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# Mathematical Programming Formulation of Rectilinear Crossing Minimization

by

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## ABSTRACT

In a *rectilinear drawing* of a simple graph $G$ each vertex is mapped to a distinct point in the plane and each edge is represented by a straight-line segment with appropriate ends. The goal of rectilinear crossing minimization is to find a rectilinear drawing of $G$ with as few edge crossings as possible. A new approach to rectilinear crossing minimization is presented including a formulation of the problem as a mathematical program with a linear objective function and simple quadratic constraints. Then we size the problem for various graph families. These sizings provide a mechanism for ranking crossing number problems according to their apparent complexity.

# 1 Introduction

This paper concerns **drawings** of a simple graph $G = (V, E)$ with vertex set $V$ and edge set $E$ in the plane where the vertices are mapped to distinct points. We are mainly concerned with *good drawings*; these are drawings where each edge is mapped to a homeomorphic image of the closed interval $[0, 1]$ with its ends as endpoints. The interior (called an **arc**) contains no endpoint, arcs incident with a common endpoint do not intersect, and no two arcs intersect in more than one point. Any drawing where the edges are straight-line segments is called a **rectilinear drawing**. The goal of (rectilinear) crossing minimization is to find a (rectilinear) drawing of $G$ with as few edge crossings as possible. This minimum value is called the **(rectilinear) crossing number** $cr(G)$ (respectively, $\overline{cr}(G)$) of $G$. Both of the corresponding decision problems are known to be NP-hard. Moreover, the crossing number problem is known to be NP-complete [16], but so far no one has determined whether the rectilinear crossing number problem is in NP. This is somewhat surprising, since it is much easier to determine if two straight-line segments intersect than it is to determine if two curved lines do.

This paper presents some new results on rectilinear crossing minimization. In Section 2 we construct a mathematical formulation of the problem in terms of a linear objective function with simple quadratic constraints. In Section 3 we give a comparative sizing of the problem for various graph families and show that the formulation appears to be too large for existing solvers. As more advanced techniques are developed and applied to this formulation, it is likely that some of these problems will be solved. Finally, in Section 4 we conclude with some related open problems.

# 2 Quadratic Constraints Formulation

In this section we construct a mathematical programming formulation of the rectilinear crossing number problem where, for a given graph $G$ on $n$ vertices, most of the variables are binary, they appear only in linear form, and the $2n$ continuous variables appear in quadratic form. Of course, many mathematical problems have useful mathematical programming formulations (for example, see [28] and [36] for related results), but until now no such formulation has appeared for the rectilinear crossing number problem.

For any rectilinear drawing of a graph $G = (V, E)$ with $V = \{1, 2, \ldots, n\}$ we define $\text{Area2}(i, j, k) = x_i y_j - y_i x_j + y_i x_k - x_i y_k + x_j y_k - x_k y_j$ where $(x_i, y_i), (x_j, y_j), (x_k, y_k)$ are the coordinates for the vertices $i, j, k$, respectively. Geometrically, $\text{Area2}(i, j, k)$ is twice the signed area determined by the triangle $ijk$ [31]; in particular, $\text{Area2}(i, j, k)$ is positive if vertex $k$ lies to the left of line $ij$, negative if it lies to the right, and zero if $k$ lies on the line $ij$. When we speak of a line $ij$ (not a line segment) we mean the two-way infinite directed line determined by the ordered pair of vertices $i, j$.

For any graph $G$, let $I(G)$ denote the set of independent (i.e., disjoint) edge sets of size two. We wish to solve the following problem which we call the *Quadratic Constraints Formulation* QCF(G) for $G$, or simply QCF when there is no particular graph $G$ under consideration.

$$\text{Minimize} \sum_{\substack{1\le i<j, i<k<l,\\ \{ij,kl\}\in I(G)}} c_{ijkl}$$

subject to

$$
\begin{aligned}
\text{Area2}(i,j,k) &\le M(1-c_{ijkl}) + Mt_{ijkl} - 1 && (1)\\
-\text{Area2}(i,j,l) &\le M(1-c_{ijkl}) + Mt_{ijkl} - 1 && (2)\\
-\text{Area2}(i,j,k) &\le M(1-c_{ijkl}) + M(1-t_{ijkl}) - 1 && (3)\\
\text{Area2}(i,j,l) &\le M(1-c_{ijkl}) + M(1-t_{ijkl}) - 1 && (4)\\
\text{Area2}(k,l,i) &\le M(1-c_{ijkl}) + Mp_{ijkl} - 1 && (5)\\
-\text{Area2}(k,l,j) &\le M(1-c_{ijkl}) + Mp_{ijkl} - 1 && (6)\\
-\text{Area2}(k,l,i) &\le M(1-c_{ijkl}) + M(1-p_{ijkl}) - 1 && (7)\\
\text{Area2}(k,l,j) &\le M(1-c_{ijkl}) + M(1-p_{ijkl}) - 1 && (8)\\
\text{Area2}(i,j,k) &\le Mc_{ijkl} + Mt_{ijkl} + Mp_{ijkl} - 1 && (9)\\
\text{Area2}(i,j,l) &\le Mc_{ijkl} + Mt_{ijkl} + Mp_{ijkl} - 1 && (10)\\
-\text{Area2}(i,j,k) &\le Mc_{ijkl} + M(1-t_{ijkl}) + Mp_{ijkl} - 1 && (11)\\
-\text{Area2}(i,j,l) &\le Mc_{ijkl} + M(1-t_{ijkl}) + Mp_{ijkl} - 1 && (12)\\
\text{Area2}(k,l,i) &\le Mc_{ijkl} + Mt_{ijkl} + M(1-p_{ijkl}) - 1 && (13)\\
\text{Area2}(k,l,j) &\le Mc_{ijkl} + Mt_{ijkl} + M(1-p_{ijkl}) - 1 && (14)\\
-\text{Area2}(k,l,i) &\le Mc_{ijkl} + M(1-t_{ijkl}) + M(1-p_{ijkl}) - 1 && (15)\\
-\text{Area2}(k,l,j) &\le Mc_{ijkl} + M(1-t_{ijkl}) + M(1-p_{ijkl}) - 1 && (16)
\end{aligned}
$$

where the variables $i,j,k,l$ satisfy $\{ij,k\ell\} \in I(G)$, $1 \le i < j, i < k < l$, the functions $c,t,p$ are 0-1 binary variables, and $M$ is a sufficiently large integer constant.

Now we explain why QCF is equivalent to rectilinear crossing minimization. The function $c_{ijkl}$ counts the number of times edges $ij$ and $kl$ cross (i.e., 0 or 1). The objective function counts the total number of crossings for the drawing. The $-1$ in the constraints insures that $|\text{Area2}(i,j,k)| \ge 1$, $|\text{Area2}(i,j,l)| \ge 1$, $|\text{Area2}(k,l,i)| \ge 1$, and $|\text{Area2}(k,l,j)| \ge 1$. Rescaling can be used to make all nonzero areas satisfy this condition, and so the $-1$ really only ensures that no vertex lies on a line to which it does not belong; in particular, we only consider good drawings.

Suppose $c_{ijkl} = 1$. Then constraints (9)-(16) become irrelevant, and constraints (1)-(4) ensure that points $k$ and $l$ lie on opposite sides of the line $ij$. This holds because $t_{ijkl} = 1$ implies that $k$ and $l$ lie to the left and right of line $ij$ (respectively), and $t_{ijkl} = 0$ implies that $k$ amd $l$ lie to the right and left of line $ij$ (respectively). Similarly, constraints (5)-(8) ensure that points $i$ and $j$ lie on opposite sides of the line $kl$. The assignment $p_{ijkl} = 1$ implies $i$ is on the left and $j$ is on the right of $kl$, and the assignment $p_{ijkl} = 0$ implies $i$ on the right and $j$ is on the left. Therefore, the line segments $ij$ and $kl$ must cross.

Now suppose $c_{ijkl} = 0$. Then constraints (1)-(8) are irrelevant, and either $p_{ijkl} = 0$ which ensures that points $k$ and $l$ lie on the same side of the line $ij$ (see constraints (9)-(12)) or

$p_{ijkl} = 1$ which ensures that points $i$ and $j$ lie on the same side of the line $kl$ (see constraints (13)-(16)). Thus, $ij$ and $kl$ do not cross.

# 3 Comparative Sizing for Graph Families

Research in integer and nonlinear programming provides some hope that some useful instances of QCF can be solved. One breakthrough was made by Crowder, Johnson and Padberg [11] whose paper won the Lanchester Prize for solving binary integer programming (BIP) problems with up to 2,756 variables and no special structure. Roy and Wolsey [33] succeeded in solving mixed BIPs with nearly 1,000 binary variables and an even larger number of real variables. Their paper won the Orchard-Hays Prize. In this section we provide data to show that, for the graphs of most interest, the number of binary and real variables in QCF is evidently still larger than the instances that can be solved by the best solvers. On the other hand, we hope that experts in the field can somehow take advantage of the special structure or any new algorithmic developments that can be applied to solve these problems.

Let $r(G), b(G), f(G)$ denote the number of real variables, the number of binary variables, and the number of functional constraints in QCF(G), respectively. Also, let $i(G)$ denote the number of matchings of size two in $G$ (i.e., $i(G) = |I(G)|$). Then $r(G) = 2n$, $b(G) = 3i(G)$ and $f(G) = 16i(G)$. Letting $e$ denote the number of edges we have

$$i(G) = \frac{1}{2}(e^2 + e - \sum_{x \in V} d^2(x)) \tag{17}$$

from which it follows that

$$i(K_n) = n(n-1)(n-2)(n-3)/8 = 3\binom{n}{4}.$$

Drawings of these graphs have been investigated extensively in the literature because of various theoretical implications and applications to VLSI and computer network design.

Some believe that $\overline{cr}(K_n) > cr(K_n)$ for $n \geq 10$; however, no proof has been published. Of course, putting the vertices of $K_n$ on the circumference of a circle creates the maximum number $\binom{n}{4}$ of crossings. The best upper bounds have been established by producing a drawing, as Singer did in an unpublished manuscript (see [15]) to show $\overline{cr}(K_{10}) \leq 62$. In the same paper he showed $\overline{cr}(K_{10}) \geq 61$. Recently, Brodsky, Durocher and Gethner [10] proved $\overline{cr}(K_{10}) = 62$. The value of $\overline{cr}(K_n)$ for $n \geq 11$ is unknown. It is known that the number of crossings in all drawings of $K_n$ have the same parity whenever $n$ is odd (see [3], [23], and [26]), and so $\overline{cr}(K_{11}) \neq 101$. Guy[20] proved that $cr(K_m) \geq \binom{m}{n} cr(K_n) \Big/ \binom{m-4}{n-4}$ for $m \geq n \geq 5$, and this result provides the lower bounds for $K_{11}$ and $K_{12}$ in the following table.

| $n$ | $r(K_n)$ | $b(K_n)$ | $f(K_n)$ | Best Bound for $\overline{cr}(K_n)$ | |
|---|---|---|---|---|---|
| 4 | 8 | 9 | 48 | 0 | |
| 5 | 10 | 45 | 240 | 1 | |
| 6 | 12 | 135 | 720 | 3 | [18] |
| 7 | 14 | 315 | 1680 | 9 | [18] |
| 8 | 16 | 630 | 3360 | 19 | [18] |
| 9 | 18 | 1134 | 6048 | 36 | [18] |
| 10 | 20 | 1890 | 10080 | 62 | [10] |
| 11 | 22 | 2970 | 15840 | {96,98,100,102} | [23],[26],[12] |
| 12 | 24 | 4455 | 23760 | [144,154] | [12] |

The following equation known as Zarankiewicz's Conjecture has been solved only for $\min\{p, q\} \leq 6$ by Kleitman [25] and for $K_{7,7}$ and $K_{7,9}$ by Woodall [37].

$$\overline{cr}(K_{p,q}) = cr(K_{p,q}) = \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor$$

The question "$\overline{cr}(K_{p,q}) = cr(K_{p,q})$?" could be treated as a separate problem. Because of a counting argument this leaves $K_{7,11}$ and $K_{9,9}$ as the two smallest unsolved cases. From Equation 17 we see that

$$i(K_{p,q}) = pq(p-1)(q-1)/2 = 2 \binom{p}{2} \binom{q}{2}$$

which is used to construct the following table.

| $p$ | $q$ | $r(K_{p,q})$ | $b(K_{p,q})$ | $f(K_{p,q})$ | Best Bounds on $\overline{cr}(K_{p,q})$ | |
|---|---|---|---|---|---|---|
| 3 | 3 | 12 | 54 | 288 | 1 | |
| 4 | 4 | 16 | 216 | 1152 | 4 | |
| 5 | 5 | 20 | 600 | 3200 | 16 | [25] |
| 6 | 6 | 24 | 1350 | 7200 | 36 | [25] |
| 7 | 7 | 28 | 2646 | 14112 | 81 | [37] |
| 8 | 8 | 32 | 4704 | 25088 | 144 | [37] |
| 7 | 11 | 36 | 6930 | 36960 | $\leq 175$ | |
| 9 | 9 | 36 | 7776 | 41472 | $\leq 256$ | |

The main conjecture on the crossing number of the product $C_p \times C_q$ is due to Harary, Kainen and Schwenk [22] and states that $cr(C_p \times C_q) = (p-2)q$ for $p \leq q$. For a survey on this problem including a discussion of the main proof techniques, see Myers [30]. The conjecture has been verified for $p \leq 7$. The drawings used to establish the upper bound for the conjecture are easy to transform into a rectilinear drawing, and so $\overline{cr}(C_p \times C_q) \leq (p-2)q$ for $p \leq q$. Recently, both Bruce Richter and Gelasio Salazar commented (personal communication) that the case $C_8 \times C_8$ can be settled using existing techniques, but the proof would require at least 100 pages! Our sizing of this problem is based on the formula

$$i(C_p \times C_q) = pq(2pq - 7)$$

from Equation 17 and appears in the following table.

| $p$ | $r(C_p \times C_p)$ | $b(C_p \times C_p)$ | $f(C_p \times C_p)$ | Best Bounds on $\overline{cr}(C_p \times C_p)$ |
|---|---|---|---|---|
| 3 | 18 | 297 | 1584 | 3 [22] |
| 4 | 32 | 1200 | 6400 | 8 [13] |
| 5 | 50 | 3225 | 17200 | 15 [32] |
| 6 | 72 | 7020 | 37440 | 24 [1] |
| 7 | 98 | 13377 | 71344 | 35 [2] |
| 8 | 128 | 23232 | 123904 | $\leq 48$ [22] |
| 9 | 162 | 37665 | 200880 | $\leq 63$ [22] |

Let $Q_p$ denote the $p$-dimensional cube; this is the graph whose vertex set consists of all possible $p$-tuples of 0's and 1's where two vertices are adjacent if and only if they differ in exactly one coordinate. Eggleton and Guy [14] announced the inequality

$$cr(Q_p) \leq \frac{5}{32}4^p - \left\lfloor \frac{p^2+1}{2} \right\rfloor 2^{p-2},$$

but an error was found in the construction [19]. So this upper bound is now only a conjecture. In fact, they conjecture that equality holds, and (if true) this would imply $cr(Q_5) = 56$ and $cr(Q_p) = 352$. So far, Madej [27] has proved the best upper bound $cr(Q_p) < 4^p/6$, and Sýkora and Vrto [35] have proved the best lower bound $cr(Q_p) > 4^p/20 - (p^2+1)2^{p-1}$. These bounds are useless for small $p$. Equation 17 gives

$$i(Q_p) = 2^{p-2}p(2^{p-1}p - 2p + 1)$$

from which we construct the following table.

| $p$ | $r(Q_p)$ | $b(Q_p)$ | $f(Q_p)$ | Best Bounds on $\overline{cr}(Q_p)$ |
|---|---|---|---|---|
| 3 | 16 | 126 | 672 | 0 |
| 4 | 32 | 1200 | 6400 | 8 [13],[14] |
| 5 | 64 | 8520 | 45440 | |
| 6 | 128 | 52128 | 278016 | |

Let $\overline{C}_n$ denote the graph obtained from $K_n$ by deleting a hamiltonian cycle; that is, $\overline{C}_n$ is the complement of $C_n$. Guy and Hill [21] determined the exact values of $cr(\overline{C}_n)$ for $n \leq 10$ and $\overline{cr}(\overline{C}_n)$ for $n \leq 9$. In general, they showed that

$$cr(\overline{C}_n) \leq \frac{1}{64}(n-3)^2(n-5)^2 \text{ for } n \text{ odd}$$

and

$$cr(\overline{C}_n) \le \frac{1}{64}n(n-4)(n-6)^2 \text{ for } n \text{ even.}$$

Equality is conjectured in both cases. Application of Equation 17 gives the formula

$$i(\overline{C}_n) = \frac{1}{8}n(n-3)(n^2-7n+14)$$

which is used to construct the following table.

| $n$ | $r(\overline{C}_n)$ | $b(\overline{C}_n)$ | $f(\overline{C}_n)$ | Best Bound for $\overline{cr}(\overline{C}_n)$ |
|---|---|---|---|---|
| 5 | 10 | 15 | 80 | 0 |
| 6 | 12 | 54 | 288 | 0 |
| 7 | 14 | 147 | 784 | 1 [21] |
| 8 | 16 | 330 | 1760 | 2 [21] |
| 9 | 18 | 648 | 3456 | 9 [21] |
| 10 | 20 | 1155 | 6160 | $\ge 15$ [21] |
| 11 | 22 | 1914 | 10208 | $\ge 36$ [21] |

Jendrol' and M. Ščerbov [24] proved that $cr(K_{1,3} \times P_n) = n$ for $n \ge 2$. Both Beineke and Ringeisen (see [7]) and Jendrol' and Scerbov [24] proved that $cr(K_{1,3} \times C_n) = 1$ for $n = 3$, 2 for $n = 4$, 4 for $n = 5$, and $n$ for $n \ge 6$. The crossing numbers of the products of cycles with all other graphs of orders 3 and 4 were determined earlier by Beineke and Ringeisen (see [6]).

Of the numerous unsolved instances of crossing number problems we have presented in the previous five tables only two instances from five different families of graphs. It seems reasonable that the formulations with fewer binary variables should be easuer to solve. Based on this criterion we obtain the following ordering of these ten graphs from easiest to hardest.

$$\overline{C}_{10}, \overline{C}_{11}, K_{11}, K_{12}, K_{7,11}, K_{9,9}, Q_5, C_8 \times C_8, C_9 \times C_9, Q_6$$

# 4 Final Remarks

It may seem that the crossing number and the rectilinear crossing number are very closely related; however in this section we summarize by pointing out some differences between these two parameters and some open problems on the rectilinear crossing number. Along the way we examine some other more specialized crossing number problems.

First, here's a question we can answer. Is the difference $\overline{cr}(G) - cr(G)$ unbounded? Bienstock and Dean [9] showed that, for every integer $k \ge 4$, there is an infinite family of graphs of crossing number $k$, but unbounded rectilinear crossing number. On the other hand, they prove that any graph $G$ for which $cr(G) \le 3$ must also satisfy $cr(G) = \overline{cr}(G)$, an extension of the classic theorem of Steinitz and Wagner (also called Fary's theorem) that *every planar graph has a rectilinear embedding in the plane.*

Fixing the binary variables in QCF(G) reduces to the problem of deciding whether, for the given graph $G$ and a given set of crossings (and non-crossings), a set of coordinates exists for a rectilinear drawing of $G$ that is consistent with these constraints. Call this problem the **QCF Realization Problem**. In a certain physical sense, this problem turns out to be NP-hard as explained below. A pseudoline is a homeomorph in $\Re^2$ of the closed unit interval. In an *arrangement* of pseudolines every two lines meet at exactly one interior point where they must cross. A *rectilinear realization* of an arrangement is an arrangement where each pseudoline is a straight-line segment, and an arrangement is said to be stretchable if such a realization exists. Shor [34] proved that the problem of determining whether a pseudoline arrangement is stretchable is NP-hard. This also follows from a paper by Mnev [29] which implies the stronger result that deciding stretchability is equivalent to the existential theory of the reals. Bienstock [8] proved that any given arrangement of $p$ pseudolines can be forced to occur in every crossing minimal drawing of an appropriate graph, with $O(p^3)$ edges and $5p(p-1)$ crossings by using some results of Goodman, Pollack, and Sturmfels [17] on arrangements whose straight-line realizations require vertex coordinates with exponentially many bits. Therefore, assuming that *the coordinates must be stored and that drawing a graph consumes time proportional to the size of the drawing*, this yields that there is no polynomial-time algorithm for producing a rectilinear drawing of a graph which achieves the rectilinear crossing number.

For the following questions we do not make the assumptions mentioned above.

**Open Problem 1** *Is the QCF Realization Problem NP-hard?*

**Open Problem 2** *Is the QCF Realization Problem NP-hard, even when restricted to complete graphs?*

**Open Problem 3** *Is testing whether* $cr(G) = \overline{cr}(G)$ *NP-hard?*

Our final question (Problem 1 in [5]) concerns an approach to simplifying QCF in an attempt to make it more practical.

**Open Problem 4** *Does every rectilinear drawing of a graph* $G$ *with exactly* $\overline{cr}(G)$ *crossings have a triangle as its convex hull?*

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