}_p * \mathcal{C}(2)$ is α_w-polynomial. It contains all the classes $\mathcal{C}(p, q)$ with $q \leq 2$.

The statement (ii) is a particular case of Theorem 2(ii) (see also Remark 1), since the class $\mathcal{C}(q)$ (for every $q \geq 3$) contains an α_w-complete class of all 3-colorable graphs. □

References

[1] V. E. Alekseev and V. V. Lozin, Independent sets of maximum weight in (p, q)-colorable graphs, RUTCOR Research Report RRR 12-2002 (Rutgers University, 2002) 7 pp.

[2] I. E. Zverovich, r-Bounded k-complete bihypergraphs and generalized split graphs, Discrete Math. (accepted in 2000)