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Expected rank in antimatroids

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ABSTRACT

We consider a probabilistic antimatroid A on the ground set E, where each element $e \in E$ may succeed with probability p_e. We focus on the expected rank $ER(A)$ of a subset of E as a polynomial in the p_e. General formulas hold for arbitrary antimatroids, and simpler expressions are valid for certain well-studied classes, including trees, rooted trees, posets, and finite subsets of the plane. We connect the Tutte polynomial of an antimatroid to $ER(A)$. When S is a finite subset of the plane with no three points collinear, we derive an expression for the expected rank which has surprising symmetry properties. Corollaries include new formulas involving the beta invariant of subsets of S and new proofs of some known formulas.

1 Introduction and mathematical preliminaries

There are many situations in reliability theory in which elements of a finite set E (frequently the edge set of a graph) are assumed to succeed or fail with certain probabilities. In this paper, we will assume each element $e \in E$ is *successful* or *operational* with probability p_e, and these probabilities operate independently. While these assumptions are not always realistic in applications, they can still be very useful in modeling complex systems.

The computation of the *reliability of a network* has a long and varied histroy. Standard references are [11, 24]. Relatively less attention as been payed to the question of the expected number of surviving components in a system. This is of interest in real-world applications, but it also gives rise to some interesting combinatorics. We will not consider the applications here, but indicate in Section 5 some possible ways to apply the invariants considered here to models of physical systems.

Expected rank in graphs (for various rank functions) have been considered in [4, 5, 25, 26] as the *pair connected reliability* and as the *resilience* in [12]. Consideration of expected rank also appears in [2, 6, 7, 27], although most of the structures considereed are graphs. For trees, this topic is explored in [3]. Like the reliability, the expected rank is a polynomial (in one or several variables) and this polynomial encodes combinatorial information about the graph, finite point set, poset, and so on.

Our goal in this work is to unify some of the different approaches in the literature by concentrating on the general class of antimatroids. While ordinary graphs do not give rise to antimatroids, there are several interesting combinatorial structures that are antimatroids. These include the main application treated here, finite subsets of the plane, as well as posets (in two different ways), trees and rooted trees.

The probabilistic approach allows short proofs of two results (Theorem 4.1 of [1] and Corollary 4.4 of [13]), and several new identities involving free sets in restrictions of antimatroids. The main theorem concerning finite subsets of the plane gives an expression for this polynomial in terms of half-planes associated with the set.

The paper is organized as follows. This section includes the relevant definitions and some old and new formulas for expected rank. Section 2 gives a relation between a one-variable expected rank function and the Tutte polynomial of the antimatroid. Section 3 gives the key probabilistic expansion for the polynomial (Proposition 3.1) and applications to trees and posets. Section 4 develops the theory for the application of these ideas to finite point sets, concentrating especially on finite subsets of the plane. Finally, there are several possibilities for research projects based on this work; we outline a few in Section 5.

Let $G = (E, r)$ be an ordered pair, where $E = \{1, \ldots, n\}$ and $r : 2^E \to Z^+ \cup \{0\}$ is a function (called the *rank* fuction) from the subsets of E to the non-negative integers. For each element $e \in E$, we assign an indeterminate p_e, which we interpret as the probability that the element e is successful or operational (so the probability that e is not operational is simply $1 - p_e$). We assume elements operate independently, although we make no assumptionas about the indeterminates p_e. Although this approach is probabilistic, most of the results given here can be considered purely combinatorially.

Our object of study in this paper is the *expected rank polynomial*, which we define as follows:

Definition 1.1

$$ER(G) = \sum_{S \subseteq E} r(S) \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j).$$

When the rank function satisfies certain conditions, we can obtain alternate formulations for $ER(G)$. In this paper, we will concentrate on *antimatroids*. More extensive introductions to the subject can be found in [8] and [21].

Definition 1.2 *An antimatroid is a pair $A = (E, \mathcal{F})$, where E is a finite set and $\mathcal{F}$ is a family of subsets of E, called the feasible sets, satisfying:*

F0 $\mathcal{F} \neq \emptyset$*;*

F1 *If $X \in \mathcal{F}$, then $X - \{x\} \in \mathcal{F}$ for some $x \in X$;*

F2 *If $X \in \mathcal{F}$, then $X \cup \{x\} \in \mathcal{F}$ for some $x \notin X$.*

The rank $r(S)$ of a subset S is the size of the largest feasible subset of S: $r(S) = \max\{|F| : F \in \mathcal{F}\}$. Note that $r(E) = |E|$. Throughout the paper, we assume A is an antimatroid on the ground set E with $|E| = n$, with feasible sets $\mathcal{F}$.

We define the *continuations* or *boundary* of the feasible set F as $\Gamma(F) = \{e \in E - F : F \cup \{e\} \in \mathcal{F}\}$. The next lemma is trivial, but useful.

Lemma 1.3 *If $S \subseteq E$, there is a unique $F \subseteq S$ with $r(S) = |F|$.*

Lemma 1.3 allows us to collect terms in the definition, which gives the next proposition. We omit the proof, which is essentially the same as the proof of Proposition 2.2 of [3].

Proposition 1.4 *Let A be an antimatroid with feasible sets $\mathcal{F}$. Then*

$$ER(A) = \sum_{F \in \mathcal{F}} |F| \prod_{e \in F} p_e \prod_{e \in \Gamma(F)} (1 - p_e).$$

We will also need the following property of antimatroids.

Lemma 1.5 (Definition 8.2.6 of [8]) *Let A be an antimatroid and suppose $F \subseteq G$, where $F, G \in \mathcal{F}$. If $F \cup \{x\} \in \mathcal{F}$ for some $x \in E$, then $G \cup \{x\} \in \mathcal{F}$.*

Deletion and contraction are very important operations in matroid theory, especially as they relate to invariants. We can also define these operations for antimatroids.

Let A be an antimatroid and let $\{e\}$ be a feasible singleton. Then F is feasible in the *deletion* $A - e$ iff $e \notin F$ and F is feasible in A. F is feasible in the *contraction* A/e iff $e \notin F$ and $F \cup \{e\}$ is feasible in A. (The requirement that $\{e\}$ is feasible guarantees $\emptyset$ will be feasible in A/e.)

Proposition 1.6 (Deletion-Contraction) *Let A be an antimatroid and let $\{e\} \in \mathcal{F}$. Then*

$$ER(A) = (1 - p_e)ER(A - e) + p_e ER(A/e) + p_e.$$

Proof. We write $ER(A) = S_1 + S_2$, where $S_1 = \sum_{S:e\in S} r_A(S) \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j)$ and $S_2 = \sum_{S:e\notin S} r_A(S) \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j)$.

Now $S_2 = (1 - p_e)ER(A - e)$ because $r_A(S) = r_{A-e}(S)$ whenever $e \notin S$, where $r_B(S)$ denotes the rank of S in the antimatroid B.

When $e \in S$, we have $r_A(S) = r_{A/e}(S - e) + 1$ by definition of the feasible sets in the contraction A/e. Then

$$\begin{aligned} S_1 &= \sum_{S:e\in S} r_A(S) \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j) \\ &= \sum_{S:e\in S} \Big(r_{A/e}(S - e) + 1\Big) \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j) \\ &= p_e ER(A/e) + p_e \sum_{S:e\notin S} \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j) \\ &= p_e ER(A/e) + p_e, \end{aligned}$$

where the term $\sum_{S:e\notin S} \prod_{i\in S} p_i \prod_{j\notin S}(1 - p_j) = \prod_{a\in S, a\neq e} (p_a + (1 - p_a)) = 1.$

$\square$

The deletion-contraction recurrence can also be proved using a simple conditional probability argument and Proposition 3.1 below. For the reliability polynomial, this is referred to as the *factoring theorem*. See [11] for more information about the reliability polynomial.

For an antimatroid A on the ground set E, define a set $C \subseteq E$ to be *convex* if the complement $E - C$ is feasible. A convex set K is *free* if every subset of K is also convex. Let *Free* denote the collection of all free convex sets of A.

The β-invariant of an antimatroid A can be defined as follows:

$$\beta(A) = \sum_{K\in Free} (-1)^{|K|-1}|K|.$$

See [13] and [16] for combinatorial interpretations of $\beta(A)$ for several classes of antimatroids, but especially when A is a finite subset of $\Re^n$.

The β invariant will allow us to interpret the coefficients of the expected rank polynomial. If F is feasible in the antimatroid A, we write $A|F$ for the antimatroid obtained by restriction to F; equivalently, $A|F$ is obtained by deleting the convex set $E - F$.

Proposition 1.7 *Let A be an antimatroid with feasible sets $\mathcal{F}$. Then*

$$ER(A) = \sum_{\emptyset\neq F\in\mathcal{F}} \beta(A|F) \prod_{e\in F} p_e.$$

Proof. For convenience, we write $ER(A) = \sum_{F\in\mathcal{F}} B_F \prod_{e\in F} p_e$ and recall that $\beta(A|F) = \sum_{K\in\mathcal{C}_F}(-1)^{|K|-1}|K|$, where $\mathcal{C}_F$ denotes the free convex sets of $A|F$. We must show $B_F = \beta(A|F)$.

By Proposition 1.4, a feasible set F' will contribute to B_F iff $F \in [F', F'\dot\cup\Gamma(F')]$, i.e., iff $F = F'\dot\cup G$ for some $G \subseteq \Gamma[F']$. But $F = F'\dot\cup G$ for $G \subseteq \Gamma[F']$ iff $F - F' \in \mathcal{C}_F$ is a free convex set of $A|F$. Therefore

$$\begin{aligned} B_F &= \sum_{F':F\in[F',F'\dot\cup\Gamma(F')]} (-1)^{|F-F'|}|F'| \\ &= \sum_{K\in\mathcal{C}_F} (-1)^{|K|}(|F|-|K|) \\ &= \sum_{K\in\mathcal{C}_F} (-1)^{|K|-1}|K| + |F| \sum_{K\in\mathcal{C}_F} (-1)^{|K|} \\ &= \beta(A|F), \end{aligned}$$

where the final equality follows from the fact that $\sum_{K\in\mathcal{C}_F}(-1)^{|K|} = 0$ by Proposition 5 of [18].

□

Formulas similar to those given in Propositions 1.6 and 1.7 hold for ordinary graphs. See Propositions 2.1 and 2.3 of [7].

As an application of Proposition 1.7, we give a very short proof of Proposition 4.6 of [16].

Proposition 1.8 (Proposition 4.6 of [16]) *Let A be an antimatroid on the ground set E with $|E| = n$, and feasible sets $\mathcal{F}$. Then*

$$\sum_{\emptyset\neq F\in\mathcal{F}} \beta(A|F) = n.$$

Proof. Set $p_e = 1$ for all $e \in E$ in $ER(A)$. Then $(ER(A)|_{p_e=1}) = n$, since the expected rank is n when every element is certain to survive. By the formula of Proposition 1.7, we get $\sum_{\emptyset\neq F\in\mathcal{F}} \beta(A|F) = n$.

□

More involved formulas involving $\beta(A)$ also hold under certain conditions. See Corollaries 2.6, 4.7 and 4.9 below.

2 The Tutte polynomial and derivatives

For applications to reliability, it is frequently true that we can assume $p_e = p$ for all e, so the probability that e succeeds does not depend on e. This allows us to obtain a one-variable polynomial, which we denote $er(A;p)$. In this section, we consider this polynomial and the Tutte polynomial.

The *Tutte polynomial* is a two-variable polynomial invariant which has been studied extensively for graphs, matroids and greedoids. See [9] for much more information about its application to graphs and matroids. The Tutte polynomial for antimatroids and greedoids was introduced in [17].

Definition 2.1 *Let A be an antimatroid with rank function r and ground set E. Then the Tutte polynomial is defined by*

$$f(A;t,z) = \sum_{S\subseteq E} t^{r(E)-r(S)} z^{|S|-r(S)}.$$

The Tutte polynomial can also be written as a sum over all feasible sets (instead of all subsets):

Proposition 2.2 (Thm. 2.2 in [15]) *Let A be an antimatroid with ground set E and feasible sets $\mathcal{F}$. Then*

$$f(A;t,z) = \sum_{F\in\mathcal{F}} t^{|E|-|F|}(z+1)^{|E|-|F|-|\Gamma(F)|}.$$

The antimatroid Tutte polynomal also satisfies a deletion-contraction recursion. We write $f(A)$ instead of $f(A;t,z)$ for simplicity.

Proposition 2.3 (Prop. 3.2 of [17]) *Let A be an antimatroid and let $\{e\} \in \mathcal{F}$. Then*

$$f(A) = f(A/e) + t^{r(A)-r(A-e)} f(A-e).$$

This result holds for all greedoids (not just antimatroids), and generalizes the matroid recursion. When the antimatroid is a rooted tree T, the Tutte polynomial completely determines the rooted tree (Theorem 2.8 of [17]): $f(T_1) = f(T_2)$ iff T_1 and T_2 are isomorphic.

The next result connects the one-variable expected rank polynomial $er(A;p)$ with the Tutte polynomial $f(A;t,z)$.

Proposition 2.4 *Assume A is an antimatroid on the ground set E, with $|E| = n$. Then*

$$er(A;p) = np^n f\left(\frac{1-p}{p}, \frac{p}{1-p}\right) - p^{n-1}(1-p)\frac{\partial f}{\partial t}\left(\frac{1-p}{p}, \frac{p}{1-p}\right).$$

Proof. We analyze the two terms separately. By Definition 2.1, $f\left(\frac{1-p}{p}, \frac{p}{1-p}\right) = \sum_{S\subseteq E} p^{|S|-n}(1-p)^{n-|S|}$ since $r(E) = |E| = n$. Thus,

$$np^n f\left(\frac{1-p}{p}, \frac{p}{1-p}\right) = n\sum_{S\subseteq E} p^{|S|}(1-p)^{n-|S|}. \quad (1)$$

For the other term, we have $\frac{\partial f}{\partial t}\left(\frac{1-p}{p}, \frac{p}{1-p}\right) = \sum_{S\subseteq E}(n-r(S))p^{|S|-n+1}(1-p)^{n-|S|}$. Thus,

$$p^{n-1}(1-p)\frac{\partial f}{\partial t}\left(\frac{1-p}{p}, \frac{p}{1-p}\right) = \sum_{S\subseteq E}(n-r(S))p^{|S|}(1-p)^{n-|S|}. \quad (2)$$

Subtracting equation (2) from equation (1) yields $\sum_{S\subseteq E} r(S)p^{|S|}(1-p)^{n-|S|}$, as desired. □

The next result is another formula involving derivatives:

Proposition 2.5 *Let A be an antimatroid in which $r(A-e) = n-1$ for all feasible singletons $\{e\}$. Then $er'(A;1) = n$.*

Proof. We let e be a feasible element and proceed using induction on n. By Proposition 1.6, we have

$$er(A;p) = (1-p) \cdot er(A-e;p) + p \cdot er(A/e;p) + p.$$

Differentiating yields

$$er'(A;p) = (1-p) \cdot er'(A-e;p) - er(A-e;p) + er(A/e;p) + p \cdot er'(A/e;p) + 1.$$

By hypothesis, the antimatroid $A-e$ has no loops, so evaluating at $p = 1$ gives $er(A-e;1) = n-1$. Further, $er(A/e;1) = n-1$ for any antimatroid, and $er'(A/e;1) = n-1$ by induction. □ There

are several classes of antimatroids for which $r(A-e) = n-1$ for all feasible e. For example, finite point sets, chordal graphs, trees, rooted trees (with pruned feasible sets), and posets (using double shelling to define feasible sets) all satisfy this proposition.

We conclude this section by applying Proposition 2.5 to the formula from Proposition 1.7, giving a new relation satisfied by $\beta(A)$. The proof of the next result follows immediately from Proposition 2.5.

Corollary 2.6 *Let A be an antimatroid in which $r(A-e) = n-1$ for all feasible singletons $\{e\}$. Then*

$$\sum_{F \in \mathcal{F}} |F|\beta(A|F) = n.$$

3 A probabilitic expansion of $ER(A)$ and some antimatroid classes

Let A be an antimatroid on the ground set E and let $e \in E$. Assume $S \subseteq E$ is the surviving subset of elements. Now define an indicator random variable $I(e)$ to be 0 or 1 depending on whether or not e contributes to the rank of S. Thus, $I(e) = 1$ if $r(S-e) < r(S)$, and $I(e) = 0$ if $r(S-e) = r(S)$. Write $Pr(e)$ for the probability that $I(e) = 1$. Then $E(I(e)) = Pr(e)$, where $E(I(e))$ is the expected value of the random variable $I(e)$. The linearity of expected value immediately gives the next proposition.

Proposition 3.1

$$ER(A) = \sum_{e \in E} Pr(e).$$

Proposition 3.1 appears explicitly for graphs in the work of Colbourn [11] and Amin, Siegrist and Slater [4, 5, 25, 26].

The remainder of this section is devoted to interpreting the expected rank polynomial for trees (both rooted and unrooted) and posets (which can form an antimatroid in two different ways).

Rooted trees: Let T be a rooted tree with a distinguished vertex, and define an antimatroid on the edge set E so that $F \subseteq E$ is feasible if the edges of F form a rooted subtree (with the same root as T). Then the convex sets are the *complements* of the rooted subtrees. Rooted trees are important in several applications to network design.

The next result appears in [3]. The proof follows immediately from 3.1.

Corollary 3.2 (Theorem 2.4 in [3]) *Let T be a rooted tree. Then*

$$ER(T) = \sum_{v \in V} \prod_{e \in P(v)} p_e.$$

Unrooted trees: When no root vertex is specified in a tree T, we can still define an antimatroid on the edge set E. The feasible sets in this antimatroid are the edges of the *complements* of subtrees of T. The resulting antimatroid is called the *pruning* antimatroid, as the feasible sets are precisely those sets that can be successively *pruned* from the tree, leaving a connected subtree at each step. The convex sets are the subtrees themselves.

When an edge e that is incident to vertices v and w is deleted from a tree T, the tree is separated into two components. Call these components $C_e(v)$ and $C_e(w)$ and note that one of these components will have no edges when e is a leaf of T.

We now use Proposition 3.1 to give a short proof of Theorem 3.3 of [3].

Corollary 3.3 (Theorem 3.3 of [3]) *Let T be an unrooted tree with $|E| = n$. Then*

$$ER(T) = \left(\sum_{e \in E(T)} p_e \left(\prod_{b \in C_e(v)} p_b + \prod_{b \in C_e(w)} p_b \right) \right) - n \prod_{e \in E} p_e.$$

Proof. Let $e \in E$ be an edge of T, and suppose S is the set of edges of T which are operational. Then $I(e) = 1$ iff e is operational and either $S \supseteq C_e(v)$ or $S \supseteq C_e(v)$. ($C_e(v)$ and $C_e(w)$ are precisely the two minimal sets of edges that e requires to be operational in order for $I(e) = 1$.) Let E_S denote the probabilistic event that the edges S are operational. Then

$$\begin{aligned} Pr(e) &= p_e Pr\left(E_{C_e(v)} \vee E_{C_e(w)}\right) \\ &= p_e \left(\prod_{b \in C_e(v)} p_b + \prod_{b \in C_e(w)} p_b \right) - \prod_{e \in E} p_e. \end{aligned}$$

This remains valid even if $C_e(v) = \emptyset$ or $C_e(w) = \emptyset$ (which occurs when e is a leaf), as $Pr(e) = p_e$ in this case. The formula now follows from Proposition 3.1.

□

Partially ordered sets: We briefly review some definitions. Let E be the ground set of the poset P. F is an *order ideal* in P if $x \in F$ and $y \leq x$ implies $y \in F$. G is an *order filter* if $x \in G$ and $y \geq x$ implies $y \in G$. (Some authors refer to order ideals as *downsets* and order filters as *upsets*.)

Posets give rise to antimatroids in at least two ways. The *ideal poset antimatroid* $A_I(P)$ has its feasible sets the order ideals of P. (This antimatroid is simply called the *poset greedoid* in [8] and [21].)

The *double shelling poset antimatroid* $A_D(P)$ has feasible sets

$$\mathcal{F} = \{F \cup G : F \text{ is an ideal and } G \text{ is a filter}\}.$$

These antimatroids arise in the study of bottleneck functions.

Corollary 3.4 *Let P be a poset on the ground set E.*

1.
$$ER(A_I(P)) = \sum_{e \in E} \prod_{a \leq e} p_a.$$

2.
$$ER(A_D(P)) = \sum_{e \in E} p_e \left(\prod_{a<e} p_a + \prod_{b>e} p_b - \prod_{c<e \vee c>e} p_c \right).$$

Proof.

1. Since the feasible sets of $A_I(P)$ are the order ideals, we have $I(e) = 1$ precisely when the operational subset of E contains the order ideal induced by e, i.e., when $\{a : a \leq e\} \subseteq S$. Thus $Pr(e) = \prod_{a \leq e} p_a$, and the result follows from Proposition 3.1.

2. Let F_e be the probabilistic event that the elements less than e are operational and G_e the event that the elements greater than e are operational. Then, as in the proof of 3.3, $Pr(e) = p_e Pr\,(F_e \vee G_e)$, and the rest of this proof is the same as the proof of 3.3.

□

There are striking similarities between the formulas for trees (Corollaries 3.2 and 3.3) and the corresponding formulas for the two antimatroids associated with a poset (Corollary 3.4). The first such similarity between 3.2 and 3.4(1) is not a coincidence; the pruning antimatroid on rooted trees is an example of a ideal poset antimatroid, so the rooted tree formula of 3.2 is simply a special case of the poset formula 3.4(1). For unrooted trees 3.3 and double shelling antimatroids 3.4(2), the correspondence does not have a simple interpretation, since these antimatroids are independent. Nevertheless, $Pr(e) = p_e Pr\,(A \vee B)$ for disjoint events A and B in both cases.

As an example of the differences between the two antimatroids associated with a poset, we compute $ER(A_I(P)$ and $ER(A_D(P)$ for the posets P_1 and P_2 of Figure 1.

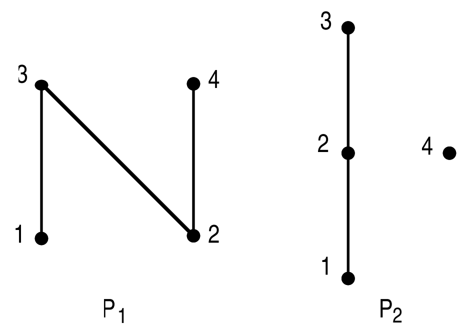


FIGURE 1.

Then $ER(A_I(P_1)) = p_1+p_2+p_1p_2p_3+p_2p_4$ and $ER(A_I(P_2)) = p_1+p_4+p_1p_2+p_1p_2p_3$, but note that the one-variable polynomial $er(A_I(P_1)) = er(A_I(P_2)) = 2p + p^2 + p^3$. This follows from Example 3.1 of [15], which shows that these two posets have the Tutte polynomials, and Proposition 2.4.

For the double shelling posets, we get $ER(A_D(P_1)) = p_1 + p_2 + p_3 + p_4$ (since each element is feasible) and $ER(A_D(P_2)) = p_1 + p_2 + p_4 + p_1p_2 + p_2p_3 - p_1p_2p_3$. In this case, $er(A_D(P_1)) = 4p \neq 3p + 2p^2 - p^3 = er(A_D(P_2))$.

We conclude this section by using Corollary 3.4(2) to compute $\beta(A_D(P))$ for a poset P giving rise to a double shelling antimatroid $A_D(P)$. A *bottleneck* e in a poset is an element which is not maximal or minimal, but is comparable to every element of P.

We now give a short proof of a result of Edelman and Reiner [13].

Proposition 3.5 (Corollary 4.4 of [13]) *Let P be a poset with b bottlenecks and associated double shelling antimatroid $A_D(P)$. Then $\beta(A_D(P)) = -b$.*

Proof. By Proposition 1.7, $\beta(A_D(P))$ is the coeffiecient of $\prod_{e\in P} p_e$ in $ER(A_D(P))$. For $e \in P$, let F_e be the ideal generated by e and G_e be the filter generated by e. By Corollary 3.4(2),

$$Pr(e) = \prod_{a\in F_e} p_a + \prod_{b\in G_e} p_b - \prod_{c\in F_e\cup G_e} p_c,$$

so we have a contribution of (-1) to this coefficient precisely when $F_e \cup G_e = P$, and e is neither maximal nor minimal. (When e is the maximum or minimum element of a poset, then $Pr(e) = p_e$.) But this condition is equivalent to e being a bottleneck in P.

□

4 Finite subsets of the plane

Let A be a finite subset of $\Re^2$, and for $C \subseteq A$, let $\overline{C}$ denote the convex hull of C in $\Re^2$. A has an antimatroid structure that is easiest to describe in terms of its convex sets. In particular, a set $C \subseteq A$ is *convex* if $C = \overline{C} \cap A$. Thus, a set C is not convex in A precisely when some point of $A - C$ is in the convex hull of C.

A point $x \in A$ is *extreme* if $x \notin \overline{A - \{x\}}$. Thus, x is extreme iff $A - \{x\}$ is convex. As usual, $F \subseteq A$ is feasible if $A - F$ is convex; a feasible set F can be built by successively pruning points from A such that each point is extreme at the time it is pruned.

Finite point sets in $\Re^n$ are among the most well-studied classes of antimatroids. Indeed, this class motivated the term 'antimatroid' as the convex closure operator satisfies an 'anti-exchange' axiom (compared with the Birkhoff-Steinitz exchange axiom for closure in matroids) [19]. Antimatroids can be thought of as abstract convexity structures in an analogous way that matroids are abstract dependence structures.

Our interpretation of elements $x \in A$ operating with independent probabilities p_x has an interesting interpretation for this class of antimatroids. If the points of A represent nodes in a network, then we are interested in sending a message from the extreme nodes of A to the internal nodes. Suppose no internal node can receive a message until it becomes 'visible', i.e, it is extreme. Then the expected rank polynomial measures how far the message can penetrate to the interior of the configuration.

For an element $x \in A$ and a subset $G \subseteq A$, we say that x *matters to* G if either $r(G \cup \{x\}) = r(G) + 1$ (if $x \notin G$) or if $r(G - \{x\}) = r(G) - 1$ (if $x \in G$). For a given x, computation of the subsets to which x matters is equivalent to computing $Pr(x)$, which can then be used to compute the polynomial $ER(A)$ by Proposition 3.1.

Lemma 4.1 *Let A be a finite subset of $\Re^n$ and let $x \in A$. Then x matters to a subset G iff and $G \supseteq F$, where F is minimal such that x is extreme in $A - F$.*

Proof. First note that if F is minimal such that x is extreme in $A - F$, then F must be a feasible set. Thus, the minimal sets F_i satisfying x extreme in $A - F_i$ are precisely the minimal feasible sets satisfying $F_i \cup \{x\}$ is also feasible. Now suppose $x \notin G \subseteq A$ has $r(G \cup \{x\}) = r(G) + 1$. Then, by Lemma 1.3, $G \cup \{x\}$ contains a unique maximal feasible set F_G, and $x \in F_G$. Then F_G contains a minimal feasible set F with x extreme in $A - F$. A similar argument holds if $x \in G$, applied to $G - \{x\}$.

Conversely, suppose $G \supseteq F_i$ for some minimal set F_i satisfying x extreme in $A - F_i$. Then $r(F_i \cup \{x\}) = r(F_i) + 1$, so $r(G \cup \{x\}) = r(G) + 1$ (if $x \notin G$) since $G \supseteq F_i$ by Lemmas 1.3 and 1.5. (If $x \in G$, apply this argument to $G - \{x\}$.)

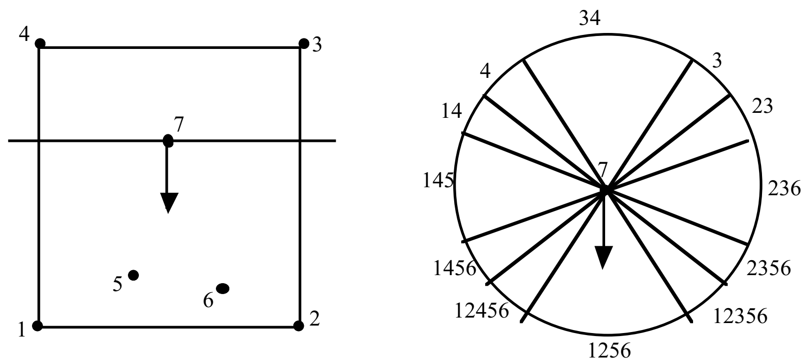
□

Note that when x is extreme, the lemma shows that x matters to all subsets S of A. In $\Re^2$, the minimal feasible sets F_i that matter to an interior point x can be cyclically ordered in a natural way. Since x is extreme in $A - F_i$, there is a line L_i through x and an open half-plane H_i determined by the line L_i such that $F_i = A \cap H_i$. We can associate to each F_i a unit normal vector v_i based at x such that v_i is normal to the line L_i and v_i is contained in the half plane H_i. We now order the F_i cyclically according to the angle v_i makes with a fixed reference line (say the horizontal line through x).

This correspondence is illustrated in Figure 2. In the example, for $x = 7$, we have the following minimal feasible sets, in counterclockwise cyclic order: $F_1 = \{3\}$, $F_2 = \{4\}$,

$F_3 = \{1, 2, 5, 6\}$. Note that these sets are disjoint (which is not true in general) and that only 3 of the 12 feasible sets determined by half-planes are minimal.

This association of a half-space with each minimal feasible set F_i remains valid in $\Re^n$ for $n > 2$, but it is no longer meaningful to order the sets cyclically.



Minimal feasible set $F_3 = \{1, 2, 5, 6\}$. All half-plane feasible sets for $x = 7$.

FIGURE 2.

If x is not an extreme or interior point of A, then x is on the boundary of A. In $\Re^2$, this means x is on a line segment L between two extreme points r_x and s_x. Let R_x and S_x be the intersections of the half-open segments $[r_x, x)$ and $(x, s_x]$ with A, so that $R_x \cup S_x = L - \{x\}$ and $R_x \cap S_x = \emptyset$.

For $S \subseteq A$, we write $Pr(S)$ to represent the probability the set S is operational. For $S, T \subseteq A$, we also write $Pr(S \vee T)$ to represent the probability that the elements of S or T are operational.

Lemma 4.2 *Let A be a finite subset of $\Re^2$, and write $p_S = \prod_{x \in S} p_x$.*

1. *If x is extreme, then*
$$Pr(x) = p_x;$$

2. *If x is not extreme but x is not an interior point of A, then*
$$Pr(x) = p_x(p_{R_x} + p_{S_x}) - p_L,$$
where $R_x = [r_x, x) \cap A$, $S_x = (x, s_x] \cap A$ and L is the boundary line segment containing x;

3. *If x is an interior point of A and $F_1, \ldots, F_k$ are the minimal feasible sets, ordered cyclically, with x extreme in $A - F_i$, then*
$$Pr(x) = p_x \left(\sum_{i=1}^{k} p_{F_i} - \sum_{i=1}^{k} p_{F_i \cup F_{i+1}} \right) + p_A,$$
where the subscripts are computed modulo k.

Proof.

1. If x is extreme, then x matters to all subsets, so x will increase the rank of any subset iff x is operational, i.e., $Pr(x) = p_x$. (This argument is valid for any feasible singleton in any antimatroid.)

2. If x is a boundary point that is not extreme, then x matters to all subsets which contain R_x or S_x. Thus, $Pr(x) = p_x Pr(R_x \vee S_x) = p_x(p_{R_x} + p_{S_x}) - p_L$, as in the proof of Corollary 3.4(2).

3. Arguing as in case (2), we have $Pr(x) = p_x Pr(F_1 \vee F_2 \vee \cdots \vee F_k)$. To compute $Pr(F_1 \vee F_2 \vee \cdots \vee F_k)$, we must show that every set containing some F_i is counted precisely once by the expression

$$p_x \left(\sum_{i=1}^{k} p_{F_i} - \sum_{i=1}^{k} p_{F_i \cup F_{i+1}} \right) + p_A. \tag{*}$$

Let $S \subseteq A$. Then each set $F_i \subseteq S$ accounts for S (with a coefficient of $+1$ for each such set) and each consecutive pair $F_i \cup F_{i+1} \subseteq S$ accounts for S (with a coefficient of -1 for each such pair and where subscripts are computed modulo k). There are several cases to consider:

Case 1. S does not contain F_i for any i, $1 \le i \le k$. Then x does not matter to S (by Lemma 4.1), so S does not contribute to $Pr(x)$. Since S contains no F_i, S will not contribute to (*), either, so S is not counted by (*).

For the remaining cases, $S \supseteq F_i$ for some i, $1 \le i \le k$. Then, by Lemma 4.1, x matters to S, so we must show S is accounted for precisely once in (*).

Case 2. $S = A$. Then S is counted by every term in (*), so each of the k terms p_{F_i} contributes $+1$, each of the k terms $p_{F_i \cup F_{i+1}}$ (computed modulo k) contributes -1, and the term p_A contributes $+1$. Thus, S is accounted for precisely once.

Case 3. $S \supseteq F_i$ for some i, but $S \ne A$. Let $y \notin S$ and renumber the indices of the minimal feasible sets (if necessary) so that $y \in F_1$. (Every element of $A - \{x\}$ is in at least one minimal feasible set F_i.) Continue the renumbering in counterclockwise cyclic order. Then $F_1 \not\subseteq S$. Let a and b be smallest and largest integers (resp.) such that $F_a \subseteq S$ and $F_b \subseteq S$.

Claim $F_c \subseteq S$ for all c with $a \le c \le b$. (By definition of a and b, $F_i \not\subseteq S$ for $i < a$ and $i > b$.) To see why the claim is true, let c be some integer between a and b. Then if $a = b$ or $a + 1 = b$, there is nothing to prove, so assume $b - a > 1$. Let H_a and H_b be the half-planes associated to F_a and F_b (resp.), as in Fig. 3. Then $H_c \subseteq H_a \cup H_b$ is clear. But $F_i = H_i \cap A$ for all i, so $F_c \subseteq F_a \cup F_b$. Thus, $S \supseteq F_a \cup F_b \supseteq F_c$ for all c between a and b.

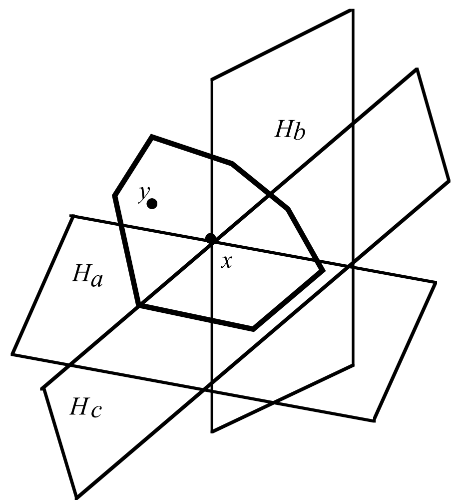


FIGURE 3.

To finish the proof of case 3, now note that S is accounted for in $b-a+1$ terms of the form p_{F_c} for c such that $a \le c \le b$ (with coefficient $+1$), and $b-a$ terms of the form $p_{F_i \cup F_{i+1}}$ for c such that $a \le c \le b-1$ (having coefficient -1). This completes the accounting for S.

□

For example, in the configuration of Fig. 2, we have $Pr(7) = p_3p_7 + p_4p_7 + p_1p_2p_5p_6p_7 - p_3p_4p_7 - p_1p_2p_3p_5p_6p_7 - p_1p_2p_4p_5p_6p_7 + p_A$. Using Proposition 3.1 then gives the entire polynomial as $ER(A) = \sum_{x \in E} Pr(x)$.

As an immediate application, we can use Lemma 4.2 to give a very short proof of the main theorem of [1], which characterizes the beta invariant for finite subsets of the plane. (This proof does not extend to higher dimensions; see Theorem 1.1 of [13].)

Corollary 4.3 (Theorem 4.1 of [1]) *Let A be a finite subset of $\Re^2$, and let $int(A)$ denote the set of interior points of A. Then*

$$\sum_{K \in Free} (-1)^{|K|-1} |K| = |int(A)|.$$

Proof. By Proposition 1.7, $\beta(A)$ is the coefficient of the monomial p_A in $ER(A)$. Let c_x be the coefficient of p_A in the polynomial $Pr(x)$. By Proposition 3.1, $\beta(A) = \sum_{x \in A} c_x$. Now, by Lemma 4.2, we have $c_x = 1$ if $x \in int(A)$ and $c_x = 0$ otherwise. This gives $\beta(A) = |int(A)|$, and the formula given follows from the definition of $\beta(A)$.

□

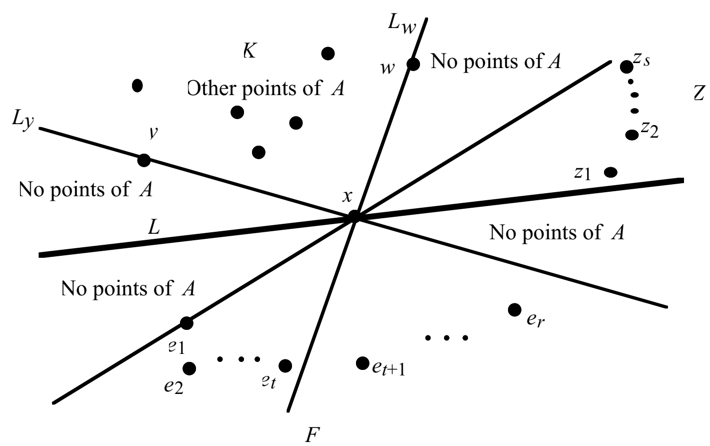
The next lemma is the key to understanding the structure of $ER(A)$ when the points of A are in general position in the plane. In this situation, the sets $F_i \cup F_{i+1}$ are simply the complements of some other minimal feasible set F_j. As a result, the resulting polynomial satisfies a striking symmetry condition.

Theorem 4.4 *Let A be a finite subset of $\Re^2$ with no three points on a line. Let $x \in A$ be an interior point. Let $F_1, \dots, F_k$ be the minimal feasible sets with x extreme in $A - F_i$. Then*

$$Pr(x) = p_x \left(\sum_{i=1}^{k} p_{F_i} \right) - \sum_{i=1}^{k} p_{A-F_i} + p_A.$$

Proof. We must show, for i given, that $A - (F_i \cup F_{i+1}) = F_j$ for some j, and conversely, $A - F_i = F_m \cup F_{m+1}$ for some m. (As usual, all subscripts are between 1 and k and are computed modulo k.). The result then follows by Lemma 4.2(3).

We first show that $A - (F_i \cup F_{i+1})$ is a minimal feasible set. Suppose F is a minimal feasible set, and let L be a corresponding line through x, and H the half-space corresponding to L. Order the points of $F = \{e_1, \dots, e_r\}$ so that they are encountered in this order as L rotates counterclockwise about x through the region determined by F.



Configuration for the proof of Theorem 4.4.
FIGURE 4.

Now rotate L clockwise about x, and let $y \in A$ be the first point of $A - F$ that L meets. Note that L will meet y prior to meeting any points of F, since F is minimal and no three points are collinear in A. Call this rotated line L_y.

Now we rotate L counterclockwise about x. As L sweeps through the region determined by the angle $\angle yxe_1$, L must meet some points of $A - F$ (by minimality of F). Let $Z = \{z_1, \dots, z_s\}$ be this set of points, listed in the order they are encountered.

Continue rotating counterclockwise, and let w be the first point of $A - F$ met. Let L_w be the line through x and w. Then, as the line sweeps through the angle $\angle z_s xw$, it passes through some points of F, say $e_1, \dots, e_t$ for some t with $1 \le t \le r$. The configuration is shown in Fig. 4.

Then $F - \{e_1\} \cup Z$ must contain some minimal feasible set G with x extreme in $A - G$. Note that x is interior to the triangle $\triangle e_1 y z_i$ for all $1 \le i \le s$, so $G \supseteq Z$. By definition of w, we must have $G = \{e_{t+1}, \dots, e_r\} \cup Z$. Clearly, F and G are consecutive minimal feasible sets.

Then $K = A - (F \cup G)$ is also a minimal feasible set with x extreme in $A - K$. To see this, first note there is a line L' through x separating $F \cup G$ from the rest of A. (We can construct L' by rotating clockwise by a small angle the line through e_1 and x.) Then, in rotating L' either clockwise or counterclockwise, we will encounter points of $F \cup G$ before meeting any y or w, which ensures minimality of K.

For the converse, let F be the same minimal feasible set as before, and rotate L clockwise until it passes through e_r, then rotate counterclockwise by a small angle, and call this line L'. Let H' be the half-plane containing w that is determined by L'. Then $y \notin H'$ and $Z \subseteq H'$. Arguing as above, we can find a minimal feasible set G' with x extreme in $A - G'$ and $Z \subseteq G' \subseteq H' \cap A$. Then G' and K are consecutive minimal feasible sets, and $G' \cup K = A - F$, as desired.

□

For example, in the configuration of Fig. 2, we have

$$\begin{aligned} ER(A) \;=\; & p_1 + p_2 + p_3 + p_4 + p_1p_5 + p_2p_6 + p_3p_7 + p_4p_7 + \\ & p_1p_5p_6 + p_2p_5p_6 - p_3p_4p_7 - 2p_1p_2p_5p_6 + p_3p_4p_5p_7 + p_3p_4p_6p_7 + \\ & p_1p_2p_5p_6p_7 - p_1p_3p_4p_5p_7 - p_2p_3p_4p_6p_7 - p_1p_2p_3p_5p_6p_7 - \\ & p_1p_2p_4p_5p_6p_7 - p_1p_3p_4p_5p_6p_7 - p_2p_3p_4p_5p_6p_7 + 3p_A. \end{aligned}$$

Note that the coefficient of $p_1p_2p_5p_6$ is -2. We can interpret this coefficient in two ways. From the viewpoint of Theorem 4.4, the set $\{3,4,7\}$ is a minimal feasible set for $x = 5$, so $A - \{3,4,7\} = \{1,2,5,6\}$ contributes a coefficient of -1 to the coefficient of $p_1p_2p_5p_6$. But $\{3,4,7\}$ is also a minimal feasible set for $x = 6$, so a contribution of -1 arises from this set, too.

From the viewpoint of Proposition 1.7, we have $\beta(A|\{1,2,5,6\}) = -2$. This interpretation is a bit more difficult to understand geometrically, since the free convex sets in the restriction $A|\{1,2,5,6\}$ depend on the position of the elements of $A - F$. One consequence of this interpretation is that $\beta(A|F)$ can take on any (positive or negative) integer value.

We conclude this section by giving several results for the reduced one-variable polynomial $er(A)$ obtained from $ER(A)$ by setting each $p_x = p$. As usual, this corresponds to the situation when each element has the same probability of success.

Corollary 4.5 *Let A be a subset of n points in the plane with no three points collinear, and write $er(p) = \sum_{i=1}^{n} a_i p^i$. Then*

1. *$a_1 =$ the number of extreme points;*

2. *$a_n =$ the number of interior points;*

3. *$a_i = -a_{n+1-i}$ for all i with $1 < i < n$.*

Proof.

1. From Lemma 4.2, we have $Pr(x)$ includes the term p_x iff x is extreme. The result follows immediately.

2. This is Theorem 4.1 of [1] (see Corollary 4.3).

3. Let x be an interior point. By Theorem 4.4, each minimal feasible set F with x extreme in $A - F$ gives rise to two terms in $Pr(x)$; $p_x p_F$ and $-p_{A-F}$. But, if $|F| = m$, then $p_x p_F$ has degree $m+1$, while $-p_{A-F}$ has degree $n-m$. The result now follows from Proposition 3.1.

□

This result is consistent with the observation that $er(1) = n$, since, for points in general position, every point is either interior (and thus contributes to the coefficient of p^n) or exterior (and so contributes to the coefficient of p), and the coefficients of the other terms of $er(p)$ cancel in pairs. This also implies, for example, that if n is odd, then the coefficient of $p^{\frac{n+1}{2}}$ must be zero.

We note that the antimatroid operation of deletion is troublesome for finite point sets. While $A - x$ is a well-defined antimatroid, it is not possible to associate a finite point set S to $A - x$ so that the feasible sets of $A - x$ and S coincide (so the class of finite point sets is not closed under the antimatroid deletion operation). Nevertheless, we can still use deletion and contraction (which causes no problems for antimatroids in general), provided $A - x$ is correctly interpreted. A more complete discussion of this problem and various solutions appears in [1].

The next result is analogous to Proposition 2.5.

Proposition 4.6 *Let A be a subset of n points in the plane with no three points collinear. Then*

$$er''(A;1) = 0.$$

Proof. Let x be an extreme point of A. We take the second derivative of both sides of the deletion-contraction recursion given in Proposition 1.6 and use induction. This gives

$$er''(A;p) = 2er'(A/x) + pre''(A/x) - 2er'(A-x) + (1-p)er''(A-x).$$

Now $er'(A/x;1) = n-1$ by Proposition 2.5 applied to A/x. Further, the same result applied to $A - x$ gives $er'(A-x;1) = n-1$. (The hypothesis that no three points of A are collinear ensures that $r(A-x-y) = n-2$ for all y, as required by 2.5.)

Finally, we have $er''(A/x) = 0$ by induction. Putting the pieces together gives the result.

□

This result is false for points which are not in general position. For example, if A is the 5-point configuration formed by the corners a square, together with its barycenter, we have $er(A) = 4p + 4p^3 - 4p^4 + p^5$, which gives $er''(1) = -4$.

As a corollary of Proposition 4.6, we get another formula involving the beta invariant. We omit the straightforward proof.

Corollary 4.7 *Let A be a subset of n points in the plane with no three points collinear. Then*

$$\sum_{F \in \mathcal{F}} |F|(|F|-1)\beta(A|F) = 0.$$

The next result is our final evaluation of $er(p)$.

Corollary 4.8 *Let A be a subset of n points in the plane with no three points collinear. If n is odd, then $er(-1) = -n$.*

Proof.

Write $er(p) = \sum_{i=1}^{n} a_i p^i$ and note that k and $n+1-k$ have the same parity for all k. The result follows from Corollary 4.5.

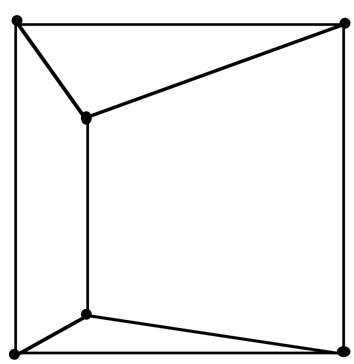
□

This result also gives rise to another identity involving the beta invariant.

Corollary 4.9 *Let A be a subset of n points in the plane with no three points collinear. If n is odd, then*

$$\sum_{\emptyset \neq F \in \mathcal{F}} (-1)^{|F|}\beta(A|F) = -n.$$

Corollary 4.8 is false for even n. As an example, consider the 6-point configuration of Fig. 5. Then $er(p) = 4p + 3p^2 + p^3 - p^4 - 3p^5 + 2p^6$, so $er(-1) = 2$.



A counterexample to Corollary 4.8 for even n.
FIGURE 5.

5 Directions for future research

We conclude by indicating a few possible research projects based on this work.

5.1 Other classes of antimatroids

There are several important classes of antimatroids that we did not consider in this paper. For example, simplicial shelling in chordal graphs induces an antimatroid structure on the vertices. For this class, there is a charcterization of $\beta(A)$ [16]; it is possible that a detailed examination of the structure of $ER(A)$ or the one-variable evaluation $er(A)$ for this class could give information about the combinatorial significance of $\beta(A|F)$ for feasible sets F.

Other classes of interest include vertex search in graphs and digraphs, edge search in graphs and vertex pruning in trees. The search antimatroids have some interesting polynomial invariants, studied in [22]. Edge pruning in trees and rooted trees is treated in [3].

5.2 Uniform expected rank and integrals

When p is uniformly distributed, it makes sense to compute the expected rank as a real number. In [2], the following operation is introduced:

$$EV(A) = \int_0^1 er(A)dp.$$

For example, for the configuration of Fig. 2, we have $EV(A) = \int_0^1 4p + 4p^2 + p^3 - p^5 - 4p^5 + 3p^7 dp \approx 3.22\ldots$. This corresponds to the situation when there is no information about the distribution of p considered as a random variable.

It would be interesting to explore this real invariant as a combinatorial exercise. In particular, among all n-point configurations having k interior points, what configuration maximizes the integral? Does moving a point toward the boudary of the configuration always increase this value? Are there two configurations A_1 and A_2 on the same number of points with $EV(A_1) = EV(A_2)$, but $A_!$ and A_2 not combinatorially equivalent?

It should also be interesting to apply this invariant to some of the other classes of antimatroids mentioned above. Trees and rooted trees are considered in [3], and rooted graphs are treated in [6] (although the edges of general rooted graphs do not form antimatroids).

5.3 Other probabilistic distributions

For 'real-world' applications, the assumption of uniform distribution on p is almost surely wrong. It makes more sense to assume some density function $f(p)$ on $[0, 1]$, and then compute $EV(A; f) = \int_0^1 er(A)f(p)dp$. For example, the *Beta* distribution gives a 2-parameter family (specifying the mean and standard deviation). See [10] (or any standard text on statistics) for descriptions of this distribution and others. A serious study of this topic should include real data on the reliability of components in the system being modeled.

5.4 Finite subsets in higher dimensions

In Sec. 4, we concentrated on planar configurations. It would be interesting to extend the characterizations of $Pr(x)$ given in Lemma 4.2 and Theorem 4.4 to higher dimensions. For example, the main theorem (Theorem 1.1) of [13] extends Corollary 4.3 to higher dimensions, proving a conjecture of [3] (also independently proven by Klain in [20], and extended to oriented matroids in [14]). It may be possible to give a relatively short proof of this general result (similar to the proof we give of Corollary 4.3).

This approach may be promising, since it is straightforward to generalize parts (1) and (2) of Lemma 4.2 to any dimension. In particular, if we could show that $Pr(x)$ contributed a coefficient of $(-1)^n$ whenever x is interior, the proof would follow immediately from Proposition 3.1 and the generalization of parts (1) and (2) of 4.2. Such a proof would require a deeper understanding of the geometry of minimal feasible sets in $\Re^n$.

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